Power Spectral Density (PSD)

- In communication systems, we have to know how the transmitted power is distributed over the frequencies.

- The question is what the power distribution of the signal $x(t)$ is.

Parseval's Theorem

Consider the function $y(t)=x(t) \times x(t)$. By the convolutional theorem , the Fourier transform of $y(t)$ is $X(f)*X(f)$; that is

$$\int_{-\infty}^{\infty} x^2(t)e^{-j2\pi\tau t}dt = \int_{-\infty}^{\infty} X(f)X(\tau - f)df$$

Setting τ=0 in the above expression yields

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} X(f)X(-f)df = \int_{-\infty}^{\infty} |X(f)|^2 df \qquad (1)$$

Proof:

Let $X(f) = R(f) + j\,I(f)$, then $X(-f) = R(-f) + j\,I(-f)$. Since $x(t)$ is real, $R(f)$ is even and $I(f)$ is odd. Consequently, $R(-f) = R(f)$ and $I(-f) = -I(f)$; and $X(-f) = R(f) - j\,I(f)$.
Hence, $X(f)\,X(-f) = R^2(f) + I^2(f)$ which is the square of the Fourier spectrum $|X(f)|$.

Equation (1) is the Parseval's theorem; it states that the energy in a waveform $x(t)$ computed in the time domain must equal the energy of $X(f)$ as computed in the frequency domain.

Case 1: Suppose $x(t)$ is time-limited, $\frac{-T}{2} \le t \le \frac{T}{2}$. Then, (1) becomes as

$$\int_{\frac{-T}{2}}^{\frac{T}{2}} x^2(t)dt = \int_{-\infty}^{\infty} |X(f)|^2 df \qquad (2)$$

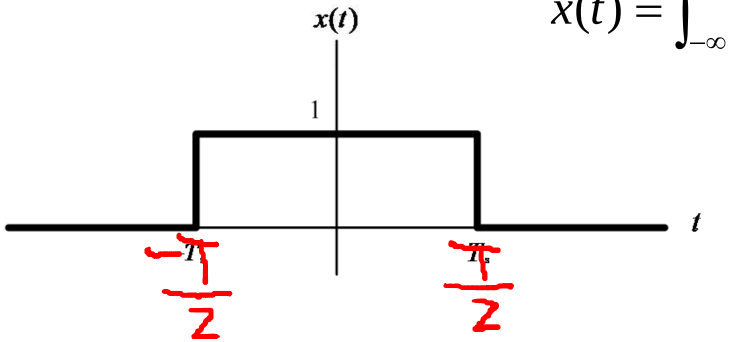
i.e. $\frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} x^2(t)dt = \int_{-\infty}^{\infty} \frac{1}{T}|X(f)|^2 df$

$\Rightarrow$ average power of $x(t)$: $P_x = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} x^2(t)dt = \int_{-\infty}^{\infty} \frac{1}{T}|X(f)|^2 df$

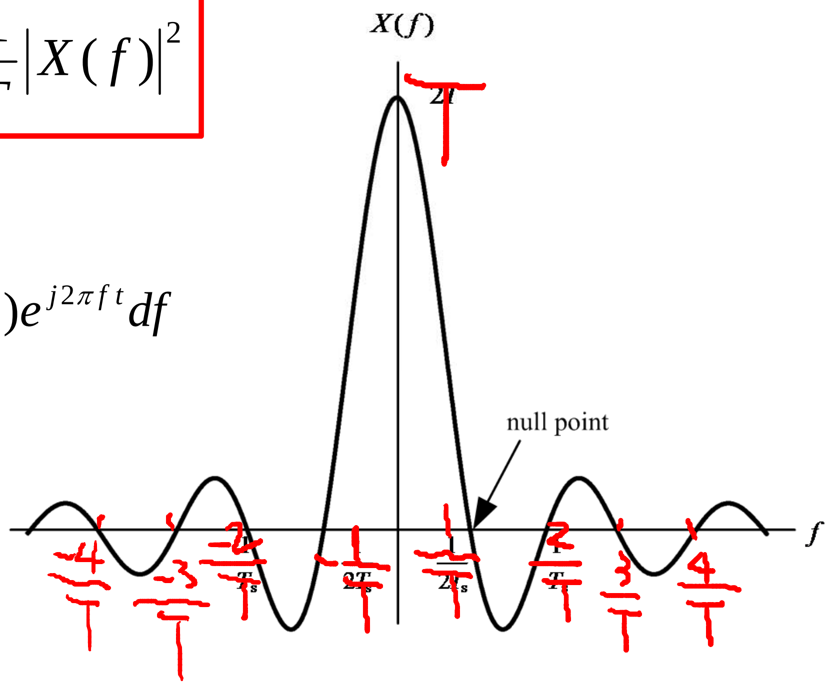
Hence, the PSD of $x(t)$ is defined as $P_x(f) = \frac{1}{T}|X(f)|^2$

i.e. $P_x = \int_{-\infty}^{\infty} P_x(f)df$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi f t}df$$



(a)



(b)

Example 1: *T*=2

$$X(f)=\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt=\int_{-1}^{1} e^{-j2\pi ft}dt \qquad X(0)=2$$

$$x(t)=\begin{cases}1 & -1\le t\le 1\\ 0 & \text{others}\end{cases}$$

$$=\frac{1}{-j2\pi f}\left[e^{-j2\pi f}-e^{j2\pi f}\right]$$

$$=\frac{\sin(2\pi f)}{\pi f}=2\,\text{sinc}(2f) \qquad \text{sinc}(x)=\frac{\sin(\pi x)}{\pi x}$$

$$\int_{-\infty}^{\infty}\text{sinc}^2 x dx=1$$

the average power of $x(t)$: $P_x=\frac{1}{2}\int_{-1}^{1}x^2(t)dt=1$

the PSD of $x(t)$: $P_x(f)=\frac{1}{2}\left|X(f)\right|^2=2\,\text{sinc}^2(2f)$

$$\Rightarrow 1=P_x=\int_{-\infty}^{\infty}P_x(f)df=2\int_{-\infty}^{\infty}\text{sinc}^2(2f)df=2\times\frac{1}{2}\int_{-\infty}^{\infty}\text{sinc}^2(x)dx=1$$

Example 1: T=2

$$X(f)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt=\int_{-1}^{1}e^{-j2\pi ft}dt \qquad X(0)=2$$

$$x(t)=\begin{cases}1 & -1\le t\le 1\\ 0 & \text{others}\end{cases}$$

$$=\frac{1}{-j2\pi f}\left[e^{-j2\pi f}-e^{j2\pi f}\right]$$

$$=\frac{\sin(2\pi f)}{\pi f}=2\,\mathrm{sinc}(2\pi f) \qquad \mathrm{sinc}(x)=\frac{\sin(\mathrm{x})}{\mathrm{x}}$$

$$\int_{-\infty}^{\infty}\mathrm{sinc}^2\,x\,dx=\pi$$

the average power of $x(t)$: $P_x=\frac{1}{2}\int_{-1}^{1}x^2(t)dt=1$

the PSD of $x(t)$: $P_x(f)=\frac{1}{2}\left|X(f)\right|^2=2\,\mathrm{sinc}^2(2\pi f)$

$$\Rightarrow 1=P_x=\int_{-\infty}^{\infty}P_x(f)df=2\int_{-\infty}^{\infty}\mathrm{sinc}^2(2\pi f)df=2\times\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{sinc}^2(x)dx=1$$

http://www.wolframalpha.com/input/?i=plot+sin%28x%29

WolframAlpha computational knowledge engine

integral sin(2*pi*f)**2/(pi*f)**2 for f=-inf to inf

Examples Random

Definite integral:

$$\int_{-\infty}^{\infty} \frac{\sin^2(2\,\pi\,f)}{(\pi\,f)^2}\,d\,f = 2$$

WolframAlpha computational knowledge engine

integral sinc(x)**2 for x=-inf to inf

Examples Random

Definite integral: More digits

$$\int_{-\infty}^{\infty} \operatorname{sinc}(x)^2\,d\,x = \pi \approx 3.14159$$

$\operatorname{sinc}(x)$ is the sinc function http://en.wikipedia.org/wiki/Sinc_function

In mathematics, physics and engineering, the **cardinal sine function** or **sinc function**, denoted by $\mathrm{sinc}(x)$, has two slightly different definitions.[1]

In mathematics, the historical **unnormalized sinc function** is defined by

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}.$$

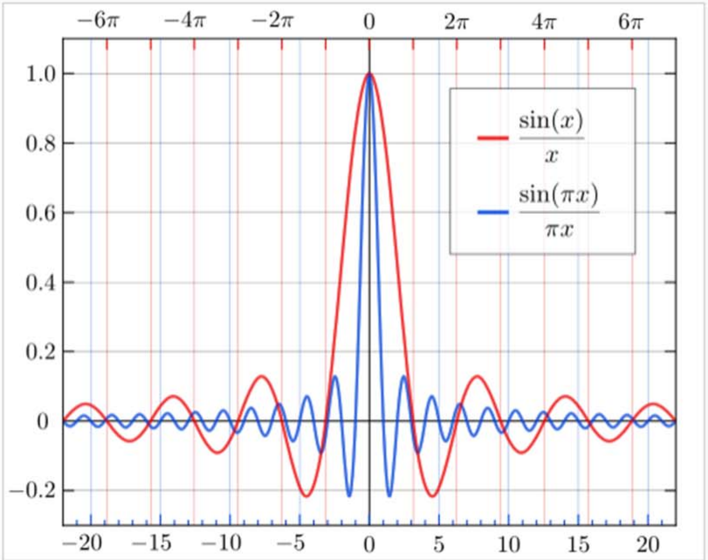
In digital signal processing and information theory, the **normalized sinc function** is commonly defined by

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$

In either case, the value at $x = 0$ is defined to be the limiting value $\mathrm{sinc}(0) = 1$.

The normalization causes the definite integral of the function over the real numbers to equal 1 (whereas the same integral of the unnormalized sinc function has a value of π). As a further useful property, all of the zeros of the normalized sinc function are integer values of x. The normalized sinc function is the Fourier transform of the rectangular function with no scaling. This function is fundamental in the concept of reconstructing the original continuous bandlimited signal from uniformly spaced samples of that signal.

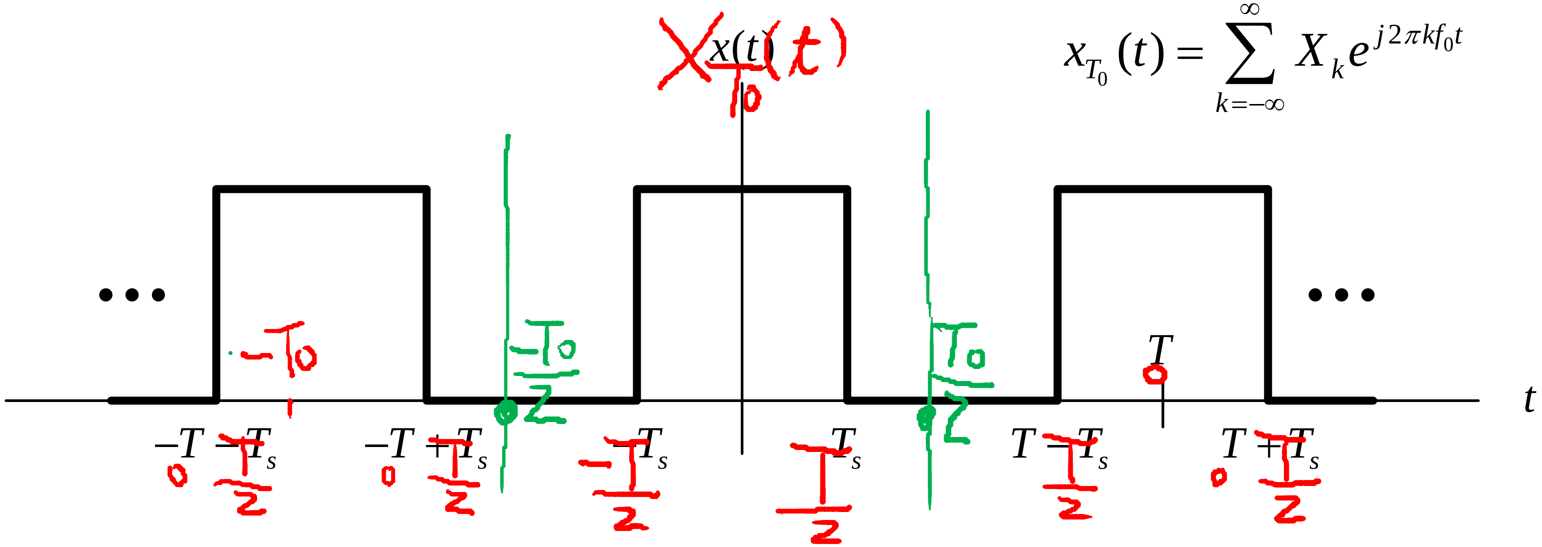
The only difference between the two definitions is in the scaling of the independent variable (the x-axis) by a factor of π. In both cases, the value of the function at the removable singularity at zero is understood to be the limit value 1. The sinc function is analytic everywhere.

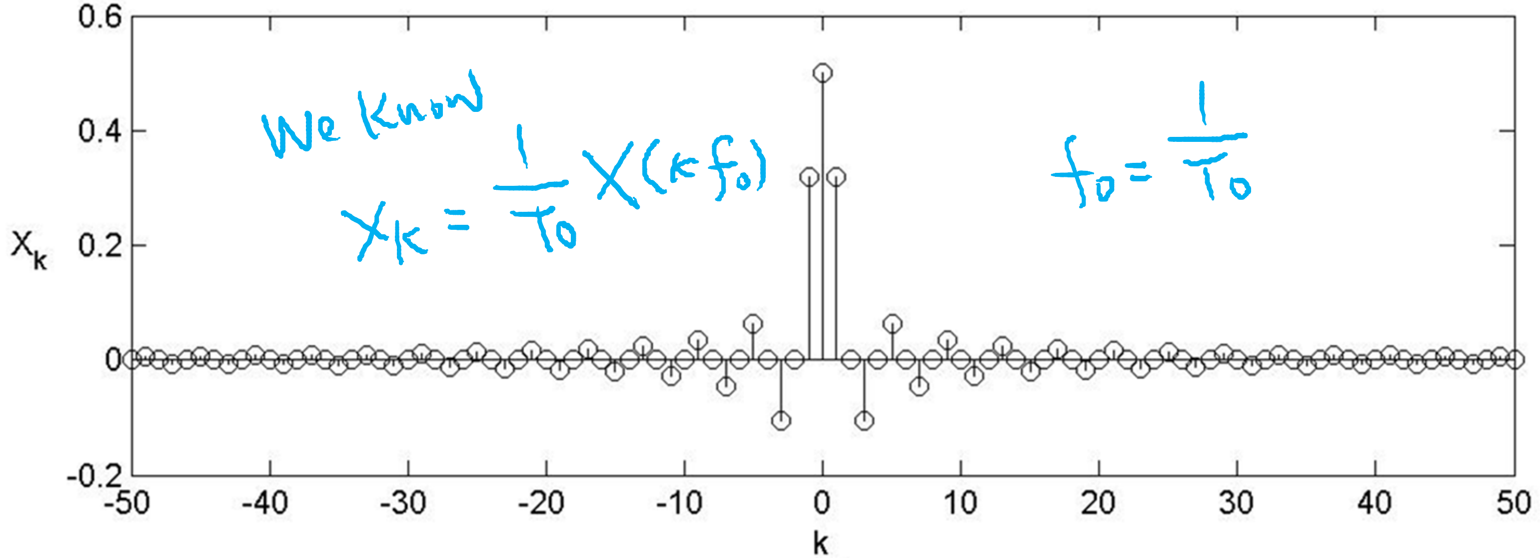


The normalized sinc (blue) and unnormalized sinc function (red) shown on the same scale.

Case 2: Suppose $x_{T_0}(t)$ is a periodic function with period T_0. (time-unlimited)

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k f_0 t}$$





$T_0 = 2T$

Since the Fourier transform of $x_{T_0}(t)$ becomes as a Fourier series.

The power of $x_{T_0}(t)$ can be obtained directly from Equation (3) by sampling the frequency (i.e. just taking those frequencies kf_0)

$$\int_{-\infty}^{\infty} x_{T_0}^2(t)dt = \int_{-\infty}^{\infty} |X_{T_0}(f)|^2 df \quad (3)$$

$$\sum_{k=-\infty}^{\infty} |X(kf_0)|^2 \times f_0 = \sum_{k=-\infty}^{\infty} T_0^2 |X_k|^2 \times f_0 = \sum_{k=-\infty}^{\infty} T_0 |X_k|^2 \qquad (X_k = \frac{1}{T_0} X(kf_0))$$

Then, we can derive that the average power ($P_{x_{T_0}}$) of $x_{T_0}(t)$ is equal to the summation of the square of all Fourier spectrums (coefficients $|X_K|$):

$$\frac{1}{T_0}\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} x_{T_0}^2(t)dt = \frac{1}{T_0}\int_{\frac{-T}{2}}^{\frac{T}{2}} x_{T_0}^2(t)dt = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} |X(kf_0)|^2 \times f_0 = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} T_0 |X_k|^2 = \sum_{k=-\infty}^{\infty} |X_k|^2$$

Parseval's Theorem for Fourier series

Hence, the PSD of $x_{T_0}(t)$ is defined as $P_{x_{T_0}}(k) = |X_k|^2$

i.e. $P_{x_{T_0}} = \sum_{k=-\infty}^{\infty} P_{x_{T_0}}(k) = \sum_{k=-\infty}^{\infty} |X_k|^2 = \frac{1}{T_0}\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} x_{T_0}^2(t)dt$

Note: The PSD of $x_{T_0}(t)$ is also defined as $P_{x_{T_0}}(f) = \sum_{k=-\infty}^{\infty} |X_k|^2 \delta(f - kf_0)$

i.e. $\int_{-\infty}^{\infty} P_{x_{T_0}}(f)df = \sum_{k=-\infty}^{\infty} |X_k|^2$

Example 2: $x_{T_0}(t) = A\cos(2\pi f_0 t) \qquad T_0 = \frac{1}{f_0}$

The average power of $x_{T_0}(t)$

$$\frac{1}{T_0}\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} A^2 \cos^2(2\pi f_0 t)dt = \frac{1}{T_0}\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} A^2 \frac{1+\cos(4\pi f_0 t)}{2}dt = \frac{A^2}{2}$$

$$X_{T_0}(f) = \frac{A}{2}\delta(f + f_0) + \frac{A}{2}\delta(f - f_0)$$

i.e. $X_{-1} = \frac{A}{2}$ and $X_1 = \frac{A}{2}$ $\qquad \Rightarrow (X_{-1})^2 + (X_1)^2 = \frac{A^2}{2}$

Note:

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_{T_0}^2(t)dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} x_{T_0}^2(t)dt = \sum_{k=-\infty}^{\infty} |X(kf_0)|^2 \times f_0$$

Now consider

$T_0 \to \infty$

$$\lim_{T_0\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_{T_0}^2(t)dt = \lim_{T_0\to\infty} \sum_{k=-\infty}^{\infty} |X(kf_0)|^2 \times f_0$$

The above equation becomes as follows :

(case 1 : time-limited)

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt = \lim_{T_0\to\infty} \sum_{k=-\infty}^{\infty} |X(kf_0)|^2 \times f_0 \overset{kf_0 \to f\,;\, f_0 \to df}{=} \int_{-\infty}^{\infty} |X(f)|^2\, df \quad (2)$$

Parseval's Theorem for Fourier transform

Case 3: Suppose $y(t)$ is a general time-unlimited and non-periodic function

Let $y_T(t)$ be a truncated signal of $y(t)$ given by:

$$y_T(t) = \begin{cases} \mathrm{y}(t) & -T/2 \le t \le T/2 \\ 0 & \text{otherwise} \end{cases}$$

Hence, the average power of the signal $y(t)$ is defined as

$$P_y = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| y_T(t) \right|^2 dt.$$

and

the PSD for $y(t)$ is defined as

$$P_y(f) = \lim_{T\to\infty} \frac{1}{T} | Y_T(f) |^2$$

where $Y_T(f)$ is the Fourier transform of $y_T(t)$

$$P_y = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| y_T(t) \right|^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \left| Y_T(f) \right|^2 df = \int_{-\infty}^{\infty} \left(\lim_{T\to\infty} \frac{\left| Y_T(f) \right|^2}{T} \right) df = \int_{-\infty}^{-\infty} P_y(f) df.$$