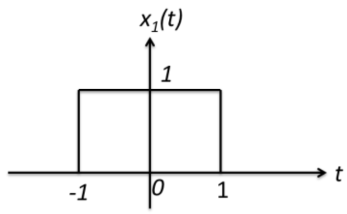
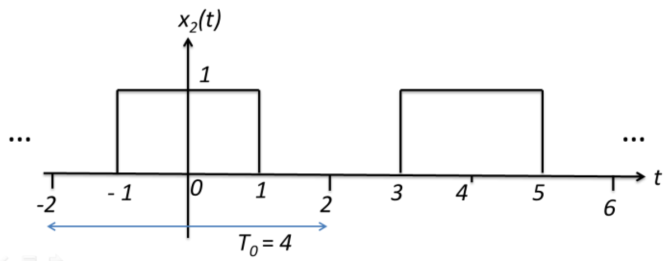
暨大資工系通訊導論　期末考　104.1.16

1. (30%)
   (a) Find the Fourier transform of $x_1(t)$.
   (b) Find the Fourier series of $x_2(t)$.
   (c) Show pictorially that $x_2(t) = \int_{-\infty}^{\infty} \frac{1}{\pi f} \sin(2\pi f) e^{j2\pi f t} df$ as $T \to \infty$, from the viewpoint of *Riemann integral*.





Solutions:

(a)

$$X_1(f) = \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi ft} dt = \int_{-T_s}^{T_s} e^{-j2\pi ft} dt$$

$$= \frac{1}{-j2\pi f}\left[e^{-j2\pi fT_s} - e^{j2\pi fT_s}\right]$$

$$= \frac{\sin(2\pi fT_s)}{\pi f} = 2T_s \operatorname{sinc}(2fT_s); \text{ where } \operatorname{sinc}(x) \triangleq \sin(\pi x)/\pi x$$

$$X_1(0) = \frac{[\sin(2\pi fT_s)]'}{[\pi f]'} = \frac{2\pi T_s \cos(2\pi fT_s)}{\pi} = 2T_s. \text{ (Here, } T_s = 1)$$

$$\text{i.e. } x_1(t) = \int_{-\infty}^{\infty} \frac{1}{\pi f} \sin(2\pi fT_s) e^{j2\pi f t} df = \int_{-\infty}^{\infty} \frac{1}{\pi f} \sin(2\pi f) e^{j2\pi f t} df \quad (1)$$

(b)

[Fourier series in standard form]

$$x_2(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t), \text{ where}$$

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)dt = \frac{1}{T_0}\int_{-T_s}^{T_s} dt = \frac{2T_s}{T_0} = \frac{1}{2}. \; (T_s = 1; T_0 = 4;\; f_0 = 1/T_0 = 0.25)$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos(2\pi n f_o t)dt = \frac{2}{T_0}\int_{-T_s}^{T_s}\cos(2\pi nkf_0 t)dt$$

$$= \frac{2}{2\pi n f_0 T_0}\sin(2\pi n f_0\, t)\Big|_{-T_s}^{T_s} = \frac{2}{\pi n f_0 T_0}\sin(2\pi n f_0 T_s) = \frac{2}{\pi k}\sin(2\pi n f_0 T_s) = \frac{2}{\pi n}\sin(2\pi n f_0).$$

$$b_k = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin(2\pi k f_o t)dt = \frac{1}{2}\int_{-1}^{1}\sin(\frac{\pi k t}{2})dt = 0.$$

$$\text{i.e. } x_2(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n}\sin(\frac{\pi n}{2})\times\cos(\frac{\pi n t}{2}).$$

[Fourier series in complex form]

$$x_2(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k f_0 t}, \text{ where}$$

$$X_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j2\pi k f_0 t}dt = \frac{1}{T_0}\int_{-T_s}^{T_s} e^{-j2\pi k f_0 t}dt$$

$$= -\frac{1}{j2\pi k f_0 T_0} e^{-j2\pi k f_0 t}\Big|_{-T_s}^{T_s} = \frac{2}{2\pi k}\left(\frac{e^{j2\pi k f_0 T_s} - e^{-j2\pi k f_0 T_s}}{2j}\right)$$

$$= \frac{1}{\pi k}\sin(2\pi k f_0 T_s) = \frac{1}{\pi k}\sin(2\pi k f_0) = \frac{1}{\pi k}\sin(\frac{\pi k}{2});\quad X_0 = \frac{1}{T_0}\int_{-T_s}^{T_s} dt = \frac{2T_s}{T_0} = \frac{1}{2}.$$

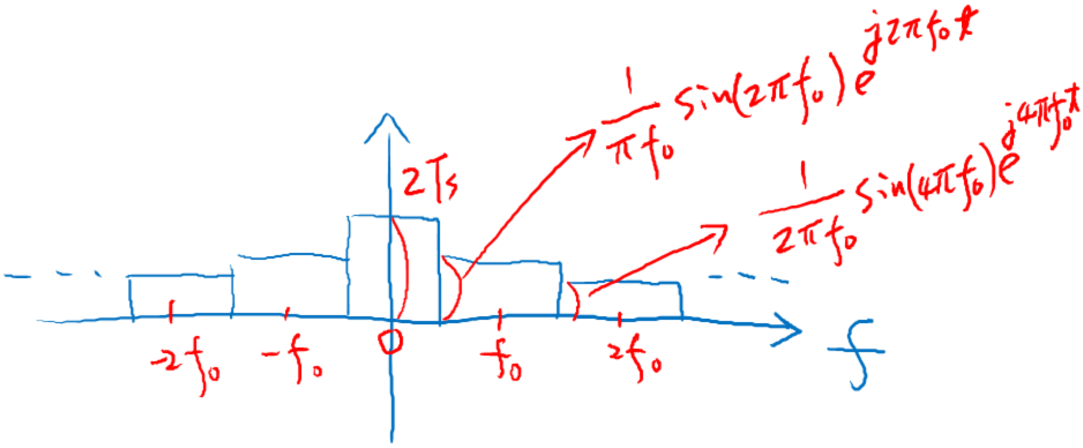
(c) From the complex form of Fourier series, we have

$$x_2(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k f_0 t} = X_0 + \sum_{k=-\infty}^{-1} X_k e^{j2\pi k f_0 t} + \sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}$$

$$= 2T_s \times f_0 + \sum_{k=-\infty}^{-1}\frac{1}{\pi k f_0}\sin(2\pi k f_0)e^{j2\pi k f_0 t} f_0 + \sum_{k=1}^{\infty}\frac{1}{\pi k f_0}\sin(2\pi k f_0)e^{j2\pi k f_0 t} f_0 \;.\;(2)$$

Hence, by *Riemann integral*, $x_2(t)$ becomes as

$$x_2(t) = \int_{-\infty}^{\infty}\frac{1}{\pi f}\sin(2\pi f)e^{j2\pi f t}df \quad \text{as } f_0 \to 0.$$



2. (10%) (Baseband Pulse Transmission; BPT) The transmitted signal is expressed as $s(t)=\sum_{k=-\infty}^{\infty} b_k h(t-0.001k)$, where $b_k \in \{+1,-1\}$ and

$$h(t)=\begin{cases}1 & 0 \le t < 0.001 \\ 0 & \text{otherwise}\end{cases}.$$

(a) For the baseband pulse transmission method, there is no carrier frequency involved, why do we still say that the transmitted digital signal has a bandwidth? What determines the bandwidth?

(b) Can the baseband pulse transmission method be used to send signals wirelessly?

Solutions: (Provided by R.C.T. Lee, M. C. Chiu, and J. S. Lin, "Communications Engineering," Wiley, 2007.)

(a) In the baseband pulse transmission system, we transmit purely pulses. Through Fourier transform, we know that a pulse contains a number of cosine functions. The narrower the pulse, the larger the bandwidth. The width of the pulse determines the bandwidth.

(b) We can't use this method to send signals wirelessly, because electromagnetic waves are sinusoidal signals.

3. (30%) (Binary Phase-Shift Keying; BPSK) For the first signal interval, the transmitted signal is expressed as

$$h(t)=\begin{cases} h_1(t)=\sqrt{\dfrac{2E_b}{T_b}}\cos(2\pi f_c t) \quad 0 \le t \le T_b, \text{ if input bit is one} \\ h_2(t)=\sqrt{\dfrac{2E_b}{T_b}}\cos(2\pi f_c t+\pi) \\ \qquad =-\sqrt{\dfrac{2E_b}{T_b}}\cos(2\pi f_c t) \; 0 \le t \le T_b, \text{ if input bit is zero.}\end{cases}$$

Suppose $E_b=5$, $T_b=2$ sec, and $f_c=100$ Hz.

(a) Find the average power P_{h_1} of $h_1(t)$.

(b) Show that the modulation property

$$F[x(t)\cos(2\pi f_c t)]=\frac{1}{2}X(f-f_c)+\frac{1}{2}X(f+f_c).$$

(c) Please derive the Fourier transform $H_1(f)$ of $h_1(t)$ directly from the above modulation property.

(d) Find the power spectral density (PSD), $P_{h_1}(f)$, of $h_1(t)$.

(e) Verify that $P_{h_1}=\int_{-\infty}^{\infty} P_{h_1}(f)df$.

(f) We all know that in digital modulation, pulses are modulated by cosine functions. And, we know that the Fourier transform of a cosine function is the delta function, why does such a modulated signal still have a bandwidth?

Solutions:

(a) $\frac{1}{T_b}\int_0^{T_b} h_1^2(t)dt = \frac{1}{2}\int_0^2 5\cos^2(200\pi\,\mathrm{t})dt = \frac{5}{2}\int_0^2 \frac{1+\cos(400\pi\,\mathrm{t})}{2}dt = \frac{5}{2}$

(b)

$$\because x(t)\cos(2\pi f_0 t) = x(t)\frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$$

$$F[x(t)e^{j2\pi f_0 t}] = \int_{-\infty}^{\infty} x(t)e^{j2\pi f_0 t}e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-f_0)t}dt = X(f-f_0)$$

$$\text{Similarly, } F[x(t)e^{-j2\pi f_0 t}] = X(f+f_0)$$

$$\therefore F\left[x(t)\cos(2\pi f_c t)\right] = \frac{1}{2}X(f-f_c) + \frac{1}{2}X(f+f_c).$$

(c) Since $h_1(t)$ can be written as $h_1(t) = \sqrt{\frac{2E_b}{T_b}}\Pi\left(\frac{t-T_b/2}{T_b}\right)\cos(2\pi f_c t)$,

by the time shift property: $F[x(t-t_0)] = e^{-j2\pi f t_0}X(f)$, we have

$$F\left[\sqrt{\frac{2E_b}{T_b}}\Pi\left(\frac{t-T_b/2}{T_b}\right)\right] = \sqrt{\frac{2E_b}{T_b}} \times T_b \mathrm{sinc}(T_b f)e^{-j\pi T_b f} = \sqrt{2E_b T_b}\mathrm{sinc}(T_b f)e^{-j\pi T_b f}$$

By the modulation property, we have

$$H_1(f) = \sqrt{\frac{E_b T_b}{2}}\mathrm{sinc}(T_b(f - f_c))e^{-j\pi T_b(f-f_c)} + \sqrt{\frac{E_b T_b}{2}}\mathrm{sinc}(T_b(f + f_c))e^{-j\pi T_b(f+f_c)}$$

$$= \sqrt{5}\mathrm{sinc}(2(f - 100))e^{-j\pi 2(f-100)} + \sqrt{5}\mathrm{sinc}(2(f + f_c))e^{-j\pi 2(f+100)}$$

(d)

$$P_{h_1}(f) = \frac{1}{T_b}|H_1(f)|^2 = \frac{E_b}{2}\mathrm{sinc}^2(T_b(f - f_c)) + \frac{E_b}{2}\mathrm{sinc}^2(T_b(f + f_c))$$

$$= \frac{5}{2}\mathrm{sinc}^2(2(f - 100)) + \frac{5}{2}\mathrm{sinc}^2(2(f + 100)).$$

(e)

$$\int_{-\infty}^{\infty} P_{h_1}(f)df = \int_{-\infty}^{\infty}[\frac{5}{2}\mathrm{sinc}^2(2(f - 100)) + \frac{5}{2}\mathrm{sinc}^2(2(f + 100))]df$$

$$= \frac{5}{2}\times\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{sinc}^2 x dx + \frac{5}{2}\times\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{sinc}^2 x dx = \frac{5}{2}.$$

(f) It is the width of the pulse which determines the bandwidth, not the frequency of the cosine function.

4. (10%) (Quadriphase-Shift Keying; QPSK) For the first signal interval, the four types of transmitted signal are expressed as:

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left(2\pi f_c t - (2i-1)\frac{\pi}{4} \right) \text{ for } 1 \le i \le 4 \text{ and } 0 \le t \le T.$$

Suppose $E = 5$, $T = 0.001$sec, and $f_c = 10000$ Hz. Find the power of the signal $s(t) = s_2(t)\text{-}s_3(t)$, $0 \le t \le 0.001$.

Solutions:

Let B = $\{\varphi_1(\text{t}), \varphi_1(\text{t})\}$ be the basis, where

$$\varphi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \text{ and } \varphi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), 0 \le t \le T.$$

$$<s_2(\text{t})>_B = (\frac{-\sqrt{2E}}{2}, \frac{\sqrt{2E}}{2}), <s_3(\text{t})>_B = (\frac{-\sqrt{2E}}{2}, \frac{-\sqrt{2E}}{2})$$

$$<s_2(\text{t})\text{-}s_3(\text{t})>_B = (\frac{-\sqrt{2E}}{2}, \frac{\sqrt{2E}}{2}) - (\frac{-\sqrt{2E}}{2}, \frac{-\sqrt{2E}}{2}) = (0, \sqrt{2E})$$

The power of $s_2(\text{t})\text{-}s_3(\text{t})$ is:

$$<s_2(\text{t})\text{-}s_3(\text{t}), s_2(\text{t})\text{-}s_3(\text{t})> = \| s_2(\text{t})\text{-}s_3(\text{t}) \|^2 = \| (0, \sqrt{2E}) \|^2 = 2E = 10.$$

Note that each of the four types of transmitted signal has power of 5.

$$\int_0^{0.001} 10\,000 \cos^2\left(20\,000\,\pi\, t - \frac{\pi}{4}\right) dt = 5.$$

$$\int_0^{0.001} 10\,000 \cos^2\left(20\,000\,\pi\, t - \frac{3\,\pi}{4}\right) dt = 5.$$

$$\int_0^{0.001} 10\,000 \cos^2\left(20\,000\,\pi\, t - \frac{5\,\pi}{4}\right) dt = 5.$$

$$\int_0^{0.001} 10\,000 \cos^2\left(20\,000\,\pi\, t - \frac{7\,\pi}{4}\right) dt = 5.$$

$<s_1(\mathrm{t})>_B=(\frac{\sqrt{2E}}{2},\frac{\sqrt{2E}}{2}),<s_2(\mathrm{t})>_B=(\frac{-\sqrt{2E}}{2},\frac{\sqrt{2E}}{2})$

$<s_3(\mathrm{t})>_B=(\frac{-\sqrt{2E}}{2},\frac{-\sqrt{2E}}{2}),<s_4(\mathrm{t})>_B=(\frac{\sqrt{2E}}{2},\frac{-\sqrt{2E}}{2})$

$\| s_1(\mathrm{t}) \|^2=\| s_2(\mathrm{t}) \|^2=\| s_3(\mathrm{t}) \|^2=\| s_4(\mathrm{t}) \|^2=E=5.$

5. (20%) For a symbol(signal) interval of an OFDM system, suppose that the transmitted signal is represented by

$s(t)=m_0\sqrt{2}\cos(2\pi t)+m_1\sqrt{2}\cos(4\pi t)+m_2\sqrt{2}\cos(6\pi t)+m_3\sqrt{2}\cos(8\pi t)$ for 0<t<1,

where $m_i=\begin{cases}1 \text{ if input bit is } 1\\ -1 \text{ if input bit is } 0\end{cases}$.

(a) What's the data rate of the OFDM system?

(b) Show that: $\begin{array}{l}\left\langle\sqrt{2}\cos(2\pi t),\sqrt{2}\cos(2\pi t)\right\rangle\triangleq\int_0^1\sqrt{2}\cos(2\pi t)\cos\sqrt{2}(2\pi t)dt=1\\ \left\langle\sqrt{2}\cos(2\pi t),\sqrt{2}\cos(4\pi t)\right\rangle\triangleq\int_0^1\sqrt{2}\cos(2\pi t)\sqrt{2}\cos(4\pi t)dt=0\end{array}$.

Hint: $\cos(\alpha)\cos(\beta)=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$

(c) Suppose the input bit sequence is "1111", find and plot the transmitted signal $s_{16}(t)$.

(d) Find $\left\langle s_{16}(t),\sqrt{2}\cos(2\pi t)\right\rangle$.

Solutions:

(a) 4 bits per second

(b)

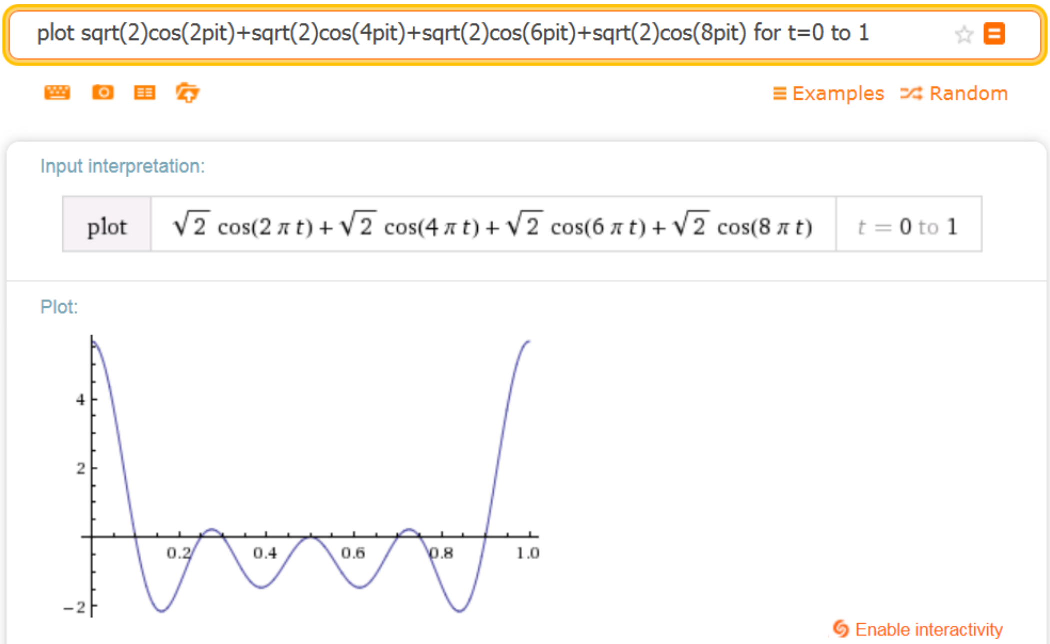
$$\left\langle\sqrt{2}\cos(2\pi t),\sqrt{2}\cos(2\pi t)\right\rangle\triangleq\int_0^1\sqrt{2}\cos(2\pi t)\sqrt{2}\cos(2\pi t)dt$$

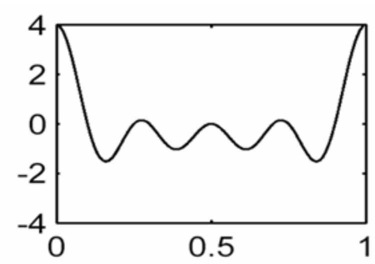
$$=2\int_0^1\frac{1+\cos(4\pi t)}{2}dt=2\times\frac{1}{2}+\frac{\cos(4\pi t)}{4\pi}\bigg|_0^1=1$$

$$\left\langle\sqrt{2}\cos(2\pi t),\sqrt{2}\cos(4\pi t)\right\rangle\triangleq\int_0^1\sqrt{2}\cos(2\pi t)\sqrt{2}\cos(4\pi t)dt$$

$$=\int_0^1\cos(6\pi t)dt+\int_0^1\cos(2\pi t)dt=\frac{\sin(6\pi t)}{6\pi}\bigg|_0^1+\frac{\sin(2\pi t)}{2\pi}\bigg|_0^1=0$$

(c) $s_{16}(t)=\sqrt{2}\cos(2\pi t)+\sqrt{2}\cos(4\pi t)+\sqrt{2}\cos(6\pi t)+\sqrt{2}\cos(8\pi t)$ for 0<t<1.





Amplitude is scaled by $\sqrt{2}$

(d)

$$
\begin{aligned}
&< s_{16}(t), \sqrt{2}\cos(2\pi t) > \\
&=< \sqrt{2}\cos(2\pi t), \sqrt{2}\cos(2\pi t) > + < \sqrt{2}\cos(4\pi t), \sqrt{2}\cos(2\pi t) > \\
&+ < \sqrt{2}\cos(6\pi t), \sqrt{2}\cos(2\pi t) > + < \sqrt{2}\cos(8\pi t), \sqrt{2}\cos(2\pi t) > \quad \text{for } 0<t<1 \\
&=1+0+0+0 = 1.
\end{aligned}
$$