

4.1 Power Series Solutions of Initial Value Problems

Def. Analytic Function

A function f is analytic at x_0 if $f(x)$ has a power series representation in some Open interval about x_0 :

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \text{ in some interval } (x_0 - h, x_0 + h)$$

EX.

$$\because \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \text{ converges for all real } x$$

$\therefore \sin(x)$ is analytic at 0

f is analytic at $x_0 \rightarrow$ f is infinitely differentiable at x_0
e.g. e^{-x^2}

Thm 4.1 Let p & q be analytic at x_0 . Then the initial value problem
 $y' + p(x)y = q(x); y(x_0) = y_0$ has a solution that is analytic at x_0

i.e.
let $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ with $a_n = \frac{y^{(n)}(x_0)}{n!}$

Thm 4.2 Let p & q be analytic at x_0 . Then the initial value problem
 $y'' + p(x)y' + q(x)y = f(x); y(x_0) = A, y'(x_0) = B$ has a unique solution
that is also analytic at x_0 .

EX.

$$y' + e^x y = x^2; y(0) = 4$$

sol:

$$\begin{aligned} \text{Let } y(x) &= \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n \\ &= y(0) + y'(0)x + \frac{y''(0)x^2}{2!} + \frac{y'''(0)x^3}{3!} + \dots \end{aligned}$$

$$(1) y'(0) + y(0) = 0 \Rightarrow y'(0) = -4$$

$$(2) y'' + e^x y' + e^x y = 2x \Rightarrow y''(0) + y'(0) + y(0) = 0 \Rightarrow y''(0) = 0$$

$$(3) y^{(3)} + 2e^x y' + e^x y'' + e^x y = 2 \Rightarrow y^{(3)}(0) = 6$$

$$\Rightarrow y(x) = 4 - 4x + x^3 + \dots$$

EX.

$$y'' - xy' + e^x y = 4 ; y(0) = 1, y'(0) = 4$$

$\because -x, e^x, & 4$ are analytic at 0

$$\therefore y(x) = y(0) + y'(0)x + \frac{y''(0)x^2}{2!} + \frac{y^{(3)}(0)x^3}{3!} + \dots$$

$$(1) y''(0) = 4 - y(0) = 3$$

$$(2) y^{(3)} - y' - xy'' + e^x y + e^x y = 0 \Rightarrow y^{(3)}(0) = y'(0) - y(0) - y'(0) = -1$$

$$\Rightarrow y(x) = 1 + 4x + \frac{3}{2}x^2 - \frac{1}{6}x^3$$

EX.

$$y'' + \cos(x)y' + 4y = 2x - 1$$

$$\text{u} \\ y'' + \cos(x)y' + 4y = 2x - 1 ; y(0) = a, y'(0) = b$$

$$\Rightarrow y(x) = a + bx + \frac{-1 - 4a - b}{2}x^2 + \frac{3 + 4a - 3b}{6}x^3 + \dots$$

4.2 Power Series Solutions Using Recurrence Relations

EX. $y'' + x^2y = 0$

Sol:

$$\begin{aligned}y(x) &= \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \& \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \\ \Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} &= 0 \\ \Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + a_{n-2} x^n &= 0 \\ \Rightarrow 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + a_{n-2}] x^n &= 0 \\ \Rightarrow a_2 = a_3 = 0 & \qquad \forall x \in (-h, h) \\ a_{n+2} &= -\frac{1}{(n+2)(n+1)} a_{n-2} \text{ for } n = 2, 3, \dots\end{aligned}$$

$$(1) \text{ 令 } x = 0 \Rightarrow a_2 = 0$$

(2) 兩邊微分 & 令 $x = 0$

$$\Rightarrow a_3 = 0$$

⋮

$$a_4 = \frac{-1}{12} a_0$$

$$a_5 = \frac{-1}{20} a_1$$

$$a_6 = \frac{-1}{30} a_2 = 0$$

$$a_7 = \frac{-1}{42} a_3 = 0$$

$$a_8 = \frac{-1}{56} a_4 = \frac{1}{672} a_0$$

$$a_9 = \frac{-1}{72} a_5 = \frac{1}{1440} a_1$$

$$\therefore y(x) = a_0 + a_1 x - \frac{1}{12} a_0 x^4 - \frac{1}{20} a_1 x^5 + \frac{1}{672} a_0 x^8 + \frac{1}{1440} a_1 x^9 + \dots$$

$$= a_0 \left(1 - \frac{1}{12} x^4 + \frac{1}{672} x^8 + \dots \right) + a_1 \left(x - \frac{1}{20} x^5 + \frac{1}{1440} x^9 + \dots \right)$$

$$\begin{aligned}\therefore y(0) &= a_0 \\ y'(0) &= a_1\end{aligned}$$

Ex.

$$y'' + xy' - y = e^{3x}$$

Sol:

$\because x, -1, \& e^{3x}$ are analytic at 0

$$\text{let } y(x) = \sum_{n=0}^{\infty} a_n x^n \& e^{3x} = \sum_{n=0}^{\infty} \left(3^n / n! \right) x^n$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left(3^n / n! \right) x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left(3^n / n! \right) x^n$$

$$\Rightarrow 2a_2 - a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} + (n-1)a_n] x^n = 1 + \sum_{n=1}^{\infty} \frac{3^n}{n!} x^n$$

$$\begin{cases} 2a_2 - a_0 = 1 \\ (n+2)(n+1)a_{n+2} + (n-1)a_n = \frac{3^n}{n!} \end{cases}$$

$$a_2 = \frac{1}{2}(1 + a_0)$$

$$a_{n+2} = \frac{(3^n/n!) + (1-n)a_n}{(n+2)(n+1)} \text{ for } n \geq 1$$

$$\therefore y(x) = a_0 + a_1x + \frac{1+a_0}{2}x^2 + \frac{1}{2}x^3 + \left(\frac{1}{3} - \frac{a_0}{24}\right)x^4 + \frac{7}{40}x^5 + \left(\frac{19}{240} + \frac{1}{240}a_0\right)x^6 + \dots$$

4.3 Singular Points and the Method of Frobenius

$$P(x)y'' + Q(x)y' + R(x)y = F(x) \quad (4.8)$$

$$y'' + p(x)y' + q(x)y = f(x) \quad (4.9)$$

Def 4.2

(1) x_0 is an ordinary point of eq. (4.8) if $P(x_0) \neq 0$

& $\frac{Q(x)}{P(x)}$ ($= p(x)$), $\frac{R(x)}{P(x)}$ ($= q(x)$), $\frac{F(x)}{P(x)}$ are analytic at x_0

(2) x_0 is a singular point of eq. (4.8) if x_0 is not an ordinary point

EX.

$$x^3(x-2)^2 y'' + 5(x+2)(x-2)y' + 3x^2y = 0$$

$$\therefore P(x) = x^3(x-2)^2 \text{ with } P(0) = P(2) = 0$$

$\therefore 0$ & 2 are singular points of the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (4.10)$$

Def. 4.3

(1) x_0 is a regular singular point of eq. (4.10) if x_0 is a singular point,

and the functions $(x - x_0) \frac{Q(x)}{P(x)}$ & $(x - x_0)^2 \frac{R(x)}{P(x)}$ are analytic at x_0

(2) A singular point that is not regular is said to be an irregular singular point.

EX.

$$\therefore (x - 0) \frac{Q(x)}{P(x)} = \frac{5x(x+2)(x-2)}{x^3(x-2)^2} = \frac{5}{x^2} \cdot \frac{x+2}{x-2} \therefore 0 \text{ irregular}$$

$$\therefore (x - 2) \frac{Q(x)}{P(x)} = 5 \frac{x+2}{x^3} \text{ & } (x-2)^2 \frac{R(x)}{P(x)} = \frac{3}{x} \text{ are analytic at 2}$$

$\therefore 2$: regular

Suppose eq. (4.10) has a regular singular point at x_0

Then, we attempt to choose number c_n & a number r

so that $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ is a solution

This series is called : Frobenius series (未必是power series)

This strategy is called : method of Frobenius

$$y'(x) = \sum_{n=0}^{\infty} (n+r)c_n (x - x_0)^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n (x - x_0)^{n+r-2}$$

EX.

$$x^2 y'' + x \left(\frac{1}{2} + 2x \right) y' + \left(x - \frac{1}{2} \right) y = 0$$

Sol:

$\because 0$ is a regular singular point

$$(i) x \frac{x \left(\frac{1}{2} + 2x \right)}{x^2} = \frac{1}{2} + 2x$$

$$(ii) x - \frac{1}{2} \text{ conv. for all } x$$

$$\therefore \text{let } y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} \frac{1}{2}(n+r)c_n x^{n+r}$$

$$+ \sum_{n=0}^{\infty} 2(n+r)c_n x^{n+r+1} + \sum_{n=0}^{\infty} c_n x^{n+r+1} - \sum_{n=0}^{\infty} \frac{1}{2}c_n x^{n+r} = 0$$

$$\left[r(r-1)c_0 + \frac{1}{2}c_0r - \frac{1}{2}c_0 \right] x^r + \sum_{n=1}^{\infty} \left[(n+r)(n+r-1)c_n + \frac{1}{2}(n+r)c_n \right]$$

$$+ 2(n+r-1)c_{n-1} + c_{n-1} - \frac{1}{2}c_n \right] x^{n+r} = 0$$

$$\begin{cases} r(r-1)c_0 + \frac{1}{2}c_0r - \frac{1}{2}c_0 = 0 \\ (n+r)(n+r-1)c_n + \frac{1}{2}(n+r)c_n + 2(n+r-1)c_{n-1} + c_{n-1} - \frac{1}{2}c_n = 0 \end{cases}$$

assume $c_0 \neq 0 \Rightarrow r(r-1) + \frac{1}{2}r - \frac{1}{2} \Rightarrow r = 1 \text{ & } \frac{1}{2}$

$$c_n = -\frac{1+2(n+r-1)}{(n+r)(n+r-1) + \frac{1}{2}(n+r) - \frac{1}{2}} c_{n-1}$$

$$(1) r = 1$$

$$c_n = -\frac{1+2n}{n\left(n+\frac{3}{2}\right)} c_{n-1} \text{ for } n=1, 2, \dots$$

$$c_1 = \frac{-6}{5}c_0 \quad \therefore y_1(x) = c_0 \left(x - \frac{6}{5}x^2 + \frac{6}{7}x^3 - \frac{4}{9}x^4 + \dots \right)$$

$$c_2 = \frac{6}{7}c_0 \quad c_3 = \frac{-4}{9}c_0$$

$$\begin{aligned}
 (2) \quad r &= \frac{-1}{2} \\
 c_n &= -\frac{1 + 2\left(n - \frac{3}{2}\right)}{\left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right) + \frac{1}{2}\left(n - \frac{1}{2}\right) - \frac{1}{2}} c_{n-1} \\
 &= -\frac{2n-2}{n\left(n - \frac{3}{2}\right)} c_{n-1} \\
 \because c_1 = 0 \quad \therefore c_2 = c_3 = \dots = 0 \quad \therefore y_2(x) &= \sum c_n x^{n-\frac{1}{2}} = c_0 x^{\frac{-1}{2}}
 \end{aligned}$$

for $x > 0$

Thm 4.3 Suppose x_0 is a regular singular point of $p(x)y'' + Q(x)y' + R(x)y = 0$.

Then there exists at least one Frobenius solution $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ with $c_0 \neq 0$.

Further, if the Taylor expansions of $(x - x_0)Q(x)/P(x)$ & $(x - x_0)^2 R(x)/P(x)$ about x_0 converge in an open interval $(x_0 - h, x_0 + h)$, then this Frobenius series also converges in this interval, except perhaps at x_0 itself

EX.

$$x^2 y'' + 5xy' + (x+4)y = 0$$

Sol:

$$\because x\left(\frac{5x}{x^2}\right) = 5 \text{ & } x^2 \frac{x+4}{x^2} = x+4 \text{ converge for all } x$$

$\therefore 0$ is a regular point

$$\text{let } y(x) = \sum_{n=0}^{\infty} c_n x^{n+r} \Rightarrow r = -2 \quad \therefore y(x) = c_0 \left[x^{-2} - x^{-1} + \frac{1}{4} - \frac{1}{36}x + \frac{1}{576}x^2 + \dots \right]$$

$$\Rightarrow c_n = (-1)^n \frac{1}{(n!)^2} c_0$$

$$= c_0 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} x^{n-2}$$

Converges for all x except 0

EX.

Bessel Function of the First kind of order ν

$$(1) x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

is called Bessel's equation of order ν for $\nu \geq 0$

(2) Solutions of Bessel's equation are called : Bessel functions

$$\because 0 \text{ is a regular singular point} \quad \text{let } y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$\Rightarrow r = \pm \nu$$

$$(i) \text{ let } r = \nu \Rightarrow c_1 = 0 \quad c_3 = c_5 = \dots = c_{\text{odd}} = 0$$

$$\& c_{2n} = \frac{(-1)^n}{2^{2^n} n! (1+\nu)(2+\nu)\dots(n+\nu)} c_0$$

$$\therefore y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2^n} n! (1+\nu)(2+\nu)\dots(n+\nu)} x^{2n+\nu}$$

4.4 Second Solutions and Logarithm Factors

Thm 4.4 A second Solution in the Method of Frobenius suppose 0 is a regular singular point of $P(x)y''+Q(x)y'+R(x)y=0$. Let r_1 & r_2 be roots of the indicial equation. If these are real. suppose $r_1 \geq r_2$. Then

(1) if $r_1 - r_2$ is not an integer, \exists two linearly independent

$$\text{Frobenius solutions: } y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \quad \& \quad y_2(x) = \sum_{n=0}^{\infty} c_n^* x^{n+r_2}$$

with $c_0 \neq 0$ & $c_0^* \neq 0$. These solutions are valid in some interval $(0, h)$ or $(h, 0)$

$$(2) \text{if } r_1 - r_2 = 0 \text{ there is a Frobenius solution } y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$$

with $c_0 \neq 0$ as well as a second solution

$$y_2(x) = y_1(x) \ln(x) + \sum_{n=1}^{\infty} c_n^* x^{n+r_1}$$

Further, y_1 & y_2 form a fundamental set of solutions on some interval $\underline{(0, h)}$

(3) if $r_1 - r_2$ is a positive integer, then \exists a Frobenius

solution $y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$ & a second solution of

the form $y_2(x) = k y_1(x) \ln(x) + \sum_{n=0}^{\infty} c_n^* x^{n+r_2}$

(i) $k = 0 \Rightarrow$ a second Frobenius solution

(ii) y_1 & y_2 forms a fundamental set of solutions on
some interval $(0, h)$

EX.

$$x^2y'' + 5xy' + (x+4)y = 0 \cdots (*) \quad (r+2)^2 = 0 \Rightarrow r = \pm 2$$

Sol:

$$y_1(x) = c_0 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} x^{n-2} \quad \text{取 } c_0 = 1$$

$$\text{let } y_2(x) = y_1(x) \ln(x) + \sum_{n=1}^{\infty} c_n^* x^{n-2} \text{ 代入 (*)}$$

$$4y_1 + 2xy_1' + \sum_{n=1}^{\infty} (n-2)(n-3)c_n^* x^{n-2} + \sum_{n=1}^{\infty} 5(n-2)c_n^* x^{n-2}$$

$$+ \sum_{n=1}^{\infty} c_n^* x^{n-1} + \sum_{n=1}^{\infty} 4c_n^* x^{n-2} + \ln(x) [x^2 y_1'' + 5xy_1' + (x+4)y_1] \stackrel{||}{=} 0$$

$$\Rightarrow -2x^{-1} + c_1^* x^{-1} + \sum_{n=2}^{\infty} \left[\frac{4(-1)^n}{(n!)^2} + \frac{2(-1)^n}{(n!)^2} (n-2) \right]$$

$$+ (n-2)(n-3)c_n^* + 5(n-2)c_n^* + c_{n-1}^* + 4c_n^*] x^{n-2} = 0$$

$$\Rightarrow \begin{cases} c_1^* = 2 \\ \frac{2(-1)^n}{(n!)^2} n + n^2 c_n^* + c_{n-1}^* = 0 \Rightarrow c_n^* = \frac{-1}{n^2} c_{n-1}^* - \frac{2(-1)^n}{n(n!)^2} \end{cases}$$

for $n = 2, 3, \dots$

$$\therefore y_2(x) = y_1(x) \ln(x) + \frac{2}{x} - \frac{3}{4} + \frac{11}{108} x - \frac{25}{3456} x^2 + \dots$$

\Rightarrow the general solution is

$$y(x) = [c_1 + c_2 \ln(x)] \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} x^{n-2} + c_2 \left[\frac{2}{x} - \frac{3}{4} + \frac{11}{108} x - \frac{25}{3456} x^2 + \dots \right]$$