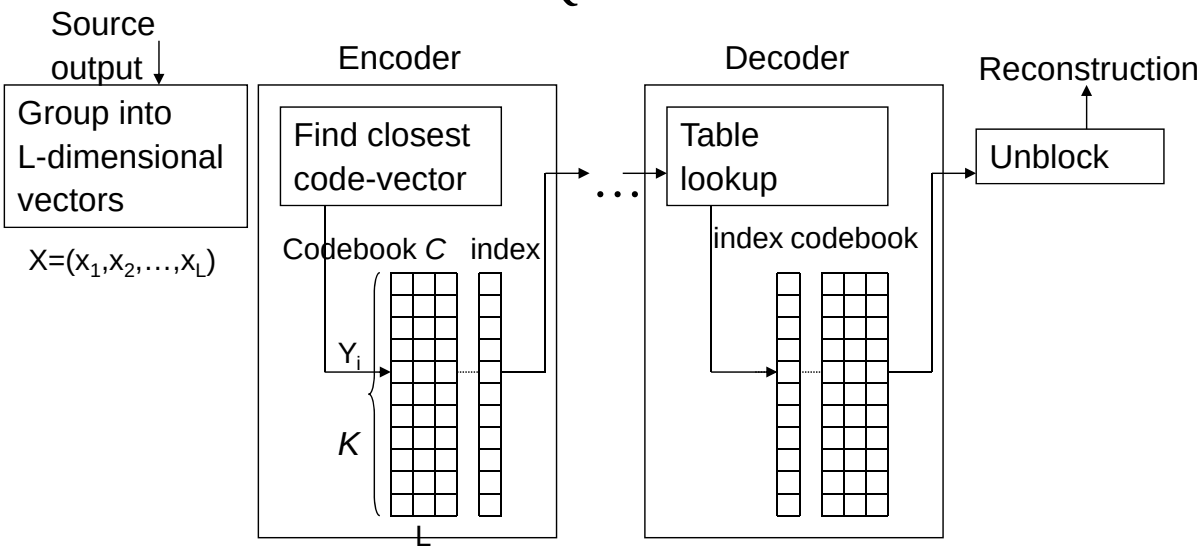
Vector Quantization



- For an L-dimensional vector quantizer with a codebook of size K

  $\because \lceil \log_2 K \rceil$ bits/vector

  $\Rightarrow$ Rate $R = \dfrac{\lceil \log_2 K \rceil}{L}$ bits/sample

- K-level quantizer

- $\|X - Y_i\|^2 \le \|X - Y_j\|^2 \ \forall \ Y_j \in C$, where $\|X\|^2 \equiv \sum_{i=1}^{L} X_i^2$

Example

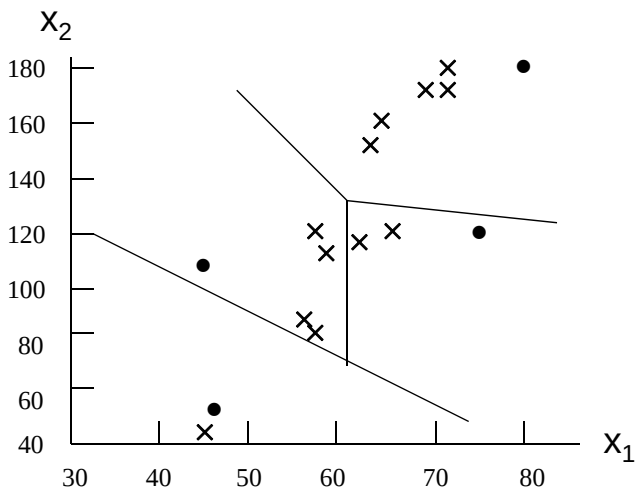
- Input : 72 180 72 175 65 120 44 41 59 119 62 114
  64 150 60 110 65 162 56 91 57 88 70 172
  - $L = 2$
  - $K = 4$

Source vectors

X	(x_1, x_2)
X_1	(72,180)
X_2	(72,175)
X_3	(65,120)
X_4	(44,41)
X_5	(59,119)
X_6	(62,114)
X_7	(64,150)
X_8	(60,110)
X_9	(65,162)
X_{10}	(56,91)
X_{11}	(57,88)
X_{12}	(70,172)

Codebook

Y_1	(45,50)
Y_2	(45,117)
Y_3	(75,117)
Y_4	(80,180)



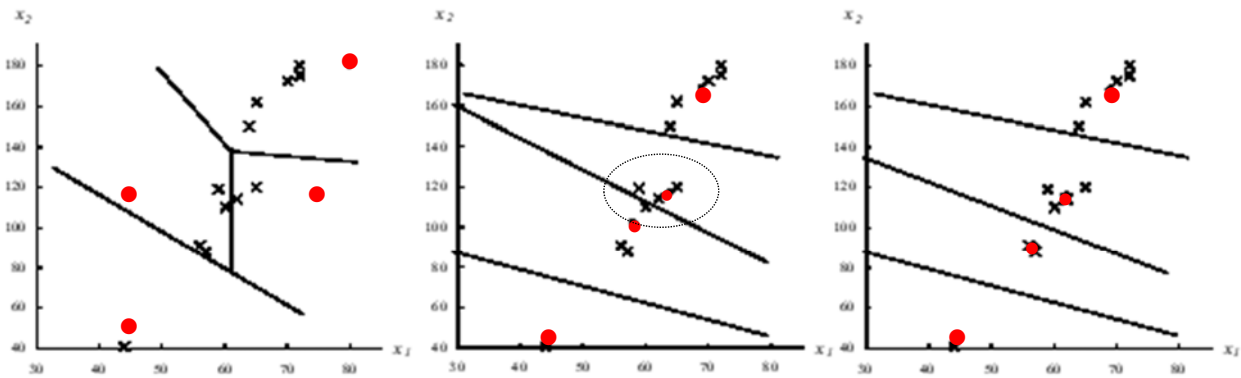
Distortion $D = \dfrac{1}{12} \sum_{i=1}^{4} \sum_{X \in V_i} \|X - Y_i\|^2 = 387.25$

LBG Algorithm (1)

- Linde-Buzo-Gray (LBG) algorithm

1. Start with an inital set of reconstruction vectors $\{Y_i^{(1)}\}_{i=1}^K$ and a set of training vectors $\{X_n\}_{n=1}^N$.
   Set $m=1$, $D^{(0)}=0$, and thresold $=\varepsilon$
2. Quantization regions $\{V_i^{(m)}\}_{i=1}^K$ are given by
   $V_i^{(m)} = \{X_n \mid d(X_n, Y_i) < d(X_n, Y_j) \ \ \forall j \neq i\} \quad i = 1,2,\ldots,K$ (Q: $V_i^{(m)}$ is empty for some i and m)
3. Compute the average distortion $D^{(m)} = \dfrac{1}{N}\sum_{i=1}^{k}\sum_{X \in V_i^{(m)}} \left\| X - Y_i^{(m)} \right\|^2$
4. If $\dfrac{D^{(m)} - D^{(m-1)}}{D^{(m)}} < \varepsilon$ stop; Otherwise, continue.
5. $m = m+1$
   Find new $\{Y_i^{(m)}\}_{i=1}^K$ that are the centroids of $\{V_i^{(m-1)}\}_{i=1}^K$
   i.e. $Y_i^{(m)} = \dfrac{1}{|V_i^{(m-1)}|}\sum_{X \in V_i^{(m-1)}} X$, *where* $|V_i^{(m-1)}|$ is the number of vectors in region $V_i^{(m-1)}$
   Go to step 2

LBG Algorithm (2)



$D = 387.25$

Codebook

Y_1	(45,50)
Y_2	(45,117)
Y_3	(75,117)
Y_4	(80,180)

$D = 89$

Codebook

Y_1	(44,41)
Y_2	(58,102)
Y_3	(64,117)
Y_4	(69,168)

$D = 60.17$

Codebook

Y_1	(44,41)
Y_2	(57,90)
Y_3	(62,116)
Y_4	(69,168)

LBG Algorithm (3)

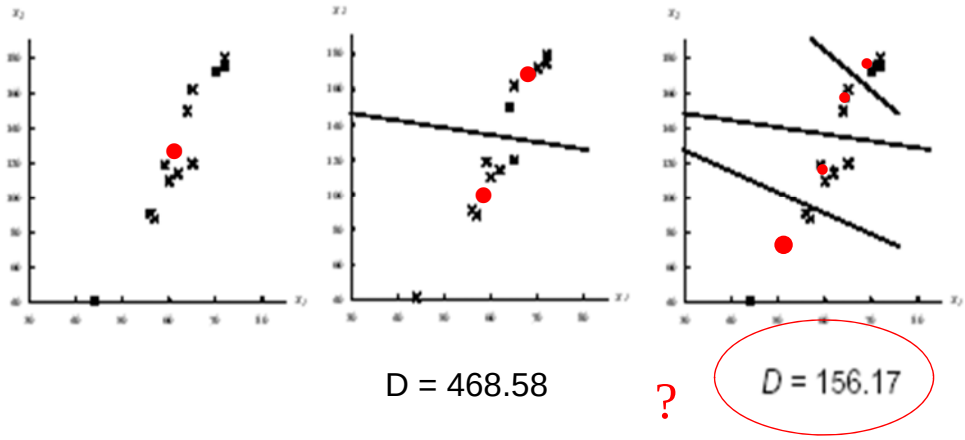
- Initializing LBG

1. Spliting technique

- split from k-level VQ to 2k-level VQ
- add a fixed perturbation vector $\mathcal{E}$ to the vector to obtain a new vector
  - Example: $\mathcal{E}$ = (10,10)
- the new codebook is at least as good as the codebook prior to splitting

Codebook

k=1	(62,127)
k=2	(62,127)
	(72,137)
	(58,98)
	(69,168)
k=4	(58,98)
	(68,108)
	(69,168)
	(79,178)
	(52,73)
	(62,116)
	(65,156)
	(71,176)



LBG Algorithm (4)

2. Pick the vectors randomly from the training set

- initialize the codebook randomly several times
- pick the one with least distortion

3. PNN (pairwise nearest neighbor) algorithm

- start with as many clusters as there are training vectors
- at each stage, combine two closest vectors into a single cluster and replace these two by their mean

  s.t. merge two clusters that would result in the smallest increase in distortion
- e.g.

$$D_i + D_j = \sum_{X \in C_i} \|X - Y_i\|^2 + \sum_{X \in C_j} \|X - Y_j\|^2$$

$$D_{ij} = \sum_{X \in C_i} \|X - Y_{ij}\|^2 + \sum_{X \in C_j} \|X - Y_{ij}\|^2, \text{ where } Y_{ij} = \frac{n_i Y_i + n_j Y_j}{n_i + n_j}$$

n_i, n_i : the number of vectors in C_i and C_j

$$D_{ij} - (D_i + D_j) = \frac{n_i n_j}{n_i + n_j} \|Y_i - Y_j\|^2$$

(72,180)	(72,180)	(71,176)(3)	(71,176) (3)	(69,168) (5)	(69,168) (5)
(72,175)	(71,174) (2)				
(65,120)	(65,120)	(65,120)	(65,120)	(65,120)	(62,116) (4)
(44,41)	(44,41)	(44,41)	(44,41)	(44,41)	(44,41)
(59,119)	(61,117) (2)	(61,117) (2)	(60,114) (3)	(61,114) (3)	
(62,114)					
(64,150)	(65,156) (2)	(65,156) (2)	(65,156) (2)		
(60,110)	(60,110)	(60,110)			
(65,162)					
(56,91)	(57,90) (2)	(57,90) (2)	(57,90) (2)	(57,90) (2)	(57,90) (2)
(57,88)					
(70,172)					

LBG Algorithm (5)

- **The Empty Cell Problem**
  - to remove an output vector that has no input vectors associated with it and replace it with a vector from the quantization region with highest population or the highest associated distortion
    - by selecting a vector randomly in this selected region
    - by applying a two-level vector quantizer to this selected region
- **Rate**

  For an L-dimensional vector quantizer with a codebook of size K, use B bits to represent each element of the vector
  - if the codebook is available to both the encoder and decoder

$$R = \frac{\lceil \log_2 K \rceil}{L} \text{ bits/sample}$$

  - otherwise
    - it need to transmit the codebook with $B \times L \times K$ bits (e.g. B=8, L=16)

e.g. 256×256 image

B=8, L=16, K=1024

(16 ×8 ×1024)/65536=2

R = 10/16 $2/R \approx 3$

RL bits/vector

$\Rightarrow$ 2^{RL} different output vectors

$\Rightarrow$ size of codebook = 2^{RL}

RL : Rate Dimension Product

Size of codebook grows exponentially with RL