Computer Science and Information Engineering
National Chi Nan University

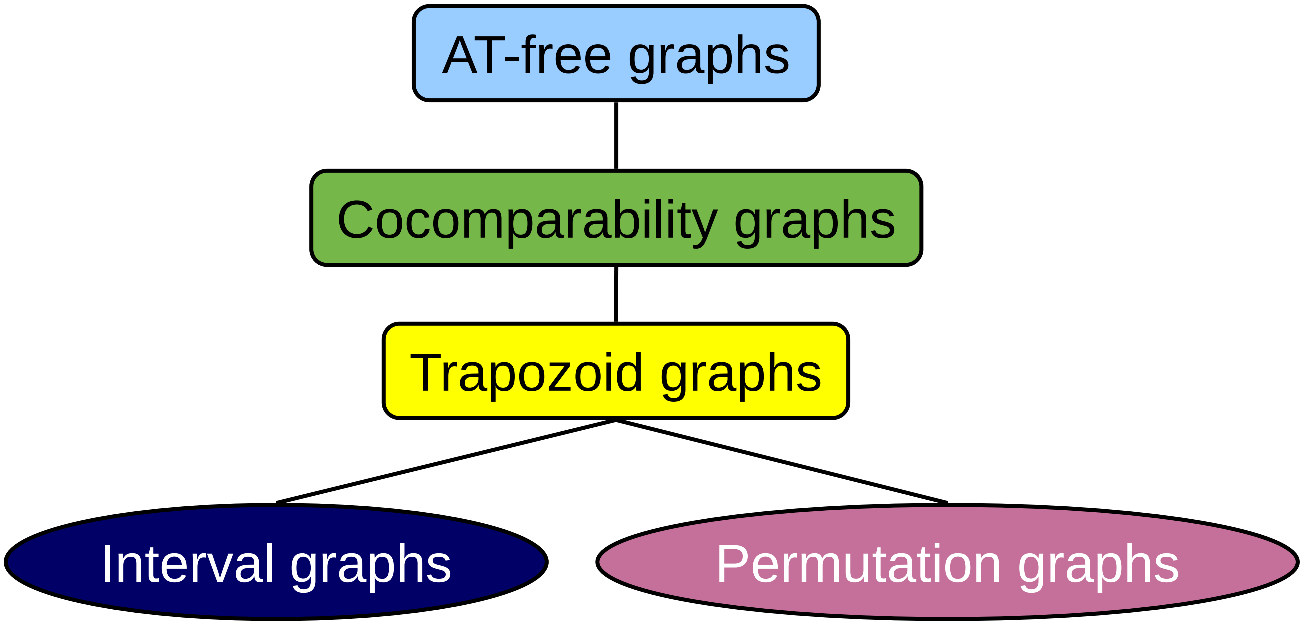
Combinatorial Optimization

Dr. Justie Su-Tzu Juan

Lecture 9 More Class of Graphs

§ 9.1 AT-Free Graphs

9.1 AT-Free Graphs



- **Def:**
  ① π is a **permutation** on $N_n = \{1, 2, \ldots, n\}$.
  ② A graph G is called a **permutation graph** if $\exists\ \pi$ on N_n s.t. $G = G(\pi) = (N_n, E(\pi))$, where $ij \in E(\pi) \Leftrightarrow (i - j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0$.

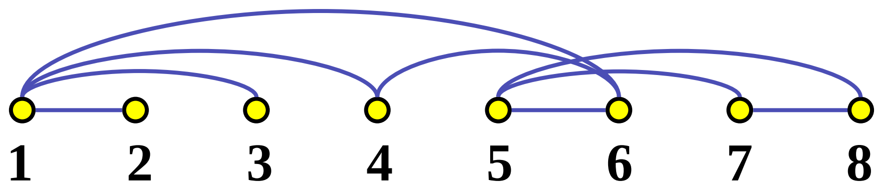
9.1 AT-Free Graphs

- **Ex:**

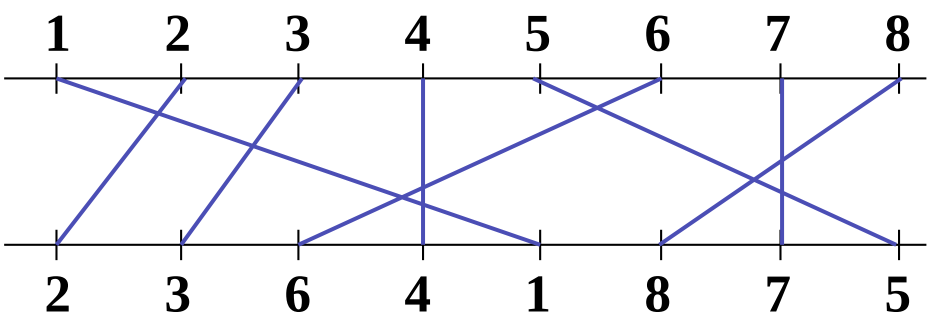
  ① $n = 8$, $\pi = (2\ 3\ 6\ 4\ 1\ 8\ 7\ 5)$

  or $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 6 & 4 & 1 & 8 & 7 & 5 \end{pmatrix}$ $\pi^{-1}(1) = 5$, $\pi^{-1}(2) = 1$,

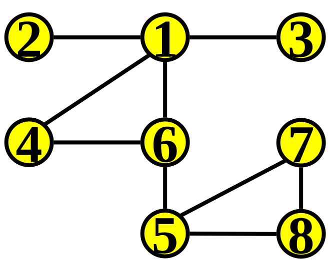
  ② $G(\pi)$



or



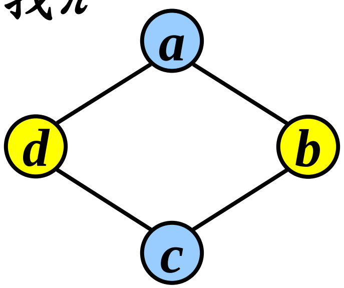
$\Rightarrow G(\pi)$

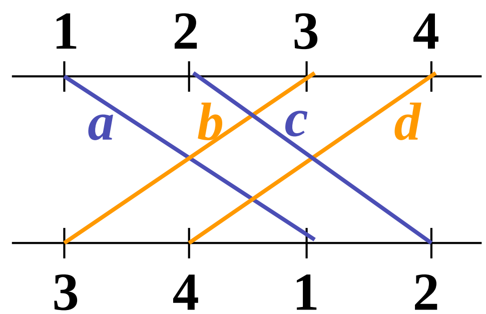


9.1 AT-Free Graphs

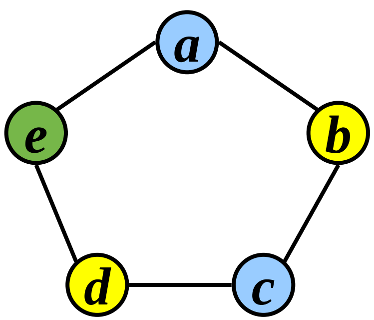
- **Ex:**

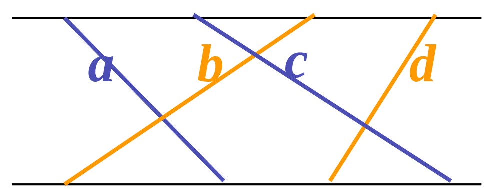
  ③自圖找π



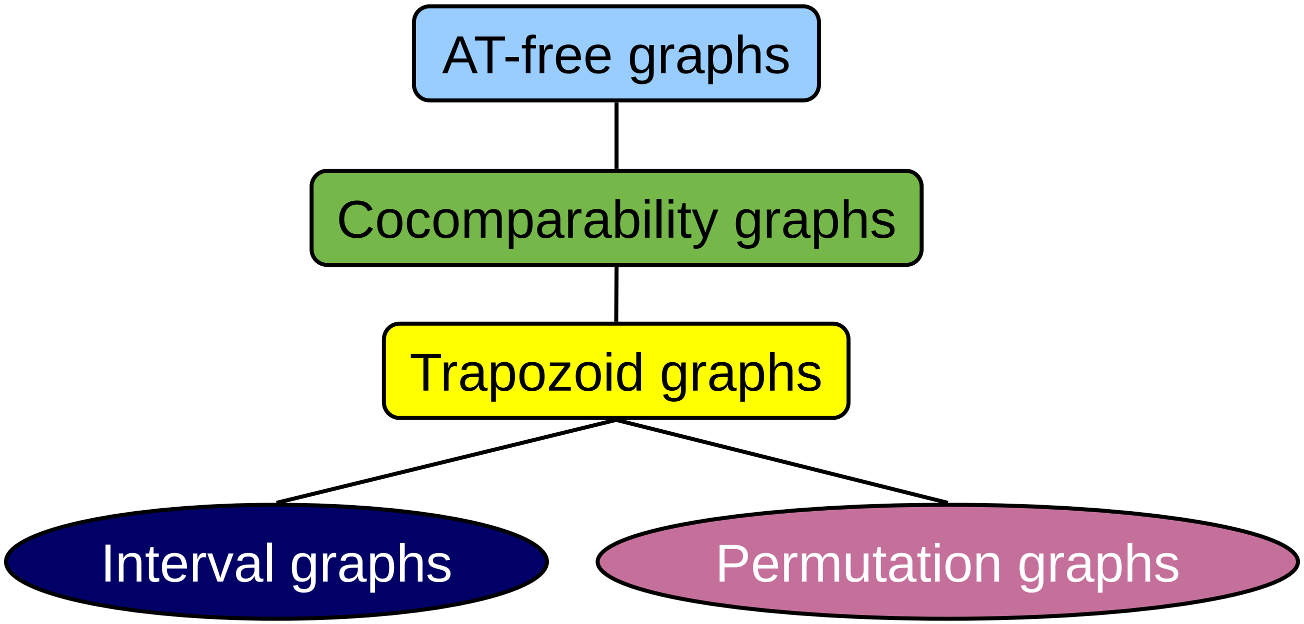


  ④





9.1 AT-Free Graphs



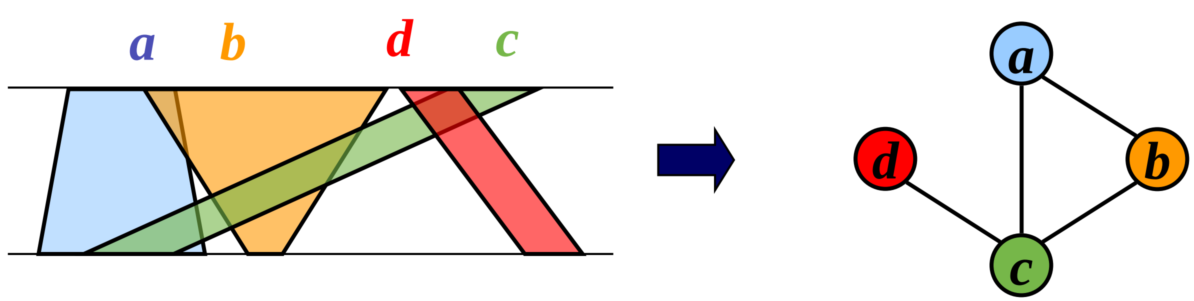
- **<u>Def</u>: A graph G is called a trapozoid graph if $\exists$ a set $\mathcal{J}$ of trapozoid with same height, and a 1-1 function f: $V(G) \rightarrow \mathcal{J}$ s.t. $xy \in E(G) \Leftrightarrow f(x) \cap f(y) \neq \phi$.**

9.1 AT-Free Graphs

- **<u>Def</u>: A graph G is called a trapozoid graph if $\exists$ a set $\mathcal{J}$ of trapozoid with same height, and a 1-1 function f: $V(G) \rightarrow \mathcal{J}$ s.t. $xy \in E(G) \Leftrightarrow f(x) \cap f(y) \neq \phi$.**

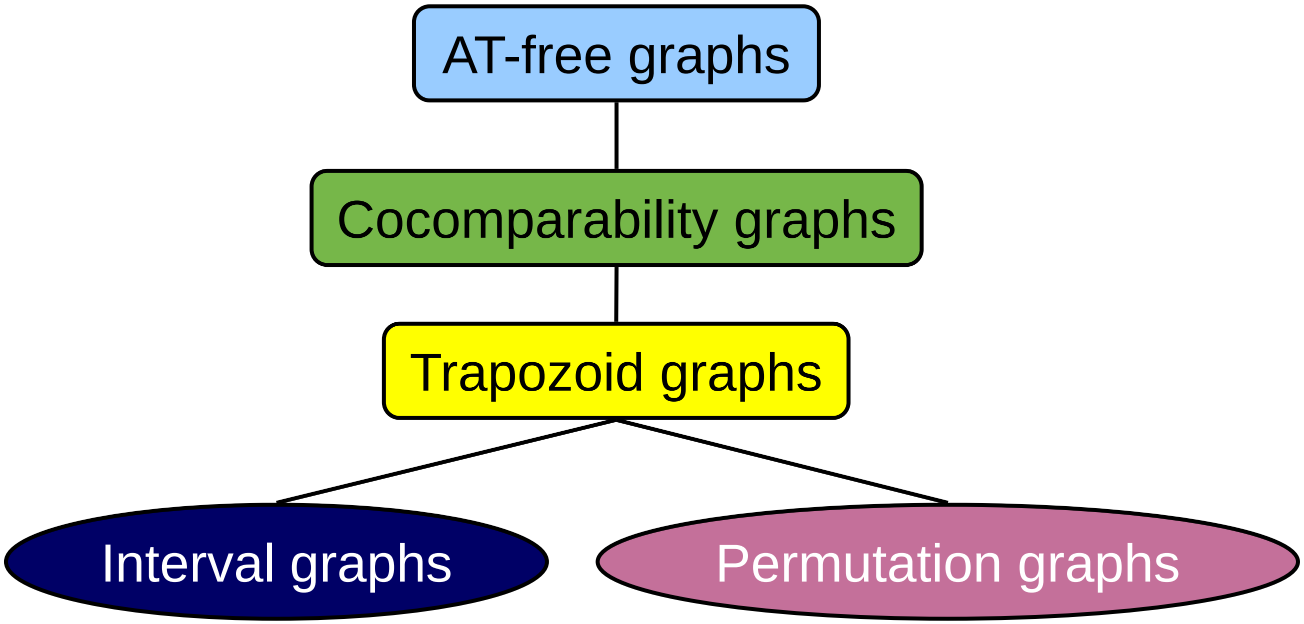
- **<u>Note</u>: Trapozoid and permutation graphs are intersection graphs.**

- **<u>Ex</u>:**



- **<u>Remark</u>: Interval graphs and permutation graphs are trapozoid graphs.**

9.1 AT-Free Graphs

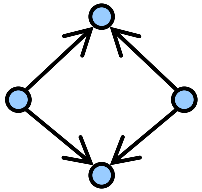


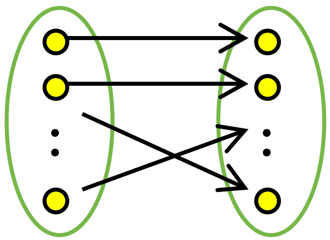
9.1 AT-Free Graphs

- **Def: A digraph $G = (V, E)$ is called transitive acyclic if**
  ① $\nexists\, v_1 \to v_2 \to \dots \to v_r \to v_1$, and
  ② if $(i, j), (j, k) \in E(G)$, then $(i, k) \in E(G)$.

- **Def:**
  ① G is a comparability graph if G has an orientation that is a transitive acyclic digraph.
  ② A co-comparability graph is the complement of a comparability graph.

- **Ex: C_4, bipartite graph are comparability graph.**





9.1 AT-Free Graphs

- **Note**: G is a permutation graph.

  $\Leftrightarrow \exists\ \pi$ such that $G = G(\pi)$

  $\Rightarrow \exists$ partially ordered set $(N_n, \preccurlyeq)$ (poset) s.t.

  $G' = (N_n, E')$ has a transitive orientation and $G' \cong \overline{G}$.

  where $E' = \{ij: i \neq j, i \preccurlyeq j\}$ and

  $\forall\ i, j \in N_n, i \preccurlyeq j$ iff $i \leq j$ and $\pi^{-1}(i) \leq \pi^{-1}(j)$.

  $\Rightarrow \overline{G}$ has a transitive orientation.

  $\Rightarrow \overline{G}$ is a comparability graph.

  $\Rightarrow G$ is a co-comparability graph.

- **Thm C**: $G = (V, E)$ is a comparability graph.

  $\Leftrightarrow \exists$ ordering $[v_1, v_2, \ldots, v_n]$ of V s.t.

  (c) $i < j < k$, $v_iv_j \in E$, $v_jv_k \in E \Rightarrow v_iv_k \in E$.

9.1 AT-Free Graphs

- **Note:** $G = (V, E)$ is a co-comparability graph.

  $\Leftrightarrow G^c = (V, E^c)$ is a comparability graph.

  $\Leftrightarrow \exists$ ordering $[v_1, v_2, \ldots, v_n]$ of V s.t.

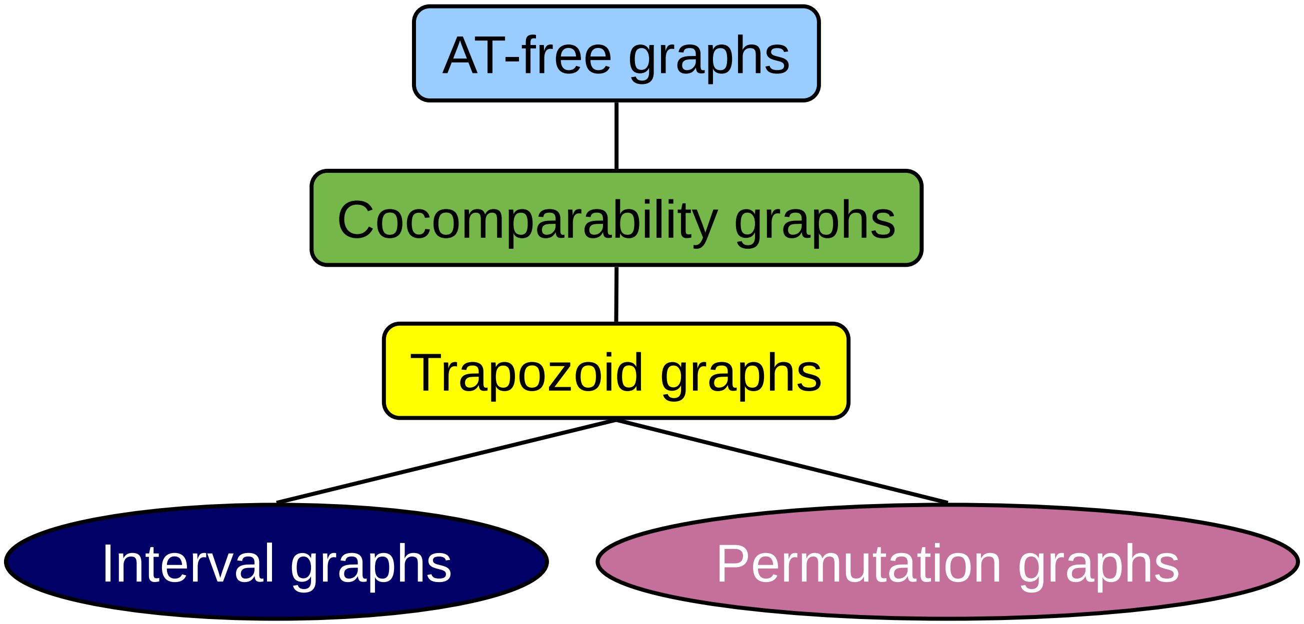
  (c*) $i < j < k,\ v_iv_j \in E^c,\ v_jv_k \in E^c \Rightarrow v_iv_k \in E^c$.

  $\Leftrightarrow$ (c**) $i < j < k,\ v_iv_j \notin E,\ v_jv_k \notin E \Rightarrow v_iv_k \notin E$.

  $\Leftrightarrow$ (cc) $i < j < k,\ v_iv_k \in E \Rightarrow v_iv_j \in E$ or $v_jv_k \in E$.

- **Problems:**
  - Weighted independent domination for co-comparability graphs (Linear).
  - Connected domination for co-comparability graphs. (Poly.)
  - (http://wwwteo.informatik.uni-rostock.de/isgci/classes/gc_147.html)

9.1 AT-Free Graphs



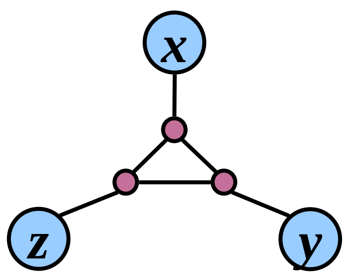
9.1 AT-Free Graphs

- **Def:**

  1) Three vertices x, y, z of graph G is called **Asteroidal triple** (**AT**) if
     ① $\{x, y, z\}$ is an independent set.
     ② $\exists$ x-y path $P \cap N[z] = \phi$, $\exists$ y-z path $Q \cap N[x] = \phi$ and $\exists$ x-z path $R \cap N[y] = \phi$.

  2) A graph G is called **AT-free** graph if $\nexists$ AT.

- **Ex:**



$\{x, y, z\}$ are AT.

- **Thm:** G is an interval graph $\Leftrightarrow$ G is a chordal graph and AT-free graph.

(cc) $i < j < k$, $v_iv_k \in E \Rightarrow v_iv_j \in E$ or $v_jv_k \in E$.

9.1 AT-Free Graphs

- **Thm**: G is co-comparability graph $\Rightarrow$ G is AT-free.

Proof.

If not, suppose $\{x, y, z\}$ is AT.

$\because$ G is co-comparability $\therefore \exists$ vertex ordering satisfy (cc).

W.L.O.G. say $x < y < z$, then $\forall$ x-z path P: $x = v_1, v_2, \ldots, v_r = z$

$\because$ $x < y < z$ $\therefore$ $\exists 1 \le i < r$ s.t. $v_i \le y < v_{i+1}$

$\because$ $v_iv_{i+1} \in E$ $\therefore$ By (cc), $v_i \in N[y]$ or $v_{i+1} \in N[y]$,

i.e. $P \cap N[y] \neq \phi$. $\rightarrow\leftarrow$

- **Remark**: D. Kratsch, "Domination and total domination in asteroidal triple-free graphs," *Discrete Appl. Math*. 99, No. 1-3, pp. 111-123 (2000).
- (http://wwwteo.informatik.uni-rostock.de/isgci/classes/gc_61.html)

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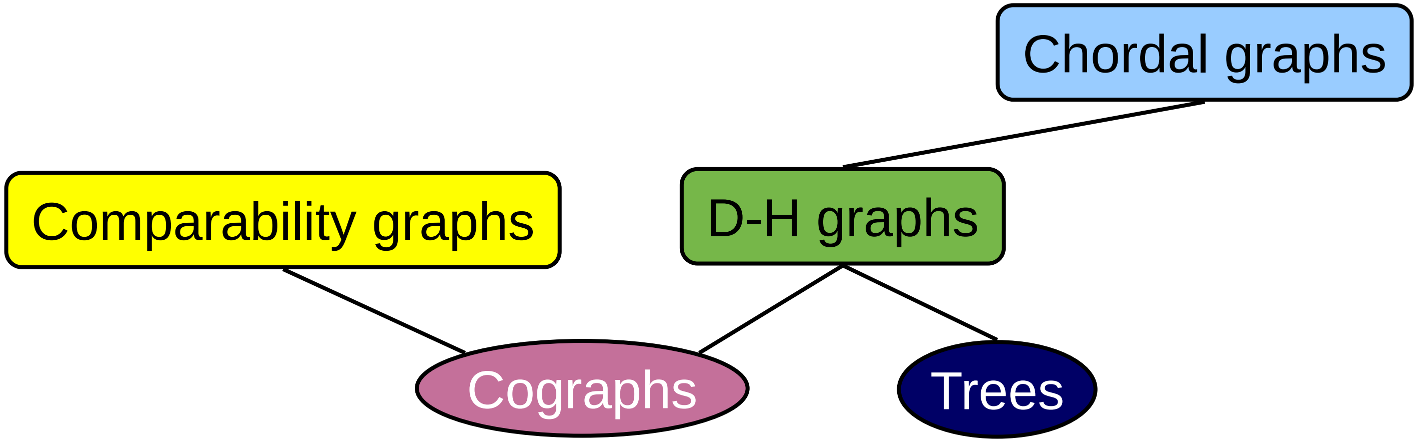
Lecture 9-1 More Class of Graphs

§ 9.2 D-H Graphs

Slides for a Course Based on the Paper

9.2 D-H Graphs



- **Def:** G **is a cograph (complement-reducible graph) if** G **can be construct from** K_1 **by disjoint union (**$\oplus$**) and join (**$\otimes$**) operations.**

- **Ex:**

K_1

$2K_1 = K_1 \oplus K_1$

$K_1 \otimes K_1$

$3K_1 = K_1 \oplus K_1 \oplus K_1$

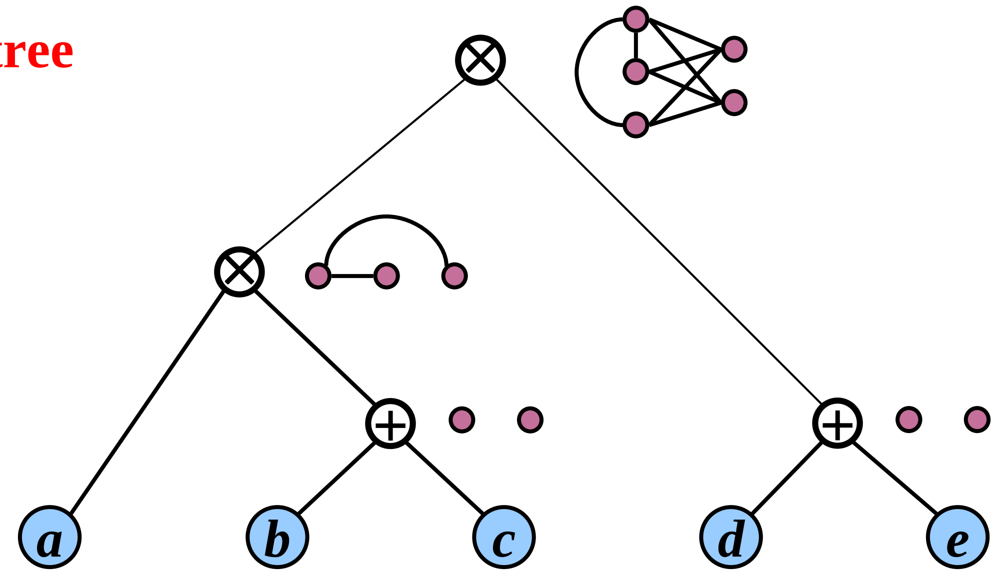
$(K_1 \otimes K_1) \otimes K_1$

$K_1 \oplus (K_1 \otimes K_1)$

$(K_1 \oplus K_1) \otimes K_1$

9.2 D-H Graphs

- **Note: 可將join ⊗換成complement.**
- **Def: cotree**



- **http://wwwteo.informatik.uni-rostock.de/isgci/classes/gc_151.html**
- **Exercise 6: 5/31 For any cograph *G* and the associated cotree, give an algorithm for finding the domination number $\gamma(G)$ of *G*.**

9.2 D-H Graphs

- **Thm**: G is cograph. ⇔ G is P_4-free.

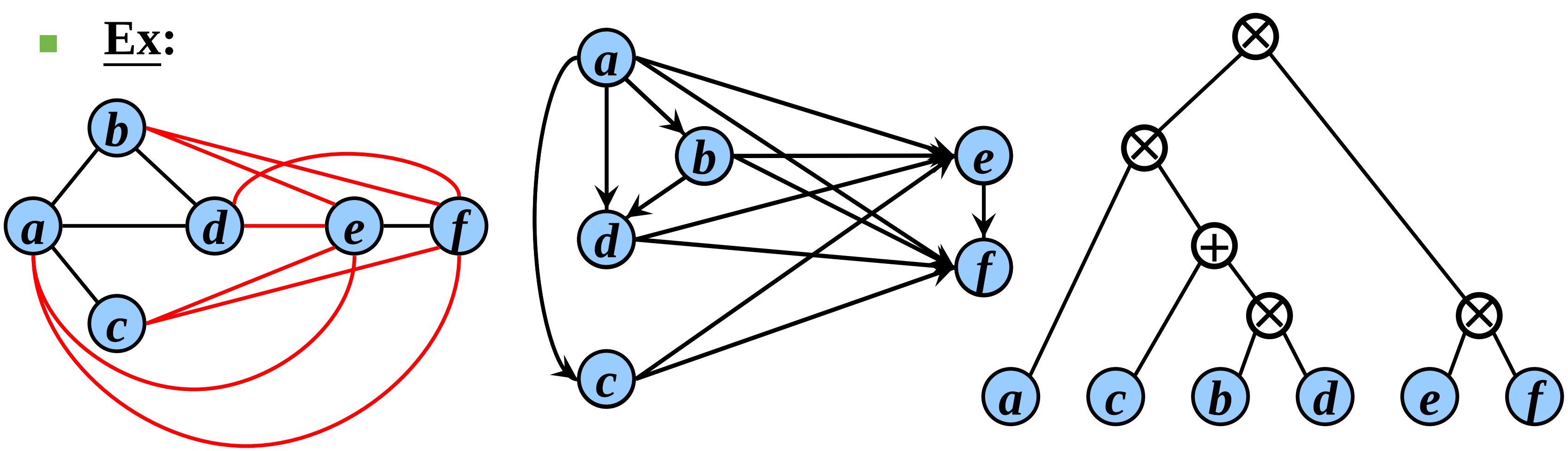
  (⇒) ∵ "join" or "union"

  ∴ $\forall\, x, y \in V(G)$, either $d_G(x, y) \le 2$ or $= \infty$.

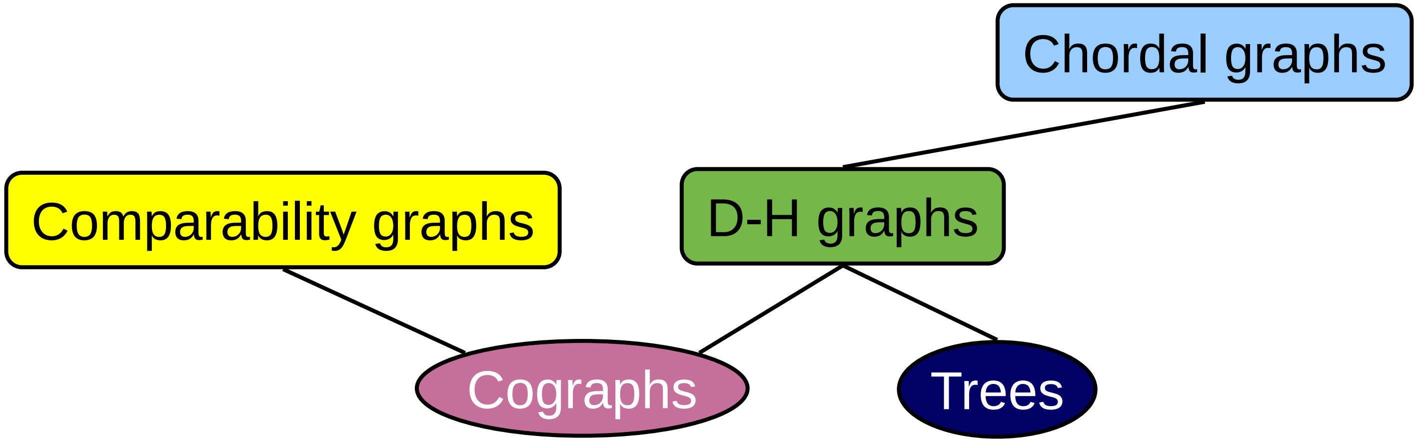
  ⇒ G has no P_4.

- **Thm**: G is cograph. ⇔ G is a comparability graph of series-parallel posets.

- **Ex**:



9.2 D-H Graphs

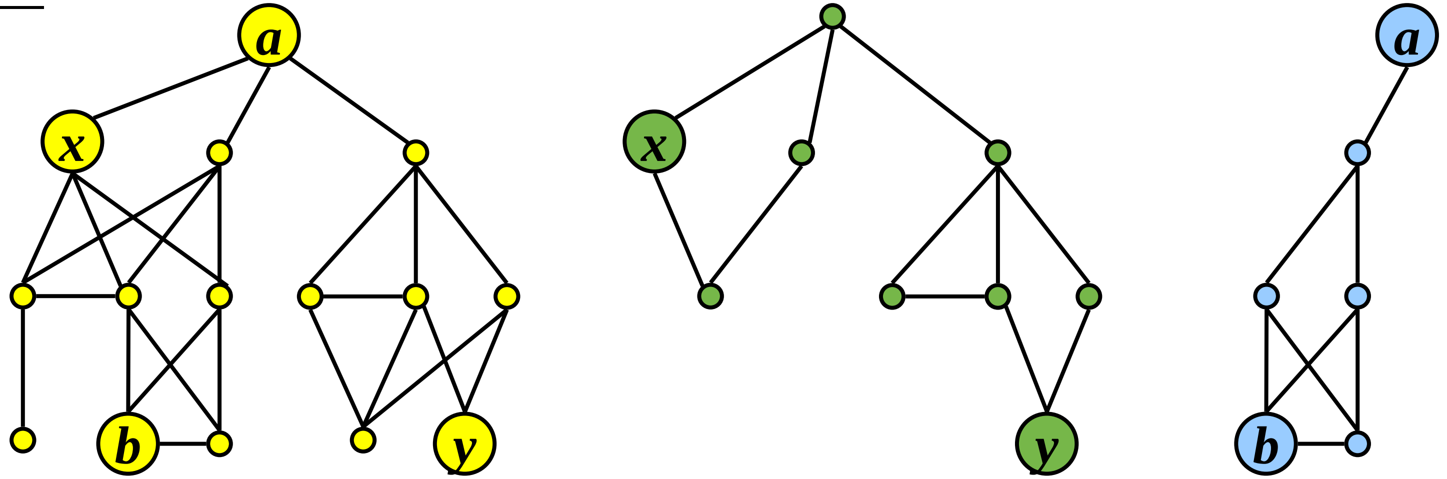


- **Def: *G* is distance-hereditary (D-H) graph if $d_H(x, y) = d_G(x, y)$ $\forall$ *x*, *y* $\in$ *V*(*H*) for any induced subgraph *H* of *G*.**

9.2 D-H Graphs

- **Def:** G is **distance-hereditary (D-H)** graph if $d_H(x, y) = d_G(x, y)$ $\forall$ $x, y \in V(H)$ for any induced subgraph H of G.

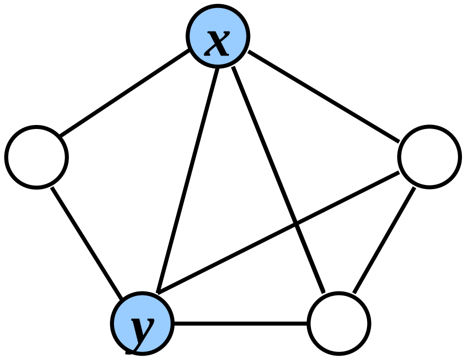
- **Ex:**

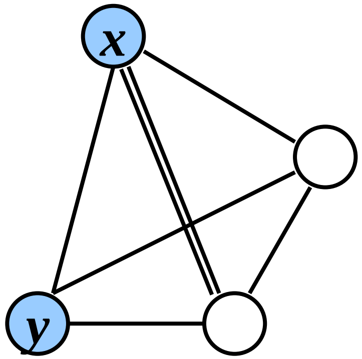


9.2 D-H Graphs

- **<u>Def</u>: *G* is a D-H graph iff *G* is (5, 2)-crossing-chordal graph.**
  i.e. for any cycle of length at least five in *G*, ∃ two crossing chords.

Two chords *uv*, *xy* are crossing iff the order in *C* of the endpoints is (*u*, *x*, *v*, *y*).





9.2 D-H Graphs

- **Problem:**
  - **Domination**
  - ***L*-Domination**
  - **Total Domination**

  } **linear-time**

  - **Weighted connected *k*-domination**
  - **Weighted *k*-dominating clique**

  } **$\mathcal{O}(|V||E|)$-time**

- **http://wwwteo.informatik.uni-rostock.de/isgci/classes/gc_80.html**