Computer Science and Information Engineering
National Chi Nan University

Combinatorial Optimization

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Lecture 8 Domatic Number Problem

§ 8.3 Cartesian Product

Slides for a Course Based on the Paper
G. J. Chang, "*The domatic number problem*," Discrete Math., 125 (1994), pp. 115-122.

8.3 Cartesian Product

- **Def:**

  ① The **Cartesian product** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \times G_2 = (V_1 \times V_2, E)$, where

  $$E = \{(x, y_1)(x, y_2): x \in V_1 \text{ and } y_1y_2 \in E_2\} \cup \{(x_1, y)(x_2, y): x_1x_2 \in E_1 \text{ and } y \in V_2\}$$

  ② P_n, the **path** of n vertices, $V(P_n) = \{1, 2, \ldots, n\}$, $E(P_n) = \{i(i+1): 1 \le i \le n-1\}$.

  ③ The **r-dimensional grid** $P_{n_1} \times P_{n_2} \times \ldots \times P_{n_r}$, where all $n_i \ge 2$.

- **Note:**

  ① $V(P_{n_1} \times P_{n_2} \times \ldots \times P_{n_r}) = \{(a_1, a_2, \ldots, a_r): 1 \le a_i \le n_i\ \forall\ 1 \le i \le r\}$
  $(a_1, a_2, \ldots, a_r)(b_1, b_2, \ldots, b_r) \in E(P_{n_1} \times P_{n_2} \times \ldots \times P_{n_r}) \Leftrightarrow$
  $\exists!\ 1 \le j \le r$, $|a_j - b_j| = 1$ and $\forall\ 1 \le i \le r$, $i \ne j$, $a_i = b_i$.

8.3 Cartesian Product

- **Note:**

  ② $d(P_{n_1} \times P_{n_2} \times \ldots \times P_{n_r}) \leq \delta(P_{n_1} \times P_{n_2} \times \ldots \times P_{n_r}) + 1 = r + 1$.

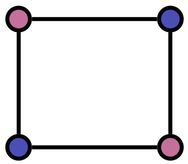
- **Remark:**

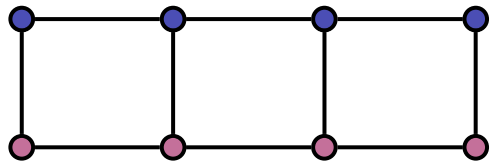
  ① P_n **is domatically full for any** $n \geq 1$.

  ② $2 \leq d(P_{n_1} \times P_{n_2}) \leq 3$.

  Let $D_1 = \{(a, b): a \text{ is odd}\}$, $D_2 = \{(a, b): a \text{ is even}\}$.
  D_1, D_2 is a domatic partition.

  ③ $d(P_2 \times P_2) = 2 = d(P_2 \times P_4) = d(P_4 \times P_2)$.



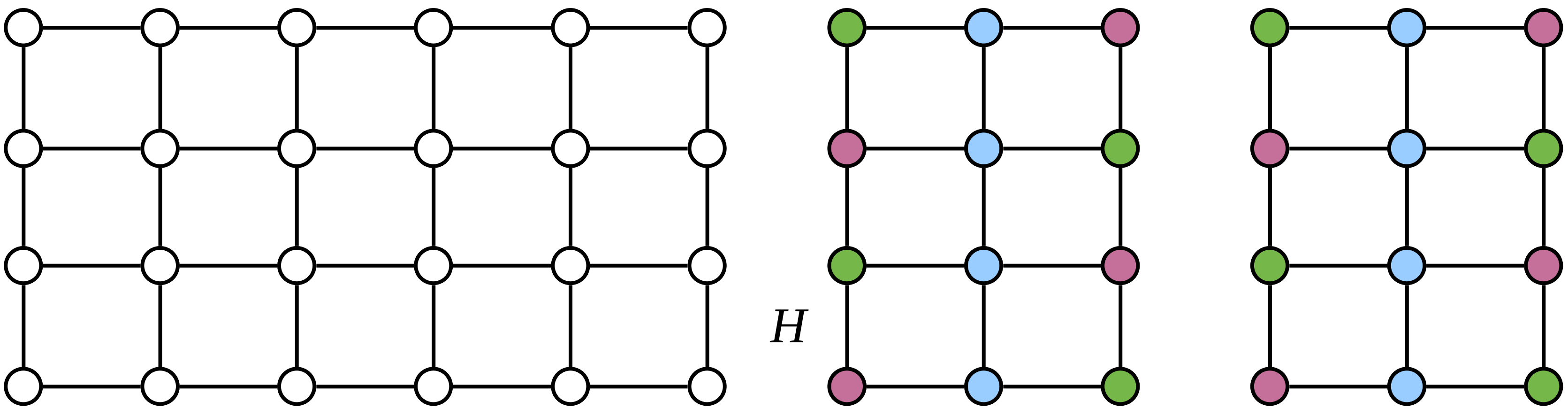


→ no dominating set with size 2

8.3 Cartesian Product

- **Proposition 3.1: For any spanning subgraph $H = (V, E')$ of $G = (V, E)$, $d(H) \le d(G)$.**

- **Ex:**



G *H*

- **Thm 3.2: $d(P_{n_1} \times P_{n_2}) = 3$ except that $(n_1, n_2) = (2, 2), (2, 4), (4, 2)$.**

8.3 Cartesian Product

- **Thm 3.2**: $d(P_{n_1} \times P_{n_2}) = 3$ except that $(n_1, n_2) = (2, 2), (2, 4), (4, 2)$.

Proof. (1/2)

Assume $(n_1, n_2) \neq (2, 2), (2, 4), (4, 2)$.

Case 1: One of n_1 and n_2 is odd, say n_1 is odd:

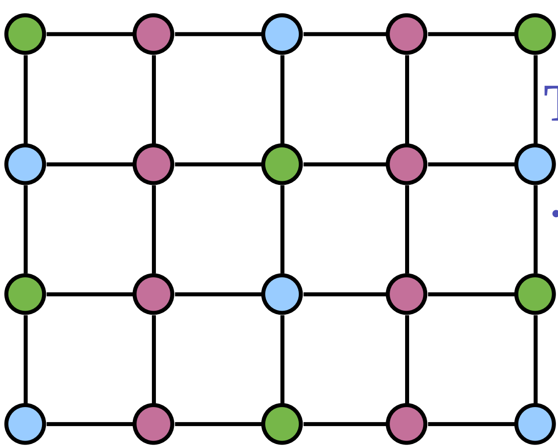
Let $D_1 = \{(a, b): a \equiv 0 \pmod 2\}$,

$D_2 = \{(a, b): a \equiv 1 \pmod 4 \text{ and } b \equiv 1 \pmod 2\} \cup$
$\{(a, b): a \equiv 3 \pmod 4 \text{ and } b \equiv 0 \pmod 2\}$,

$D_3 = \{(a, b): a \equiv 1 \pmod 4 \text{ and } b \equiv 0 \pmod 2\} \cup$
$\{(a, b): a \equiv 3 \pmod 4 \text{ and } b \equiv 1 \pmod 2\}$.

Then D_1, D_2, D_3 form a domatic partition of $P_{n_1} \times P_{n_2}$.

$\therefore d(P_{n_1} \times P_{n_2}) = 3$.



$d(P_5 \times P_4) = 3$

8.3 Cartesian Product

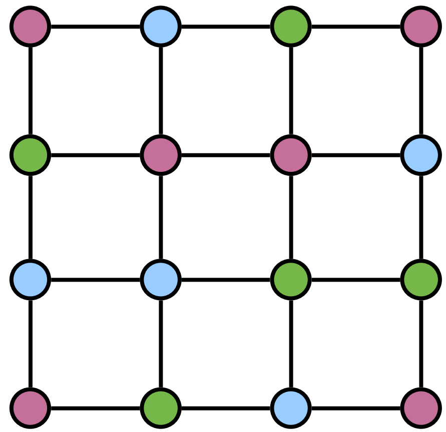
- **Thm 3.2**: $d(P_{n_1} \times P_{n_2}) = 3$ except that $(n_1, n_2) = (2, 2), (2, 4), (4, 2)$.

Proof. (2/2)

Assume $(n_1, n_2) \neq (2, 2), (2, 4), (4, 2)$.

Case 2: $(n_1, n_2) = (4, 4)$ show as follow:

$d(P_4 \times P_4) = 3$



Case 3: Both n_1, n_2 are even and at least one ≥ 6, say $n_1 \geq 6$:

$\because (P_3 \times P_{n_2}) \cup (P_{n_1-3} \times P_{n_2})$ is a spanning subgraph of $P_{n_1} \times P_{n_2}$.

$\therefore$ By Case 1 and proposition 2.1 and 3.1:

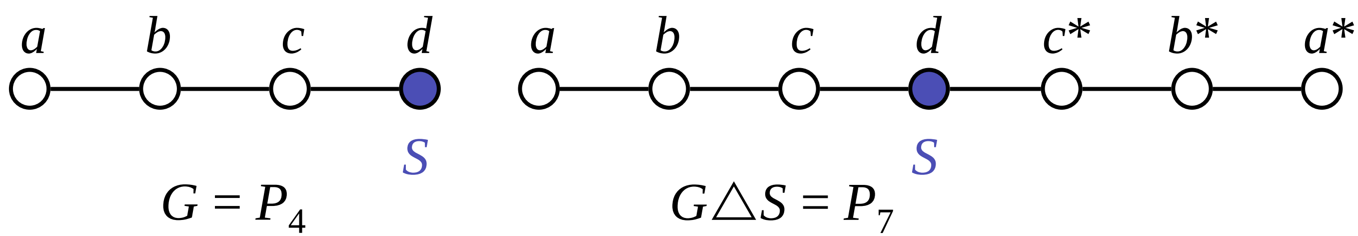
$$d(P_{n_1} \times P_{n_2}) \geq d((P_3 \times P_{n_2}) \cup (P_{n_1-3} \times P_{n_2}))$$

$$\geq \min\{d(P_3 \times P_{n_2}), d(P_{n_1-3} \times P_{n_2})\} = 3.$$

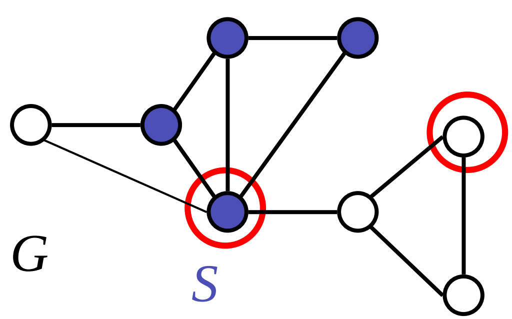
8.3 Cartesian Product

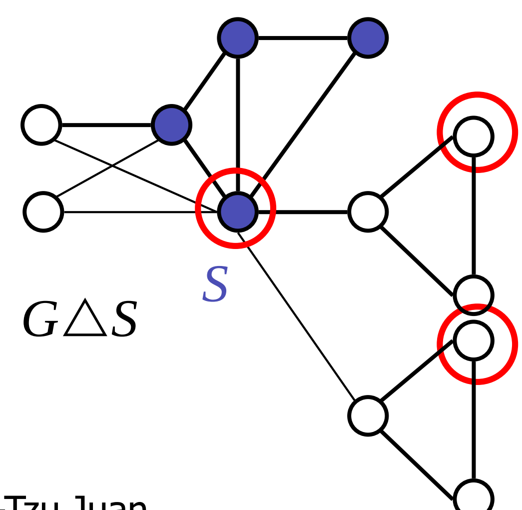
- **Def:** $G = (V, E)$ is a graph and $S \subseteq V$, let $G\triangle S = (V^*, E^*)$ with
  $V^* = V \cup \{x^*: x \in V-S\}$ and
  $E^* = E \cup \{x^*y: x \in V-S, y \in S, xy \in E\} \cup \{x^*y^*: x, y \in V-S, xy \in E\}$.

- **Ex:** ①



②





8.3 Cartesian Product

- **Lemma 3.3**: $S \subseteq V$ and $d(G \triangle S) \geq d(G)$.

Proof. $\forall$ dominating set D of G,

$D^* = D \cup \{x^*: x \in D - S\}$ is a dominating set of $G \triangle S$.

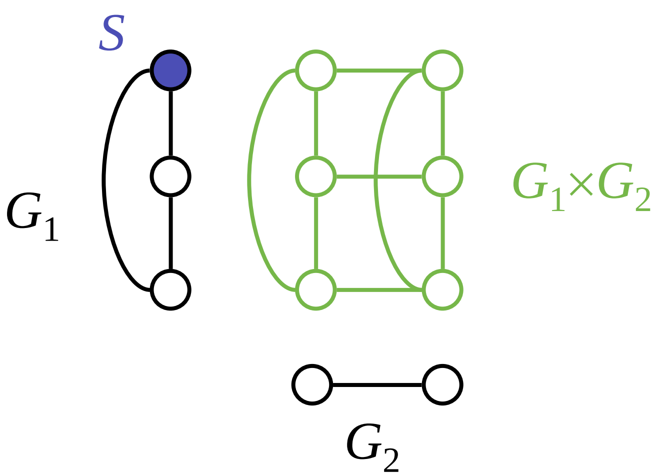
- **Lemma 3.4**: If x is an end vertex of P_n, the $P_n \triangle \{x\} \cong P_{2n-1}$.

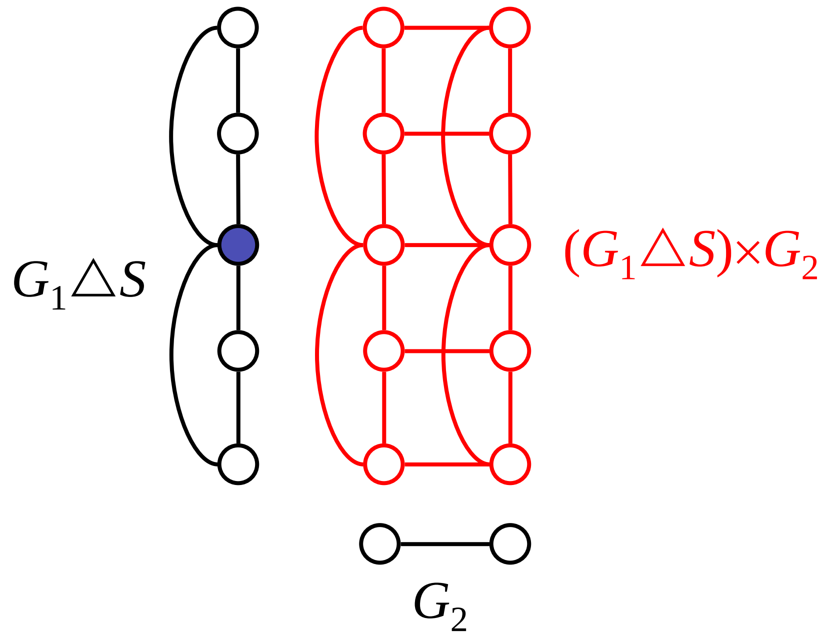
- **Lemma 3.5**: $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, $S \subseteq V_1$

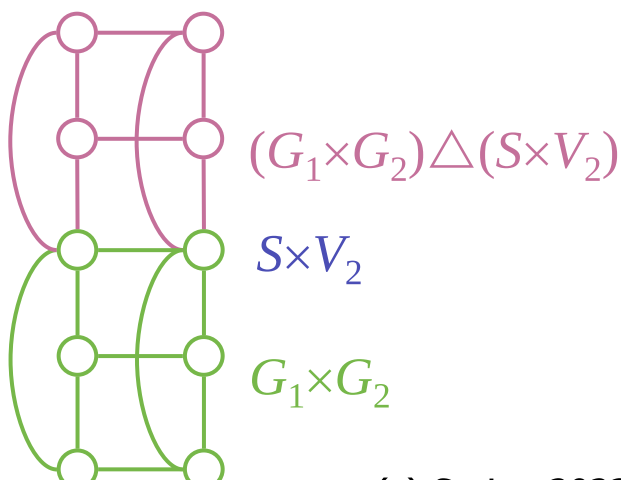
$$\Rightarrow (G_1 \triangle S) \times G_2 \cong (G_1 \times G_2) \triangle (S \times V_2).$$

8.3 Cartesian Product

- **Ex:**







8.3 Cartesian Product

- **Thm 3.6: If $r, n \in \mathbb{N}$ and $n_i \in \{n, 2n-1\}\ \forall\ 1 \le i \le r$, then**

$$d(P_{n_1} \times P_{n_2} \times \ldots \times P_{n_r}) \ge d(\underbrace{P_n \times P_n \times \ldots \times P_n}_{r}).$$

Proof. Let $h = |\{i: n_i = 2n-1\ \forall\ 1 \le i \le r\}|$.

Prove by induction on h:

(i) When $h = 0$, it's trivial.

(ii) Suppose it's true when $h < k$; when $h = k$, W.L.O.G. say $n_1 = 2n-1$

$\because$ By Lemma 3.4 and 3.5

$$\therefore P_{2n-1} \times P_{n_2} \times \ldots \times P_{n_r} \cong (P_n \triangle \{x\}) \times P_{n_2} \times \ldots \times P_{n_r}$$

$$\cong (P_n \times P_{n_2} \times \ldots \times P_{n_r}) \triangle (\{x\} \times V_2 \times \ldots \times V_r)$$

By Lemma 3.3, $d((P_n \times P_{n_2} \times \ldots \times P_{n_r}) \triangle (\{x\} \times V_2 \times \ldots \times V_r))$

$$\ge d(P_n \times P_{n_2} \times \ldots \times P_{n_r})$$

$$\Rightarrow d(P_{2n-1} \times P_{n_2} \times \ldots \times P_{n_r}) \ge d(P_n \times P_{n_2} \times \ldots \times P_{n_r}) \ge d(P_n \times P_n \times \ldots \times P_n).$$

↑ By I.H.

8.3 Cartesian Product

- **Note:**
  ① $gcd(n, 2n-1) = 1$.
  ② $\exists\, n_0$ s.t. $\forall\, m \geq n_0 \in \mathbb{Z}$, $m = rn + s(2n-1)$ for some $r, s \in \mathbb{N} \cup \{0\}$.
  ③ Denote the minimum such n_0 by $M(n)$.

- **Ex:** $M(2) = 2$, $M(3) = 8$.

- **Thm 3.7:** If $r, n \in \mathbb{Z}^+$ and $n_1, n_2, \ldots, n_r \geq M(n)$, then

$$d(P_{n_1} \times P_{n_2} \times \ldots \times P_{n_r}) \geq d(P_n \times P_n \times \ldots \times P_n).$$

Proof. $\because \forall\, n_i, \exists\, r_i, s_i \in \mathbb{N} \cup \{0\}$ s.t. $n_i = r_i n + s_i(2n-1)$.

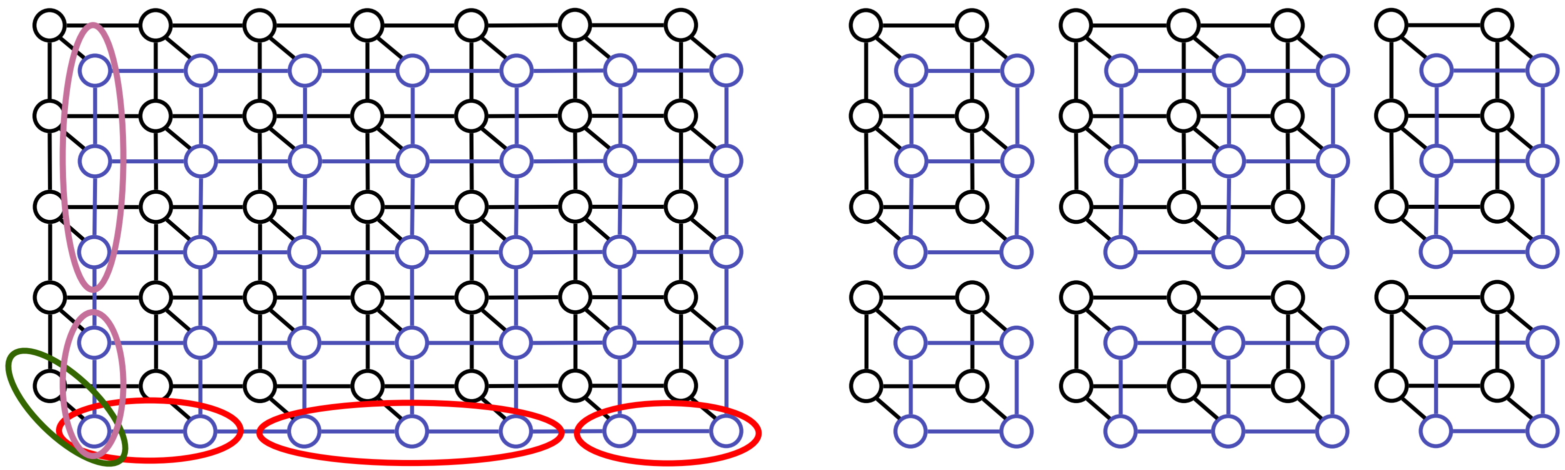
$\therefore P_{n_1} \times P_{n_2} \times \ldots \times P_{n_r}$ has a spanning subgraph which is the union of some grids $P_{m_1} \times P_{m_2} \times \ldots \times P_{m_r}$, where $m_i \in \{n, 2n-1\}\ \forall\, 1 \leq i \leq r$.

By Propositions 2.1, 3.1 and Thm 3.6,

$$d(P_{n_1} \times P_{n_2} \times \ldots \times P_{n_r}) \geq d(P_n \times P_n \times \ldots \times P_n).$$

8.3 Cartesian Product

- **Ex**: $n = 2$:



- **Thm 3.8: (Laborde 1987, Zelinka 1983)**
  If $k \in \mathbb{N}$ and $r = 2^k - 1$, then $\underbrace{P_2 \times P_2 \times \ldots \times P_2}_{r}$ is domatically full.
- **Corollary 3.9: If $k \in \mathbb{N}$, $n_i > 1$ for all i and $r = 2^k - 1$, then $P_{n_1} \times P_{n_2} \times \ldots \times P_{n_r}$ is domatically full.**

8.3 Cartesian Product

- **Conjecture: All r-dimensional grids, with finitely many exceptions, are domatically full.**

- **Note:**

  ① If we can find some n such that the r-dimensional grids, $P_n \times P_n \times \ldots \times P_n$ is domatically full for all r, then the conjecture is true.

  ② If we can find a domatically full r-dimensional grids for all r, then the conjecture is true.

Sol. ① By Thm 3.6.

② If we find $P_{n_1} \times P_{n_2} \times \ldots \times P_{n_r}$ is domatically full, then let $n = lcm(n_1, n_2, \ldots, n_r)$.

$\Rightarrow P_n \times P_n \times \ldots \times P_n$ is domatically full.