Computer Science and Information Engineering
National Chi Nan University

Combinatorial Optimization

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Lecture 8 Domatic Number Problem

§ 8.1 Strongly Chordal Graphs

Slides for a Course Based on the Paper

S.-L. Peng and M.-S. Chang, "*A Simple Linear Time Algorithm for the Domatic Partition Problem on Strongly Chordal Graphs*," Inform. Process. Lett., 43 (1992), pp. 297-300.

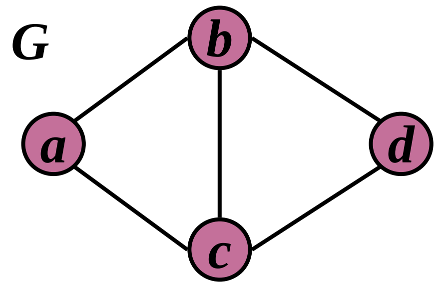
8.1 Strongly Chordal Graphs

- **Def:**
  - ① The **domatic number** of G, $d(G)$ = max. number of pairwise disjoint dominating sets in G.
  - ② The **domatic partition** problem is to partition $V(G)$ into $d(G)$ disjoint dominating sets.

- **Note:** $d(G) \leq \delta(G) + 1$

- **Def:**
  - ③ G is **domatically full** if $d(G) = \delta(G) + 1$.

- **Ex:**



$S_1 = \{b\}$
$S_2 = \{c\}$
$S_3 = \{a, d\}$

$d(G) = 3$

8.1 Strongly Chordal Graphs

- **Remark:**

  ① **K_n, $\overline{K_n}$, C_{3n}, Trees, maximal outer planar graphs, interval graph are domatically full.**

  ②

The domatic partition problem $\|V\| = n, \|E\| = m$			
General Graphs	1979	Garey and Johnson	NP-hard
Interval Graphs	1988	Bertossi	$\mathcal{O}(n^{2.5})$
	1989	Rao and Rangan	$\mathcal{O}(m+n)$
	1990	Lu, Ho, and Chang	
	1991	Peng and Chang	$\mathcal{O}(n)$ with sorted intervals
Proper Interval Graphs	1988	Bertossi	$\mathcal{O}(n\log n)$
Proper Circular-arc Graphs	1985	Bonuccell	$\mathcal{O}(n^2\log n)$
Circular-arc Graphs			NP-hard

  ③ **1989, Farber showed that strongly chordal graphs are domatically full; following the proof, it can design a polynomial time algorithm, but not simple and efficient.**

8.1 Strongly Chordal Graphs

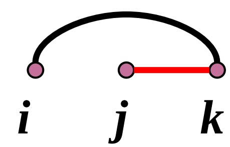
- **Remark:**

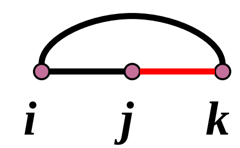
  ④ There exist an $\mathcal{O}(n\log n)$ algorithm to recognize strongly chordal graphs and determines a strong elimination ordering.

- **Recall:**

  ① Graph G is called a strongly chordal gaphs if $\exists$ ordering of $V(G)$: $[v_1, v_2, \ldots, v_n]$ satisfy $i \le j, k \le l, i \sim k, i \sim l, j \sim k \Rightarrow j \sim l$ (SEO)

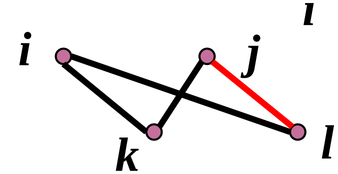
  ② $\begin{cases} \text{(IO)} \quad i < j < k, i \sim k \Rightarrow j \sim k \\ \text{(PEO)} \; i < j < k, i \sim j, i \sim k \Rightarrow j \sim k \\ \text{(SEO)} \; i \le j, k \le l, i \sim k, i \sim l, j \sim k \Rightarrow j \sim l \end{cases}$

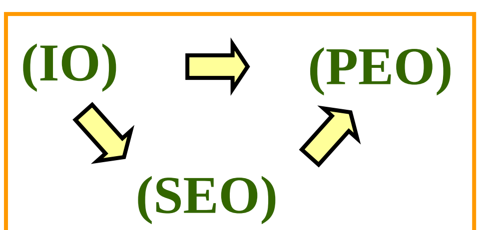




$\Updownarrow$

$\begin{cases} \text{(a) PEO} \\ \text{(b) } i < j < k < l, i \sim k, i \sim l, j \sim k \Rightarrow j \sim l \end{cases}$





8.1 Strongly Chordal Graphs

- **Def:**

  ① A vertex v is **dominated** by set S if $\exists\, u \in S$ s.t. $u \in N[v]$.

  ② A vertex v is **completely dominated** if v is dominated by $\delta + 1$ dominating sets.

- **Algorithm 8:**

  $S_i \leftarrow \phi$ for $1 \le i \le \delta + 1$.
  for $i = n$ to 1 by –1 do
  find the largest k with $v_k \in N[v_i]$ and v_k is not completely dominated;
  Let S_l be the set does not dominated v_k;
  if no such set exists then select any S_l;
  $S_l \leftarrow S_l \cup \{v_i\}$;
  end

- Time Complexity = $\mathcal{O}(|V|+|E|)$.

find the largest k with $v_k \in N[v_i]$ and v_k is not completely dominated;
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if no such set exists then select any S_l;
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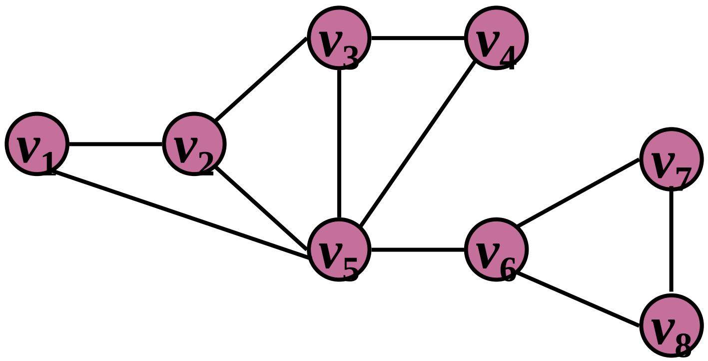
- **Ex:**

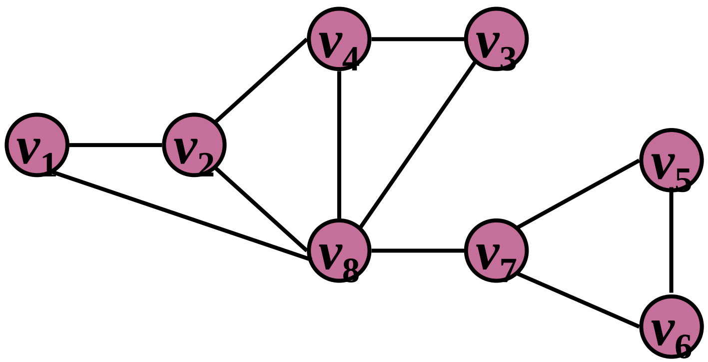


$S_1 = \{ \quad \}$

$S_2 = \{ \quad \}$

$S_3 = \{ \quad \}$

i	1	2	3	4	5	6	7	8
k								
l								



$S_1 = \{ \quad \}$

$S_2 = \{ \quad \}$

$S_3 = \{ \quad \}$

i	1	2	3	4	5	6	7	8
k								
l								

8.1 Strongly Chordal Graphs

- **Def: During execution of Algorithm 8**
  ① $R^{(j)}(v)$ = $|\{ x \in N[v]$: x not in any of $S_l \}|$ in iteration $i = j$ executed.
  ② $ndom^{(j)}(v)$ = number of S_l does not dominate v in iteration $i = j$ executed.

- **Note:** ① $R^{(n+1)}(v) = deg(v) + 1, \forall v \in V(G)$.
  ② $ndom^{(n+1)}(v) = \delta(G) + 1, \forall v \in V(G)$.
  ③ $R^{(1)}(v) = 0, \forall v \in V(G)$.

- **Goal:** $ndom^{(1)}(v) = 0, \forall v \in V(G)$.

8.1 Strongly Chordal Graphs

- **Thm: At any time, $R^{(i)}(v) \geq ndom^{(i)}(v)$ for any vertex v.**

Proof. (1/3)

By induction on i:

When $i = n + 1$ (initial): $\because deg(v) \geq \delta(G), \forall v \in V(G)$.

$\therefore R^{(n+1)}(v) \geq ndom^{(n+1)}(v), \forall v \in V(G)$.

Suppose $R^{(j)}(v) \geq ndom^{(j)}(v), \forall v \in V(G), \forall i < j \leq n+1$.

When iteration i:

Algorithm will
- **① select $v_k \in N[v_i]$ not completely dominated, and k is maximum;**
- **② select S_l does not dominate v_k;**
- **③ $S_l \leftarrow S_l \cup \{v_i\}$.**

Note: 只有 $j \sim i$ 才可能改變 $R^{(i)}(v_j)$, $ndom^{(i)}(v_j)$ 之值。
且若有改變，

$$\begin{cases} R^{(i)}(v_j) = R^{(i+1)}(v_j) - 1. \\ ndom^{(i)}(v_j) = ndom^{(i+1)}(v_j) \text{ or } ndom^{(i+1)}(v_j) - 1. \end{cases}$$

8.1 Strongly Chordal Graphs

- **Thm: At any time, $R^{(i)}(v) \geq ndom^{(i)}(v)$ for any vertex v.**

Proof. (2/3)

$\Rightarrow$ **Only need to see the cases of:** (a) $R^{(i)}(v_j) = R^{(i+1)}(v_j) - 1$

(b) $ndom^{(i)}(v_j) = ndom^{(i+1)}(v_j)$

(c) $R^{(i+1)}(v_j) = ndom^{(i+1)}(v_j)$

$\because$ (b), that means $\exists\, p > i$, $v_p \in S_l$, $p \sim j$ …(d)

Case 1: $j > k$:

$\because$ ①, k is maximum, $\therefore$ $ndom^{(i+1)}(v_j) = 0$.

$\Rightarrow ndom^{(i)}(v_j) = 0 \leq R^{(i)}(v_j)$.

Case 2: $j \leq k$:

$\because$ $i < p$ ($\because$ (d)), $j \leq k$ ($\because$ Case 2) and $i \sim j$ ($\because$ Note), $i \sim k$ ($\because$ ①), $p \sim j$ ($\because$ (d))

$\Rightarrow$ by (SEO), $p \sim k$

8.1 Strongly Chordal Graphs

- **Thm: At any time, $R^{(i)}(v) \geq ndom^{(i)}(v)$ for any vertex v.**

Proof. (3/3)

Case 2: $j \leq k$: ∵ Case 2 ∵ ①

∵ $i < p$, $j \leq k$ and $i \sim j$, $i \sim k$, $p \sim j$

∵ (d) ∵ Note ∵ (d)

⇒ by (SEO), $p \sim k$

∵ $v_p \in S_l$ (before iteration i)

but by ③, S_l does not dominate v_k →←

∴ $R^{(i)}(v) \geq ndom^{(i)}(v)$, $\forall\ v \in V(G)$ at any time.

- **Corollary: Algorithm 8 is true.**

$\forall\ v \in V(G)$, $R^{(1)}(v) \geq ndom^{(1)}(v) \Rightarrow 0 \geq ndom^{(1)}(v)$.
∴ $ndom^{(1)}(v) = 0$

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§ 8.2 Graph Union and Joint

Slides for a Course Based on the Paper
G. J. Chang, "*The domatic number problem*," Discrete Math., 125 (1994), pp. 115-122.

8.2 Graph Union and Join

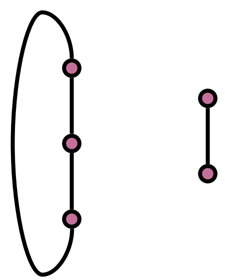
- **<u>Def</u>:** $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ **are two graphs with** $V_1 \cap V_2 = \phi$:

  ① **The union of** G_1 **and** G_2, $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

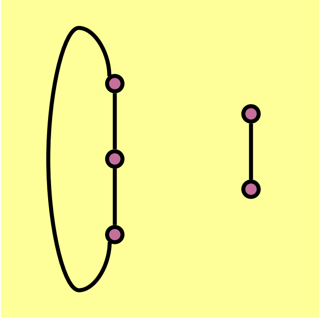
  ② **The join of** G_1 **and** G_2, $G_1 + G_2 = (G_1 \cup G_2) + \{xy : x \in V_1, y \in V_2\}$

  $= (V_1 \cup V_2, E_1 \cup E_2 \cup \{xy : x \in V_1, y \in V_2\})$

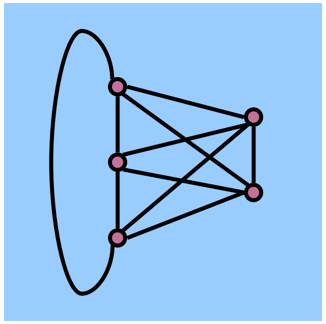
- **<u>Ex</u>:**



G_1 G_2



$G_1 \cup G_2$

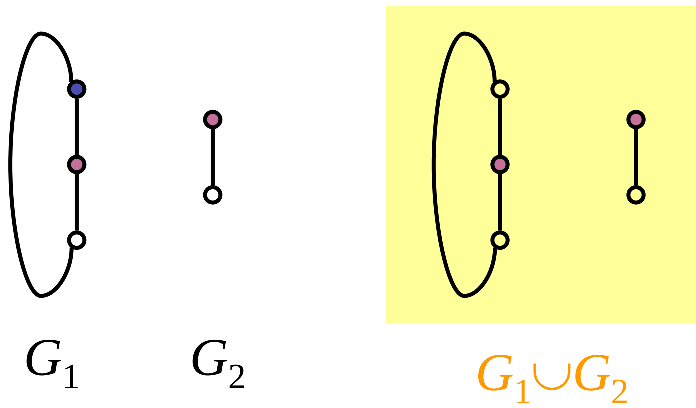


$G_1 + G_2$

8.2 Graph Union and Join

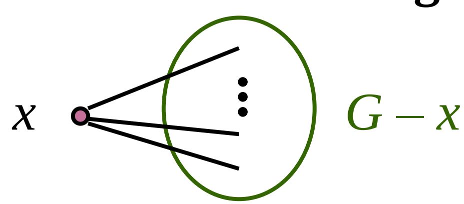
- **Proposition 2.1**: $d(G_1 \cup G_2) = \min\{d(G_1), d(G_2)\}$ for any two graphs G_1 and G_2.

- **Ex:**



- **Def**: $v \in V(G)$ is called **dominating vertex** if $\{v\}$ is a dominating set of G, i.e. $N[v] = V(G)$.

- **Note**: If x is a dominating vertex of G, then $G \cong (G - x) + K_1$.



8.2 Graph Union and Join

- **<u>Proposition 9.2</u>: If x is a dominating vertex of G, then $d(G) = d(G - x) + 1$.**

Proof.

① Let $D_1, D_2, \ldots, D_k$ be a domatic partition of $G - x$, where $k = d(G–x)$

$\Rightarrow D_1, D_2, \ldots, D_k, \{x\}$ form a domatic partition of G.

$\therefore\ d(G) \geq d(G - x) + 1$.

② Let $D_1, D_2, \ldots, D_k$ be a domatic partition of G, where $k = d(G)$

Assume $x \in D_1$, note that $D_1 \cup D_2 - \{x\}, D_3, \ldots, D_k$ is a domatic partition of $G - x$

$\therefore\ d(G) \leq d(G - x) + 1$.

$\therefore\ d(G) = d(G - x) + 1$.

8.2 Graph Union and Join

- **Note: Let $r \in \mathbb{N}$ and $r \geq 2$. If $G_1, G_2, \ldots, G_r$ are graph without a dominating vertex, the $G_1 + G_2 + \ldots + G_r$ also has no dominating vertex.**

- **Thm 2.3: Suppose $r \geq 2$ and $|V(G_i)| = n_i$, and G_i has no dominating vertex, $\forall\ 1 \leq i \leq r$. If $1 \leq n_1 \leq n_2 \leq \ldots \leq n_r$ and $n_1 + n_2 + \ldots + n_{r-1} \geq n_r$, then $d(G_1 + G_2 + \ldots + G_r) = \lfloor (n_1 + n_2 + \ldots + n_r)/2 \rfloor$.**

Proof. (1/4)

① $\because$ $G_1 + G_2 + \ldots + G_r$ has no dominating vertex

$\therefore$ each dominating set contains ≥ 2 vertices

$\Rightarrow d(G_1 + G_2 + \ldots + G_r) \leq \lfloor (n_1 + n_2 + \ldots + n_r)/2 \rfloor$.

② Claim: $G_1 + G_2 + \ldots + G_r$ has a domatic partition $D_1, D_2, \ldots, D_k$ s.t. $|D_1| = 2$ or 3, $|D_i| = 2$ $\forall\ 2 \leq i \leq k$, where $k = \lfloor (n_1 + n_2 + \ldots + n_r)/2 \rfloor$.

8.2 Graph Union and Join

- **Claim of proof ② in Thm 2.3:**

  $G_1 + G_2 + \ldots + G_r$ has a domatic partition $D_1, D_2, \ldots, D_k$ s.t. $|D_1| = 2$ or 3, $|D_i| = 2$ $\forall$ $2 \le i \le k$, where $k = \lfloor (n_1 + n_2 + \ldots + n_r)/2 \rfloor$.

Proof. (2/4)

Prove by induction on $n = n_1 + n_2 + \ldots + n_r$.

(i) $n \le 3$: it's clearly. (let $D_1 = V$)

$r = 2$: $\Rightarrow n_1 = n_2$, it's true.

(ii) Suppose $n \ge 4$, $r \ge 3$ and the assertion is true for $n' = n - 2$:

Choose $x \in V(G_{r-1})$, $y \in V(G_r)$

Consider $G' = G_1 + G_2 + \ldots + G_{r-2} + (G_{r-1} - x) + (G_r - y)$.

Case 1: $n_{r-2} < n_r$

$$\begin{cases} n_1 \le n_2 \le \ldots \le n_{r-2} \le n_r - 1,\ n_{r-1} - 1 \le n_r - 1 \\ n_1 + n_2 + \ldots + n_{r-2} + (n_{r-1} - 1) \ge n_r - 1 \end{cases}$$

8.2 Graph Union and Join

- **Claim of proof ② in Thm 2.3:**

  $G_1 + G_2 + \ldots + G_r$ has a domatic partition $D_1, D_2, \ldots, D_k$ s.t. $|D_1| = 2$ or 3, $|D_i| = 2\ \forall\ 2 \le i \le k$, where $k = \lfloor (n_1 + n_2 + \ldots + n_r)/2 \rfloor$.

Proof. (3/4)

(ii) Case 2: $n_{r-2} = n_{r-1} = n_r$

$n_1 \le n_2 \le \ldots \le n_{r-3} \le n_{r-2}$, $n_{r-1}-1 = n_r-1 < n_{r-2}$

Case 2.1: $n_r = n_{r-1} = n_{r-2} \ge 2$

$n_1 + n_2 + \ldots + n_{r-3} + (n_{r-1}-1) + (n_r-1) \ge n_r = n_{r-2}$

Case 2.2: $n_r = n_{r-1} = n_{r-2} = 1$

$\because n \ge 4 \quad \therefore r = n \ge 4$

$\Rightarrow n_1 + n_2 + \ldots + n_{r-3} + (n_{r-1}-1) + (n_r-1) \ge n_{r-3} = 1 = n_{r-2}$

By I.H., G' has a domatic partition of $\lfloor (n_1 + \ldots + (n_{r-1}-1) + (n_r-1))/2 \rfloor = k - 1$ dominating sets; say $D_1, D_2, \ldots, D_{k-1}$ with $|D_1| = 2$ or 3, $|D_i| = 2\ \forall\ 2 \le i \le k$.

8.2 Graph Union and Join

- **Claim of proof ② in Thm 2.3:**
  $G_1 + G_2 + \ldots + G_r$ has a domatic partition $D_1, D_2, \ldots, D_k$ s.t. $|D_1| = 2$ or 3, $|D_i| = 2$ $\forall$ $2 \le i \le k$, where $k = \lfloor (n_1 + n_2 + \ldots + n_r)/2 \rfloor$.

Proof. (4/4)

$\Rightarrow D_1, D_2, \ldots, D_{k-1}, \{x, y\} = D_k$ form the desired domatic partition of $G_1 + G_2 + \ldots + G_r$.

- **Def:**
  ① **Let $m \in \mathbb{N} \cup \{0\}$, an m-domatic partition of a graph $G = (V, E)$ is $\{D_1, D_2, \ldots, D_k\}$, where D_i is a dominating set of G and $D_i \cap D_j = \phi$ $\forall$ $1 \le i < j \le k$ and $|D_1 \cup D_2 \cup \ldots \cup D_k| \le m$.**
  ② **The m-domatic number $d(G|m)$ of G is the maximum k s.t. $\exists$ an m-domatic partition of k dominating sets.**

8.2 Graph Union and Join

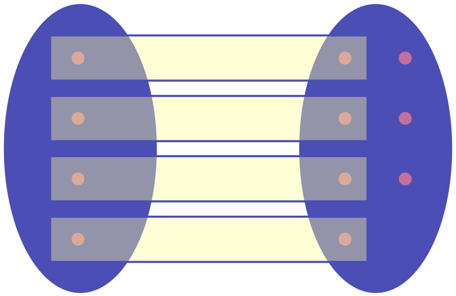
- **<u>Note</u>: For any graph G of n vertices, $d(G) = d(G|n)$.**

- **<u>Proposition 2.4</u>: For any graph G and any nonnegative integers**

  $$m \le m', \; d(G|m) \le d(G|m').$$

- **<u>Thm 2.5</u>: Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are two graphs which both has no dominating vertex and $|V_1| = n_1 \le n_2 = |V_2|$. Then**

$$d(G_1+G_2|m) = \begin{cases} \lfloor m/2 \rfloor, & \text{if } 0 \le m \le 2n_1. \\ n_1 + d(G_2|m-2n_1), & \text{if } 2n_1 < m \le n_1+n_2. \end{cases}$$



8.2 Graph Union and Join

- **Thm 2.5: Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are two graphs which both has no a dominating vertex and $|V_1| = n_1 \leq n_2 = |V_2|$. Then**

$$d(G_1+G_2|m) = \begin{cases} \lfloor m/2 \rfloor, & \text{if } 0 \leq m \leq 2n_1. \\ n_1 + d(G_2|m-2n_1), & \text{if } 2n_1 < m \leq n_1+n_2. \end{cases}$$

Proof. (1/5)

Let $V_1 = \{x_1, x_2, \ldots, x_{n_1}\}$, $V_2 = \{y_1, y_2, \ldots, y_{n_2}\}$.

(1) $0 \leq m \leq 2n_1$:

① Let $D_i = \{x_i, y_i\}$, $1 \leq i \leq \lfloor m/2 \rfloor$, D_i is a dominating set of G_1+G_2

$\Rightarrow d(G_1+G_2|m) \geq \lfloor m/2 \rfloor$.

② $\because$ G_i has no dominating vertex, $\therefore$ neither does G_1+G_2

$\therefore$ $\forall$ dominating set D of G_1+G_2, $|D| \geq 2$.

$\Rightarrow d(G_1+G_2|m) \leq \lfloor m/2 \rfloor$.

$\Rightarrow d(G_1+G_2|m) = \lfloor m/2 \rfloor$.

8.2 Graph Union and Join

- **Thm 2.5**: **Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are two graphs which both has no a dominating vertex and $|V_1| = n_1 \le n_2 = |V_2|$. Then**

$$d(G_1+G_2|m) = \begin{cases} \lfloor m/2 \rfloor, & \text{if } 0 \le m \le 2n_1. \\ n_1 + d(G_2|m-2n_1), & \text{if } 2n_1 < m \le n_1+n_2. \end{cases}$$

Proof. (2/5)

(2) $2n_1 < m \le n_1+n_2$:

① Let $D_1, D_2, \ldots, D_k$ be an $(m-2n_1)$-domatic partition of G_2 , where $k = d(G_2|m-2n_1)$.

Note D_i is also a dominating set of G_1+G_2, $\forall\ 1 \le i \le k$

$\because\ n_2 - (m-2n_1) \ge n_1$

W.L.O.G. say $\{y_1, y_2, \ldots, y_{n_1}\} \cap \{D_1 \cup D_2 \cup \ldots \cup D_k\} = \phi$.

Let $D'_i = \{x_i, y_i\}$, $1 \le i \le n_1$, D'_i is a dominating set of G_1+G_2

$\Rightarrow d(G_1+G_2|m) \ge n_1 + d(G_2|m-2n_1)$.

8.2 Graph Union and Join

- **Thm 2.5: Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are two graphs which both has no a dominating vertex and $|V_1| = n_1 \le n_2 = |V_2|$. Then**

$$d(G_1+G_2|m) = \begin{cases} \lfloor m/2 \rfloor, & \text{if } 0 \le m \le 2n_1. \\ n_1 + d(G_2|m-2n_1), & \text{if } 2n_1 < m \le n_1+n_2. \end{cases}$$

Proof. (3/5)

(2) $2n_1 < m \le n_1+n_2$:

② A dominating set D of G_1+G_2 is called standard if $D = \{x, y\}$ for some $x \in V_1$, $y \in V_2$.

Claim: ∃ an m-domatic partition of G_1+G_2, $D_1, D_2, \ldots, D_r$ where $r = d(G_1+G_2|m)$, s.t. among these r dominating sets, ∃ n_1 standard ones, and the other $r - n_1$ sets are all subsets of V_2.

Proof. **Let $D_1, D_2, \ldots, D_r$ is an m-domatic partition of G_1+G_2, $r = d(G_1+G_2|m)$, s.t.**

standard dominating sets as many as possible.

Claim: $\exists$ an m-domatic partition of G_1+G_2, $D_1, D_2, \ldots, D_r$ where $r = d(G_1+G_2|m)$, s.t. among these r dominating sets, $\exists$ n_1 standard ones, and the other $r - n_1$ sets are all subsets of V_2.

- **Thm 2.5: Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are two graphs which both has no a dominating vertex and $|V_1| = n_1 \le n_2 = |V_2|$. Then**

$$d(G_1+G_2|m) = \begin{cases} \lfloor m/2 \rfloor, & \text{if } 0 \le m \le 2n_1. \\ n_1 + d(G_2|m-2n_1), & \text{if } 2n_1 < m \le n_1+n_2. \end{cases}$$

Proof. (4/5)

(2) $2n_1 < m \le n_1+n_2$:

Proof of Claim.

Case 1: If $\exists$ D_i s.t. $\{x, y\} \subset D_i$ for some $x \in V_1$, $y \in V_2$, then replace D_i by $\{x, y\}$.

Case 2: If $\exists$ D_i, D_j s.t. $\{x_a, x_b\} \subseteq D_i$ and $\{y_c, y_d\} \subseteq D_j$ for some $x_a, x_b \in V_1$, $y_c, y_d \in V_2$, then replace D_i, D_j by $\{x_a, y_c\}$, $\{x_b, y_d\}$.

Case 3: If all nonstandard dominating set D are subsets of V_1,
$\because$ $n_1 \le n_2$
$\therefore$ we can replace each nonstandard dominating set D by $\{x, y\}$, where $x \in D$, $y \in V_2 - \{D_1 \cup D_2 \cup \ldots \cup D_r\}$.

Claim: $\exists$ an m-domatic partition of G_1+G_2, $D_1, D_2, \ldots, D_r$ where $r = d(G_1+G_2|m)$, s.t. among these r dominating sets, $\exists$ n_1 standard ones, and the other $r - n_1$ sets are all subsets of V_2.

- **Thm 2.5: Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are two graphs which both has no a dominating vertex and $|V_1| = n_1 \le n_2 = |V_2|$. Then**

$$d(G_1+G_2|m) = \begin{cases} \lfloor m/2 \rfloor, & \text{if } 0 \le m \le 2n_1. \\ n_1 + d(G_2|m-2n_1), & \text{if } 2n_1 < m \le n_1+n_2. \end{cases}$$

Proof. (5/5)

(2) $2n_1 < m \le n_1+n_2$:

Proof of Claim.

Case 4: If $\exists$ $x \in V_1 - \{D_1 \cup D_2 \cup \ldots \cup D_r\}$ then

let $y \in D_j$ for some $D_j \subset V_2$ (or $y \in V_2 - \{D_1 \cup D_2 \cup \ldots \cup D_r\}$)

$\Rightarrow$ replace D_j (or any $D_j \subset V_2$) by $\{x, y\}$.

$\therefore$ By Claim, $\exists$ $r - n_1$ nonstandard dominating set of G_1+G_2

$\Rightarrow$ $\exists$ an $(m - 2n_1)$-domatic partition of G_2 with size $r - n_1$

$\Rightarrow d(G_2|m - 2n_1) \ge r - n_1 = d(G_1+G_2|m) - n_1$

$\Rightarrow d(G_1 + G_2|m) \le n_1 + d(G_2|m - 2n_1)$.

$\therefore$ By ①, ②, $d(G_1 + G_2|m) = n_1 + d(G_2|m - 2n_1)$.

8.2 Graph Union and Join

- **Corollary 2.6: Suppose $r \geq 2$, G_i is a graph with n_i vertices and without a dominating vertex, $\forall\ 1 \leq i \leq r$. If $n_1 + n_2 + \ldots + n_{r-1} < n_r$, then $d(G_1 + G_2 + \ldots + G_r) = n_1 + n_2 + \ldots + n_{r-1} + d(G_r|n_r - n_1 - \ldots - n_{r-1})$.**

Proof.

Follows from Thm 2.5 by
$$\begin{cases} G_1 = G_1 + G_2 + \ldots + G_{r-1}, \\ G_2 = G_r. \\ m = n_1 + n_2 + \ldots + n_r. \end{cases}$$

- **Corollary 2.7: If $r \geq 2$ and $n_1 + n_2 + \ldots + n_{r-1} < n_r$, then $d(\overline{K_{n_1}} + \overline{K_{n_2}} + \ldots + \overline{K_{n_r}}) = n_1 + n_2 + \ldots + n_{r-1}$.**

Proof.

Follows from Corollary 2.6 and $d(\overline{K_a}|b) = 0$ for $a > b$.

Note: Complete k-partite graph $K_{n_1, n_2, \ldots, n_r} = \overline{K_{n_1}} + \overline{K_{n_2}} + \ldots + \overline{K_{n_r}}$