Computer Science and Information Engineering
National Chi Nan University

Combinatorial Optimization

Dr. Justie Su-Tzu Juan

Lecture 7 Applications of L. P. Duality

§7.3 The Independent Set on Chordal Graphs (Primal-Dual)

7.3 The independent set on chordal graphs (Primal-Dual)

- **Def**: Given a graph G,

  ① $S \subseteq V(G)$ is called an **independent set** if $\forall\, x \neq y$ in S, $xy \notin E(G)$.

  ② $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$ is called a **clique cover** of G if

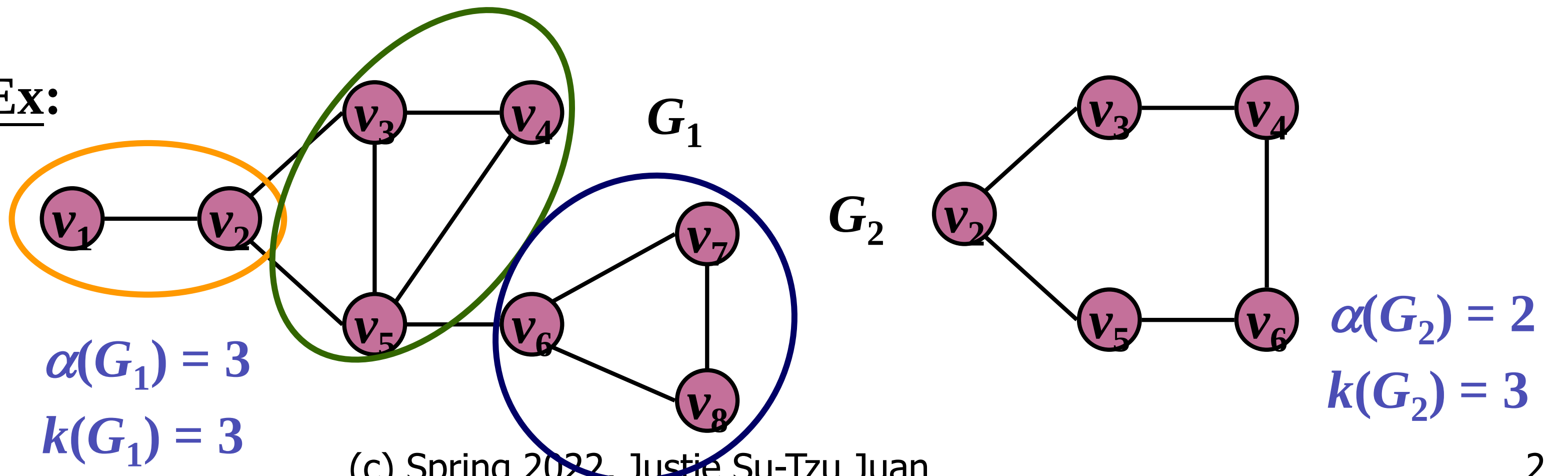
  a. $\forall\, 1 \le i \le t$, $C_i \subseteq V(G)$, $G(C_i)$ is a clique and

  b. $\bigcup_{1 \le i \le t} C_i = V$

  ③ $\alpha(G) = \max\{|S| : S$ is an independent set of $G\}$.

  ④ $k(G) = \min\{|\mathcal{C}| : \mathcal{C}$ is a clique cover of $G\}$.

- **Ex**:



7.3 The independent set on chordal graphs (Primal-Dual)

- **Thm**: $\alpha(G) \le k(G)$. (W. D. I.)

Proof.

For any independent set S and any clique cover $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$.

Let $f: S \to \mathcal{C}$ where $f(u) = C_i$ if $u \in C_i$.

Note that f is a 1 - 1 function:

$\forall\ u \ne v \in S$, if $f(u) = f(v) = C_i$ for some $1 \le i \le t$,

then $u \in C_i$ and $v \in C_i$ by definition of f.

$\because C_i$ is a clique $\quad \therefore uv \in E(G)$.

But $u, v \in S$, S is an independent set

$\Rightarrow uv \notin E(G) \to\leftarrow$

$\therefore f(u) \ne f(v)$.

$\therefore |S| \le |\mathcal{C}|$

$$\Rightarrow \max_{\forall S} |S| \le \min_{\forall \mathcal{C}} |\mathcal{C}| \Rightarrow \alpha(G) \le k(G).$$

7.3 The independent set on chordal graphs (Primal-Dual)

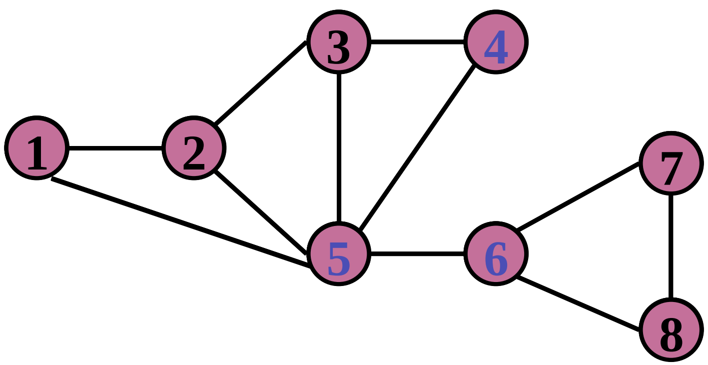
- **Remark**: $[v_1, v_2, \ldots, v_n]$ is a PEO.

  $\Leftrightarrow i < j < k,\ v_iv_j \in E,\ v_iv_k \in E \Rightarrow v_jv_k \in E$

  $\Leftrightarrow i \le j,\ i \le k,\ i \sim j,\ i \sim k \Rightarrow j \sim k$

  $\Leftrightarrow C_i = \{v_j: j \sim i, j \ge i\}$ is a clique.

- **Ex**:



7.3 The independent set on chordal graphs (Primal-Dual)

- **Algorithm:**

Given a chordal graph G with PEO $[v_1, v_2, \ldots, v_n]$

$S^* \leftarrow \phi;$

$\mathcal{C}^* \leftarrow \phi;$

$t \leftarrow 0;$

for $i = 1$ **to** n **do**

  if $N[v_i] \cap S^* = \phi$ **then**

    $S^* \leftarrow S^* \cup \{v_i\};$

    $t \leftarrow t + 1;$

    $C_t \leftarrow \{v_j: j \sim i, j \geq i\};$

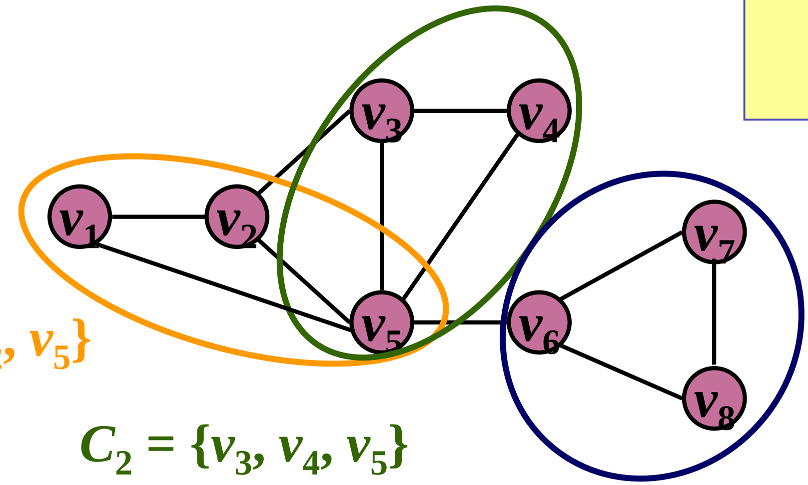
    $\mathcal{C}^* \leftarrow \mathcal{C}^* \cup C_t;$

- **Time Complexity = $\mathcal{O}(|V|+|E|)$.**

7.3 The independent set c graphs (Primal-Dual)

for $i = 1$ to n do
  if $N[v_i] \cap S^* = \phi$ then
    $S^* \leftarrow S^* \cup \{v_i\}$;
    $t \leftarrow t + 1$;
    $C_t \leftarrow \{v_j : j \sim i, j \geq i\}$;
    $\mathcal{C}^* \leftarrow \mathcal{C}^* \cup C_t$;

- **Ex:**



$C_1 = \{v_1, v_2, v_5\}$

$C_2 = \{v_3, v_4, v_5\}$

$C_3 = \{v_6, v_7, v_8\}$

$S^* = \{v_1, v_3, v_6\}$

$\mathcal{C}^* = \{C_1, C_2, C_3\}$

- **Theorem:** ① S^* is an independent set of G.
  ② $\mathcal{C}^*$ is a clique cover of G.
  ③ $|S^*| \geq |\mathcal{C}^*|$.
- **Proof. (略)**

7.3 The independent set on chordal graphs (Primal-Dual)

- **Note: ① S^* is a maximum independent set of G, i.e. $|S^*| = \alpha(G)$.**
  ② $\mathcal{C}^*$ is a minimum clique cover of G, i.e. $|\mathcal{C}^*| = k(G)$.
  ③ $\alpha(G) = k(G)$.

∵ $|S^*| \le \alpha(G) \le k(G) \le |\mathcal{C}^*| \le |S^*|$

∴ All "≤" are "=".

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§ 7.4 Weighted Independent Set on Chordal Graphs (L. P. Duality)

7.4 Weighted independent set on chordal graphs (L. P. Duality)

- **Def**: G is a weighted graph with each v_i be assigned a real weight $w_i \geq 0$,
  ① $\alpha(G, w) = \max\{ \sum_{v_i \in S} w_i \mid S$ is an independent set of $G\}$.
  ② $k(G, w) = \min\{ \sum_{C_i \in \mathcal{C}} y(C_i) \mid \mathcal{C} = \{C_1, C_2, \ldots, C_t\}$ is a clique cover of G, $\forall\, v_i, \sum_{v_i \in C_j} y(C_j) \geq w_i$, all $y(C_i) \geq 0\}$.
- **Method 1**: Primal-Dual(略)
- **Method 2**: Linear Programming
- **Notation**: ① $i \sim j \Leftrightarrow i = j$ or $ij \in E(G)$.
  ② $i \gtrsim j \Leftrightarrow i \geq j$ and $i \sim j$.
  ③ $i \lesssim j \Leftrightarrow i \leq j$ and $i \sim j$.

7.4 Weighted independent set on chordal graphs (L. P. Duality)

- **Def:** ① Let ***P*(*G*)** is the following linear program:

$$\text{Maximize } \sum_{i=1}^{n} w_i x_i$$

$$\text{Subject to } \begin{cases} \sum_{j \gtrsim i} x_j \le 1, \forall i \\ x_i \ge 0, \forall i \end{cases}$$

② Let ***P_1*(*G*)** is the linear program *P*(*G*) with $x_i \in \{0, 1\}$, $\forall$ *i*.

③ The dual problem ***D*(*G*)** of *P*(*G*) is the linear program:

$$\text{Minimize } \sum_{i=1}^{n} y_i$$

$$\text{Subject to } \begin{cases} y_i \ge 0, \forall i \\ \sum_{j \lesssim i} y_j \ge w_i, \forall i \end{cases}$$

7.4 Weighted independent set on chordal graphs (L. P. Duality)

- **Thm: (Weakly duality inequality)**

  ∀ Feasible solutions <$x_1, x_2, \ldots, x_n$>, <$y_1, y_2, \ldots, y_n$> for $P(G)$, $D(G)$,

$$\sum_{i=1}^{n} w_i x_i \le \sum_{i=1}^{n} y_i$$

- **Remark:**
  - **① ∃ 1 - 1 correspondence between feasible solutions to $P_1(G)$ and independent sets in G.**
  - **② An optimal solution to $P_1(G)$ corresponds to a maximum weighted independent set in G.**

w.d.i.

- **Note: $value(P_1(G)) \le value(P(G)) \le value(D(G))$.**

  ∵ $P_1(G)$ is a special problem of $P(G)$

7.4 Weighted independent set on chordal graphs (L. P. Duality)

- **Algorithm:**

Input: A chordal graph G with PEO $[v_1, v_2, \ldots, v_n]$ and positive vertex weights $w_1, w_2, \ldots, w_n$.

Output: Optimal solutions to $P(G)$ and $D(G)$.

Initially: $\forall\, i,\ x_i \leftarrow 0;\ \forall\, j,\ y_j \leftarrow 0.$

Stage One: for $i = 1$ to n do

$$y_i \leftarrow \max\{0,\ w_i - \sum_{j \lesssim i} y_j\}.$$

Stage Two: for $i = n$ to 1 by -1 do

if $\sum_{j \lesssim i} y_j = w_i$ and $\nexists\, j \gtrsim i$ with $x_j = 1$

then $x_i = 1$.

- **Time Complexity** = $\mathcal{O}(|V|+|E|)$.

for $i = 1$ to n do

$$y_i \leftarrow \max\{0,\ w_i - \sum_{j \lesssim i} y_j\}.$$

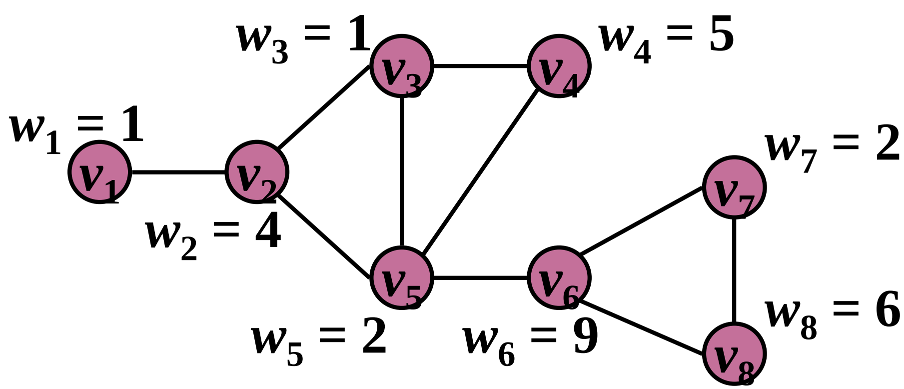
...nted ind... ...graphs (I...

for $i = n$ to 1 by -1 do

if $\sum_{j \lesssim i} y_j = w_i$ and $\nexists\, j \gtrsim i$ with $x_j = 1$

then $x_i = 1$.

- **Ex:**



$w_1 = 1$	$w_2 = 4$	$w_3 = 1$	$w_4 = 5$	$w_5 = 2$	$w_6 = 9$	$w_7 = 2$	$w_8 = 6$
$x_1 =$	$x_2 =$	$x_3 =$	$x_4 =$	$x_5 =$	$x_6 =$	$x_7 =$	$x_8 =$
$y_1 =$	$y_2 =$	$y_3 =$	$y_4 =$	$y_5 =$	$y_6 =$	$y_7 =$	$y_8 =$

Check: ① $S = \{v_2, v_4, v_6\}$, $w(S) = 18 = value(P_1(G))$

② $\forall\, i$, $\sum_{j \lesssim i} y_j \geq w_i$; and $\sum_{i=1}^{n} y_j = 18$.

7.4 Weighted independent set on chordal graphs (L. P. Duality)

- **Exercise 5 (5/17):**

  ① Prove the weakly duality inequality for *P*(*G*) and *D*(*G*).

  ② State the condition of complementary slackness ((CS1), (C*S*2))

  加分：③ Prove that the algorithm works.

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§ 7.5 Recognizing Chordal Graphs by Lexicographic BFS

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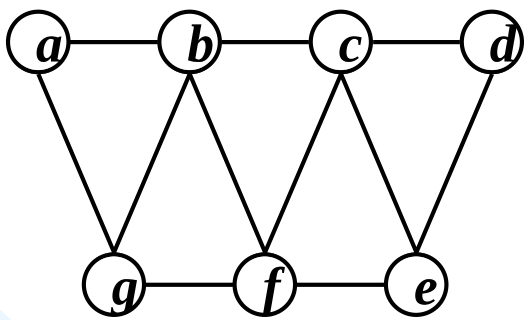
- **Def:** An undirected graph G is a **triangulated graph** (**chordal graph**) if every cycle of length > 3 possesses a chord.

- **Def:** x is **simplicial** if Adj(x) is a clique (不要求 maximal)

  (事實上, Adj(x) $\cup$ {x} is also a clique (較大))

- **Ex:**



a, *d* 是 simplicial

b, *c* 不是

- **Def:** Let $\sigma = [v_1, v_2, \ldots, v_n]$ be an ordering of the vertices of G. σ is a **perfect vertex elimination scheme** (或 **perfect scheme**)

  iff v_i is a simplicial vertex of $G_{\{v_i, v_{i+1}, \ldots, v_n\}}$.

  (i.e. $X_i = \{v_j \in \text{Adj}(v_i) \mid j > i\}$ is complete) (i.e. 對剩餘的點而言v_i是 simplicial)

§ 7.5 Recognizing Chordal Graphs by Lexicographic BFS

- **Algorithm**: **Lexicographic breadth-first search**

  Input: The adjacency sets of an undirected graph $G = (V, E)$

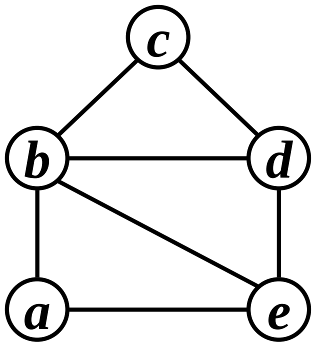
  Output: An ordering σ of the vertices (called Lex. BFS ordering)

  Method:

```
begin
    令每個 vertex 的 label 均為 the empty set ∅;
    for i ← n to 1 step –1 do
        由尚未排定 ordering 的 vertex 中選一點 v with largest label;
        σ(i) = v; (設此 ordering 中之第 i 個為 v)
        for every w ∈ Adj(v) s.t. w 尚未排定 ordering do
            label(w) ← label(w) ∪ {i};
end
```

§ 7.5 Recognizing Chordal Graphs by Lexicographic BFS

- **Ex:**



label(a) = ∅
label(b) = ∅ label(b) = {5}
label(c) = ∅ label(c) = {4} label(c) = {4, 2}
label(d) = ∅ label(d) = {4} label(d) = {4, 3}
label(e) = ∅ label(e) = {5} label(e) = {4, 5}
$\sigma(5) = a$ $\sigma(4) = b$ $\sigma(3) = e$ $\sigma(2) = d$ $\sigma(1) = c$
⇒ ordering $\sigma = c\ d\ e\ b\ a$

§ 7.5 Recognizing Chordal Graphs by Lexicographic BFS

- **Property**: **當做到 i 時, let $L_i(x)$ denote the label of x.**

  ex: $L_n(x) = \varnothing, \forall x$

  $L_{n-1}(x) = \{n\}$ iff $x \in \mathrm{Adj}(\sigma(n))$

  (L_1) $L_i(x) \le L_j(x)$ if $j \le i$

  (對同一點而言, 由 $n, n-1, \ldots,$ 至 1 時, 其 label 只會 ↑)

  (L_2) $L_i(x) < L_i(y) \Rightarrow L_j(x) < L_j(y)$ if $j < i$

  (對不同的點 x, y 而言, 由 n 做至 1 時, label 一旦比較大, 就會一直比較大!!)

§ 7.5 Recognizing Chordal Graphs by Lexicographic BFS

- **Property:** 當做到 i 時, let $L_i(x)$ denote the label of x.

  (L_3) $[\sigma^{-1}(a) < \sigma^{-1}(b) < \sigma^{-1}(c)] \wedge [c \in \text{Adj}(a) - \text{Adj}(b)]$

  $\Rightarrow \exists\, d \in \text{Adj}(b) - \text{Adj}(a)$ with $\sigma^{-1}(c) < \sigma^{-1}(d)$

  $\because \sigma^{-1}(a) < \sigma^{-1}(b) < \sigma^{-1}(c)$, 即先得 c 再 b, 再 a

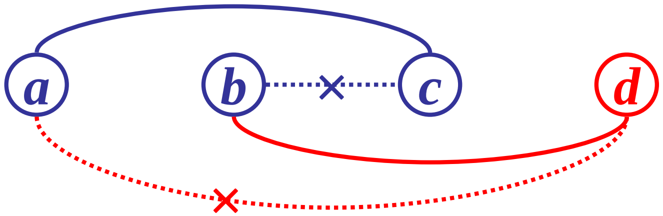
  $\because c \in \text{Adj}(a) - \text{Adj}(b)$, 即 $ac \in E$, $bc \notin E$

  $\Rightarrow$ 做到 c 時, $L_{\sigma^{-1}(c)}(a)$ 會多 $\{\sigma^{-1}(c)\}$, $L_{\sigma^{-1}(c)}(b)$ 不會得,

  可是 b 比 a 先做, 表示 $L_{\sigma^{-1}(c)}(a) > L_{\sigma^{-1}(c)}(b)$,

  $\therefore$ 必 $\exists\, d$, 其編號 $\sigma^{-1}(d) > \sigma^{-1}(c)$ 且 $bd \in E$, $da \notin E$

  i.e $\exists\, d \in \text{Adj}(b) - \text{Adj}(a)$ with $\sigma^{-1}(c) < \sigma^{-1}(d)$



§ 7.5 Recognizing Chordal Graphs by Lexicographic BFS

- **Thm 4.3:** G **is a tri. graph** $\Leftrightarrow$ **The Lex. BFS ordering** σ **is a perfect scheme.**

Proof. (1/4)

($\Leftarrow$) ok

($\Rightarrow$) $|V| = 1$, true

$|V| < n$, 設為真

$|V| = n$:

claim: $x = \sigma(1)$ is simplicial.

然後, $\because G - x$ is tri.,

在 G 中之 $\sigma(2), \ldots, \sigma(n)$ 也是 $G - x$ 之 Lex. BFS ordering

$\therefore$ By I.H., 此 ordering is perfect scheme of $G - x$

i.e. $\sigma(i)$ 是 $G_{\{\sigma(i),\sigma(i+1),\ldots,\sigma(n)\}}$ 中之 simplicial $(2 \leq i \leq n)$

$\Rightarrow$ 加上 $i = 1$ 亦滿足, $\therefore$ σ is a perfect scheme of G.

§ 7.5 Recognizing Chordal Graphs by Lexicographic BFS

- **Thm 4.3: *G* is a tri. graph ⇔ The Lex. BFS ordering σ is a perfect scheme.**

Proof. (2/4)

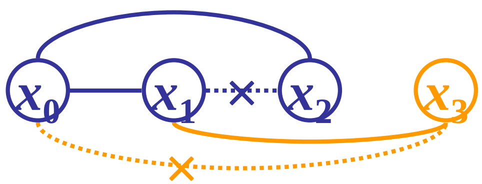
(⇒) $|V| = n$: claim: $x = \sigma(1)$ is simplicial.

Proof of claim.

Prove by 矛盾法:

若 x 非 simplicial, choose $x_1, x_2 \in \mathrm{Adj}(x)$ with $x_1, x_2 \notin E$

so that x_2 is as large as possible (w.r.t σ)



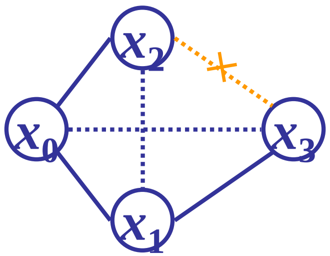
Let $x = x_0$:

由 (L_3), $\exists\, x_3 \ni x_1x_3 \in E$, 但 $x_0x_3 \notin E$

選這種 x_3 中最大者 (w.r.t. σ)

$\because\ x_1x_2 \notin E$, $x_0x_3 \notin E$ and G is tri.:

$\therefore\ x_2x_3 \notin E$.



§ 7.5 Recognizing Chordal Graphs by Lexicographic BFS

- **Thm 4.3**: G is a tri. graph $\Leftrightarrow$ The Lex. BFS ordering σ is a perfect scheme.

Proof. (3/4)

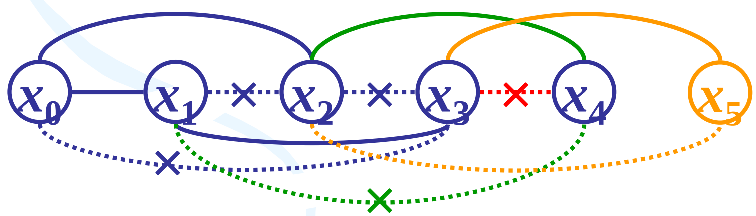
($\Rightarrow$) $|V| = n$: claim: $x = \sigma(1)$ is simplicial.

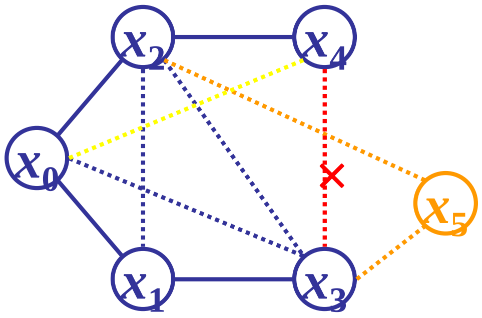
Proof of claim.

Prove by 矛盾法:

Consider x_1, x_2, x_3; 由 (L_3),

$\exists\, x_4 \ni x_2x_4 \in E$, 但 $x_1x_4 \notin E$, 選這種 x_4 中最大者 (w.r.t. σ)





§ 7.5 Recognizing Chordal Graphs by Lexicographic BFS

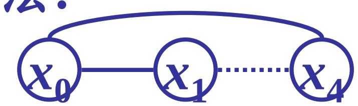
- **Thm 4.3: *G* is a tri. graph ⇔ The Lex. BFS ordering σ is a perfect scheme.**

Proof. (4/4)

(⇒) $|V| = n$: claim: $x = \sigma(1)$ is simplicial.

Proof of claim.

Prove by 矛盾法:

$x_0x_4 \notin E$, 否則 

, 可得 x_2 不是最大 →←

∴ 由於 *G* is tri., 可知 $x_4x_3 \notin E$

Consider x_2, x_3, x_4, 由 (L_3),

$\exists\, x_5 \ni x_3x_5 \in E$, 但 $x_2x_5 \notin E$.

⋮

成為 infinite →←

∴ *x* 為 simplicial.

§ 7.5 Recognizing Chordal Graphs by Lexicographic BFS

- **<u>Note</u>:**
  所以我們可以用 Lex. BFS 先產生 an ordering σ, ($O(|V| + |E|)$, Thm 4.4)
  接著檢查 σ 是不是 perfect scheme. ($O(|V| + |E|)$, Thm 4.5)
  如果是, 則 G is tri.; 否則, G is not tri.

- **<u>Corollary 4.6</u>: Tri. graphs can be recognized in $O(|V| + |E|)$ time.**
  (linear)

§ 7.5 Recognizing Chordal Graphs by Lexicographic BFS

- **Remark: Tarjan [1976]:**
  用另一個方法: called maximum cardinality search (MCS)

- **Note: MCS ordering ≠ Lex. BFS ordering ≠ 所有的 perfect scheme**
  i.e. $\exists\, \sigma, \sigma \in M \wedge \sigma \notin L$;
  $\exists\, \sigma, \sigma \in L \wedge \sigma \notin M$;
  $\exists\, \sigma, \sigma \in P \wedge \sigma \notin M \wedge \sigma \notin L$.

7.5 Recognizing Chordal Graphs by Lexicographic BFS

- 期中考(5/3)：請撰寫程式，於 5/10 demo

  Input: A graph G with PEO and positive vertex weights $w_1, w_2, \ldots, w_n$.

  Output: 1. Whether G is a chordal graph and what's its PEO?

  2. Optimal solutions to $P(G)$ and $D(G)$. (S and y_i)

輸入：圖型的點數 n、邊數 m、邊集合、每一個點的權重 w。

輸出：① 請判斷輸入的圖形是否為 Chordal Graph (Triangulated graph)，若是，請輸出其PEO；若否，請輸出說明。

② 若輸入的圖形確實為Chordal Graph，請回答 7.4 節的問題。(Find $\alpha(G, w)$ and $k(G, w)$.)

請試著讓執行時間越快越好。