Computer Science and Information Engineering
National Chi Nan University

Combinatorial Optimization

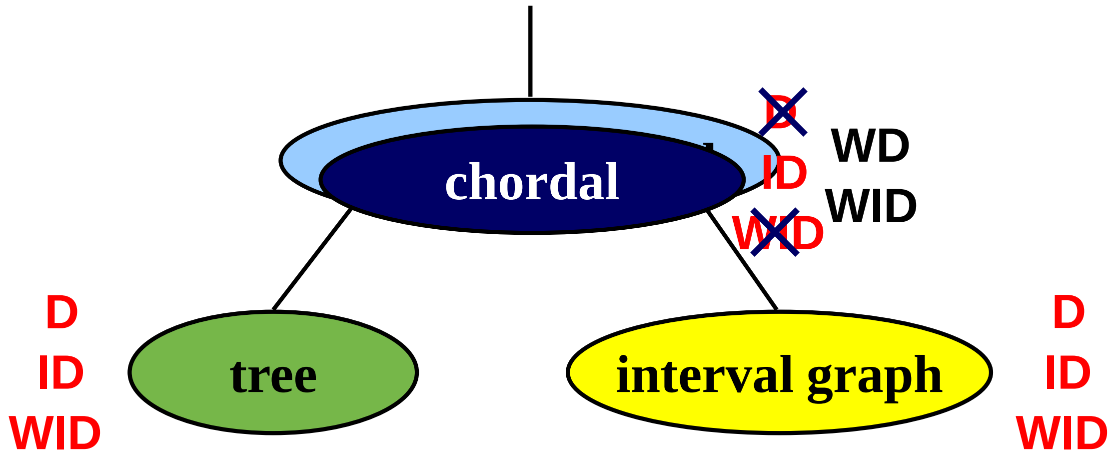
Dr. Justie Su-Tzu Juan

Lecture 7. Applications of L.P. Duality

§7.1 Definitions of Strongly Chordal Graphs

Slides for a Course Based on the Paper
M. Farber, "*Domination, independent domination, and duality in strongly chordal graphs*", Disc. Appl. Math. 7 (1984), pp. 115-130

7.1 Definitions of Strongly Chordal Graphs

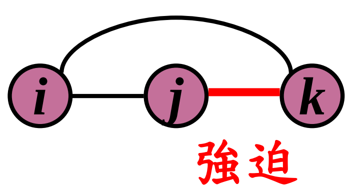


7.1 Definitions of Strongly Chordal Graphs

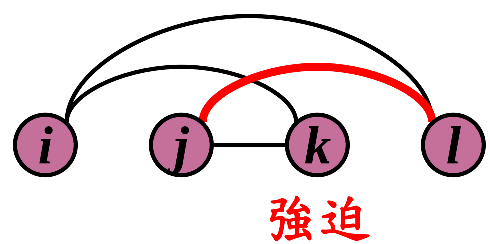
- **<u>Def</u>: A strong elimination ordering (SEO) of a graph $G = (V, E)$ is an ordering $[v_1, v_2, \ldots, v_n]$ of V such that $\forall\, i, j, k, l \in \{1, 2, \ldots, n\}$.**
  - **(a)** $i < j < k,\ v_iv_j,\ v_iv_k \in E \Rightarrow v_jv_k \in E$
  - **(b)** $i < j < k < l,\ v_iv_k,\ v_iv_l,\ v_jv_k \in E \Rightarrow v_jv_l \in E$

- **<u>Def</u>: A graph is strongly chordal graph if G has a SEO.**

- **<u>Remark</u>:**
  - **① A strong elimination ordering is a perfect elimination ordering ⇒ A strongly chordal graph is chordal graph.**
  - **②圖示: (a)**

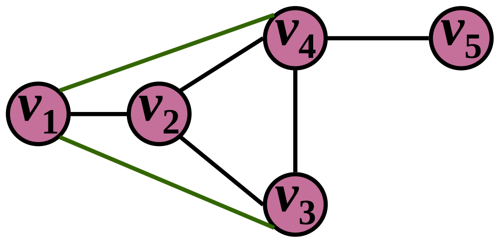


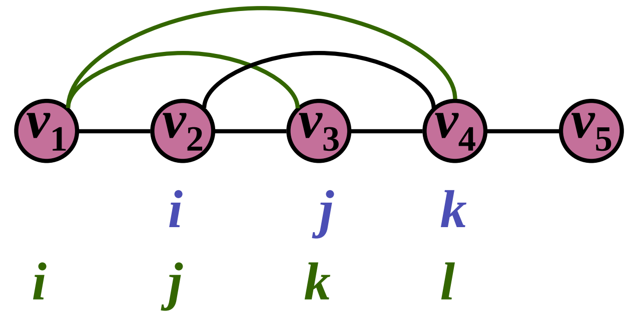
(b)



7.1 Definitions of Strongly Chordal Graphs

- **Ex:**





- **Notation: Given [v_1, v_2, …, v_n]: a vertex ordering,**

  $i \sim j$ (or $v_i \sim v_j$) 表示 $v_i \in N[v_j]$.

- **Lemma: [v_1, v_2, …, v_n] is a strong elimination ordering iff**

  $i \leq j, k \leq l, i \sim k, i \sim l, j \sim k \Rightarrow j \sim l$. (SEO2)

7.1 Definitions of Strongly Chordal Graphs

- **<u>Lemma</u>: $[v_1, v_2, \ldots, v_n]$ is a strong elimination ordering iff**

  $$i \le j, k \le l, i \sim k, i \sim l, j \sim k \Rightarrow j \sim l. \text{ (SEO2)}$$

Proof. (1/3)

($\Leftarrow$) (a) $\forall\, i < j < k,\ v_iv_j,\ v_iv_k \in E$

$\because\ i \le j, j \le k, i \sim j, i \sim k, j \sim j$

$\therefore$ By (SEO2), $j \sim k$ and $\because\ j < k, j \ne k\ \therefore\ v_jv_k \in E$

(b) $\forall\, i < j < k < l,\ v_iv_k,\ v_iv_l,\ v_jv_k \in E$

$\because\ i \le j, k \le l, i \sim k, i \sim l, j \sim k$

$\therefore$ By (SEO2), $j \sim l$ and $\because\ j < l, j \ne l\ \therefore\ v_jv_l \in E$

($\Rightarrow$) If $i = j$ or $k = l$ or $j = l$, then $j \sim l$

So we may assume $i < j, k < l, j \ne l$

W.L.O.G. we may assume $i \le k$

<u>Case 1</u>: $j < k$. <u>Case 2</u>: $j = k$. <u>Case 3</u>: $k < j$.

7.1 Definitions of Strongly Chordal Graphs

- **Lemma**: $[v_1, v_2, \ldots, v_n]$ is a strong elimination ordering iff $i \le j, k \le l, i \sim k, i \sim l, j \sim k \Rightarrow j \sim l$. **(SEO2)**

Proof. (2/3) $i < j, k < l, j \ne l$ and $i \le k$

(⇒) Case 1: $j < k$:

Then $i < j < k < l$

∴ By (b), $v_iv_k, v_iv_l, v_jv_k \in E \Rightarrow v_jv_l \in E \Rightarrow j \sim l$

Case 2: $j = k$:

Then $i < j < l$

∴ By (a), $v_iv_j, v_iv_l \in E \Rightarrow v_jv_l \in E \Rightarrow j \sim l$

Case 3: $k < j$:

Then $i \le k < j, k < l$

7.1 Definitions of Strongly Chordal Graphs

- **Lemma**: $[v_1, v_2, \ldots, v_n]$ is a strong elimination ordering iff

$$i \le j,\ k \le l,\ i \sim k,\ i \sim l,\ j \sim k \Rightarrow j \sim l. \quad \textbf{(SEO2)}$$

Proof. (3/3) $i < j,\ k < l,\ j \ne l$ and $i \le k$

(⇒) Case 3: $k < j$:

Then $i \le k < j,\ k < l$

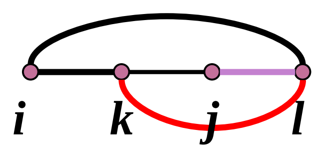
① $i = k$: $v_k v_l = v_i v_l \in E$

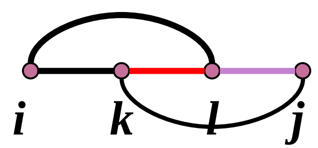
② $i < k$: ∵ $i < k < l$ and $v_i v_k,\ v_i v_l \in E$

∴ By (a), $v_k v_l \in E$

Hence $k < l,\ k < j$ and $v_k v_l,\ v_k v_j \in E$

∴ By (a), $v_l v_j \in E \Rightarrow j \sim l$.



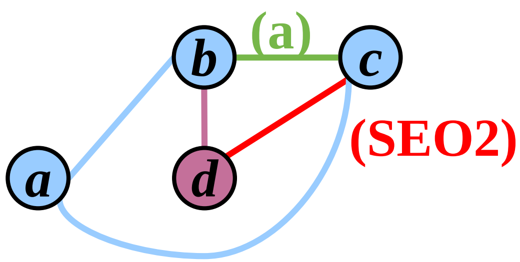


- **Notation**: For $i \le j$, define $N_i[v_j] = \{v_k : k \ge i \text{ and } j \sim k\}$.

7.1 Definitions of Strongly Chordal Graphs

- **Thm: (SEO2) $\Leftrightarrow a \leq b \leq c$, $a \sim b$ and $a \sim c \Rightarrow N_a[v_b] \subseteq N_a[v_c]$.**

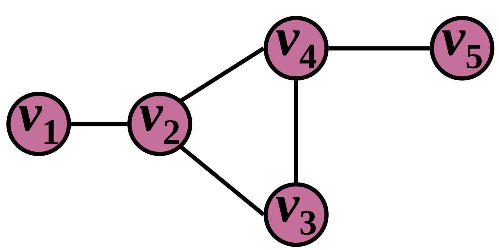
Proof. (略)



- **Note: By (SEO2), we can see an ordering is SEO or not by the adjacent matrix $A + I$; it will satisfy:**

$$\begin{array}{c} \\ i \\ j \end{array}\begin{array}{c} \begin{array}{cc} k & l \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & \end{bmatrix} \end{array} \Rightarrow \begin{array}{c} \\ i \\ j \end{array}\begin{array}{c} \begin{array}{cc} k & l \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

- **Ex:**



$$A + I = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

7.1 Definitions of Strongly Chordal Graphs

- **Note: There exist a polynomial algorithm to recognize a strongly chordal graph which constructs SEO on the vertices, if one exists.**

- **Reference:**

- **1. M. Farber, *Applications of l.p. duality to problems involving independence and domination*, Ph.D. Thesis, Rutgers University, New Brunswick, NJ (January 1982); also issued as Technical Report 81-13, Computing Science Department, Simon Fraser University, 1981.**
- **2. M. Farber, *Characterizations of strongly chordal graphs*, Discrete Math. 43 (1983) 173-189.**

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§ 7.2 The Weighted Domination Problem on Strongly Chordal Graphs

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7.2 The Weighted Domination Problem on Strongly Chordal Graphs

- **Def**: G is a weighted graph with each v_i be assigned a real weight w_i,

  ① The **weighted domination** (**WD**) problem for a weighted graph G is to find a dominating set D s.t. $w(D) = \sum_{v_i \in D} w_i$ is minimum.

  ② $\gamma(G, w) = \min_{D} w(D)$ (for the case of $w_i = 1, \forall i$, $\gamma(G, w) = \gamma(G)$.)

  ③ Let $P(G)$ is the following linear program:

$$\text{Minimize } \sum_{i=1}^{n} w_i x_i$$

$$\text{Subject to } \begin{cases} \sum_{i \sim j} x_i \geq 1, \forall j \\ x_i \geq 0, \forall i \end{cases}$$

  ④ Let $P_1(G)$ is the linear program $P(G)$ with $x_i \in \{0, 1\}$, $\forall i$.

  ⑤ $value(P)$ = the value of the optimal solution of a linear program P.

7.2 The Weighted Domination Problem on Strongly Chordal Graphs

- **Remark**:
  - ① ∃ 1 – 1 correspondence between feasible solutions to $P_1(G)$ and dominating sets in G.
  - ② An optimal solution to $P_1(G)$ corresponds to a minimum weighted dominating set in G.
  - ③ $P(G)$ is unbounded if $\exists\, i,\, w_i < 0$.

- **Lemma**: $G = (V, E)$ is a graph in which every vertex v has a weight $w(v) \in \mathbb{R}$. Let $w'(v) = \max\{0, w(v)\}$

$$\Rightarrow \gamma(G, w) = \gamma(G, w') + \sum_{\substack{x \in V \\ w(x) < 0}} w(x)$$

7.2 The Weighted Domination Problem on Strongly Chordal Graphs

- **Lemma**: $G = (V, E)$ is a graph in which every vertex v has a weight $w(v) \in \mathbb{R}$. Let $w'(v) = \max\{0, w(v)\} \Rightarrow \gamma(G, w) = \gamma(G, w') + \sum_{x \in V,\, w(x)<0} w(x)$

Proof.

Suppose D is a dominating set of G such that $\gamma(G, w) = w(D)$.

$$\gamma(G, w') \le w'(D) = w(D) - \sum_{x \in D,\, w(x)<0} w(x)$$

$$\le w(D) - \sum_{x \in V,\, w(x)<0} w(x) = \gamma(G, w) - \sum_{x \in V,\, w(x)<0} w(x)$$

Suppose D' is a dominating set of G such that $\gamma(G, w') = w'(D')$.

Then $D' \cup A$ is also a dominating set of G where $A = \{v \in V: w(v) < 0\}$.

Then $\gamma(G, w) \le w(D' \cup A) = w((D' \setminus A) \,\dot\cup\, A) = w(D' \setminus A) + w(A)$

$$= w'(D' \setminus A) + w(A) = w'(D') + w(A) = \gamma(G, w') + \sum_{x \in V,\, w(x)<0} w(x)$$

7.2 The Weighted Domination Problem on Strongly Chordal Graphs

- **Note: For the remainder of this section, we will assume that $w(v_i) \geq 0$, $\forall v_i \in V(G)$.**

- **Def: The dual problem $D(G)$ of $P(G)$ is the linear program:**

  Maximize $\sum_{j=1}^{n} y_j$

  Subject to $\begin{cases} y_j \geq 0, \forall j \\ \sum_{j \sim i} y_j \leq w_i, \forall i \end{cases}$

- **Thm: (Weakly duality inequality)**

  $\forall$ Feasible solutions $\langle x_1, x_2, \ldots, x_n \rangle$, $\langle y_1, y_2, \ldots, y_n \rangle$ for $P(G)$, $D(G)$

  $$\sum_{i=1}^{n} w_i x_i \geq \sum_{j=1}^{n} y_j$$

7.2 The Weighted Domination Problem on Strongly Chordal Graphs

- **Thm: (Weakly duality inequality)**

  $\forall$ **Feasible solutions** $\langle x_1, x_2, \ldots, x_n\rangle$, $\langle y_1, y_2, \ldots, y_n\rangle$ **for** $P(G)$, $D(G)$

$$\sum_{i=1}^{n} w_i x_i \geq \sum_{j=1}^{n} y_j$$

Proof.

$$\sum_{i=1}^{n} w_i x_i - \sum_{j=1}^{n} y_j$$

$$= \sum_{i=1}^{n} w_i x_i - \sum_{i=1}^{n}\sum_{j\sim i} y_j x_i + \sum_{j=1}^{n}\sum_{i\sim j} y_j x_i - \sum_{j=1}^{n} y_j$$

$$= \sum_{i=1}^{n} x_i (w_i - \sum_{j\sim i} y_j) + \sum_{j=1}^{n} y_j (\sum_{i\sim j} x_i - 1) \geq 0$$

$$\therefore \sum_{i=1}^{n} w_i x_i \geq \sum_{j=1}^{n} y_j$$

- **Corollary**: ***value***$(P(G)) \geq$ ***value***$(D(G))$.

7.2 The Weighted Domination Problem on Strongly Chordal Graphs

w.d.i.

- **Note:** $value(P_1(G)) \geq value(P(G)) \geq value(D(G))$.

  $\because$ $P_1(G)$ is a special problem of $P(G)$

- **Thm:** If we have feasible solutions $x^* = \langle x_1, x_2, \ldots, x_n \rangle$, $y^* = \langle y_1, y_2, \ldots, y_n \rangle$ satisfy (CS1) $\forall j, y_j > 0 \Rightarrow \sum_{i \sim j} x_i = 1$

  (CS2) $\forall i, x_i > 0 \Rightarrow \sum_{j \sim i} y_j = w_i$

  ((CS1) and (CS2) called the condition of complementary slackness)

  Then $\sum_{i=1}^{n} w_i x_i = \sum_{j=1}^{n} y_j$.

- **Notation:** ① $h(i) = w_i - \sum_{j \sim i} y_j$, $\forall i$.

  ② $T_i = \{k : k \sim i \text{ and } y_k > 0\}$.

7.2 The Weighted Domination Problem on Strongly Chordal Graphs

- **AlgorithmI:**

Input: A strongly chordal graph G with strong elimination ordering $v_1, v_2, \ldots, v_n$ and positive vertex weights $w_1, w_2, \ldots, w_n$.

Output: Optimal solutions to $P(G)$ and $D(G)$.

Initially: $T = \{1, 2, \ldots, n\}$, $\forall\, i$, $x_i = 0$, $y_i = 0$, $h(i) = w_i$, $T_i = \{\}$.

Stage One:

for $j = 1$ to n do
    $y_j \leftarrow \min\{h(k): k \sim j\}$;
    $\forall\, i \sim j$, modify $h(i)$, T_i;

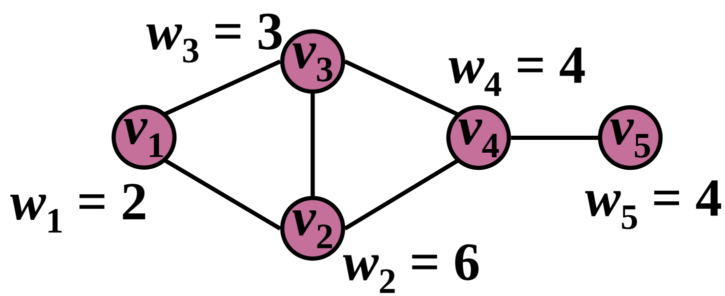
Stage Two:

for $i = n$ to 1 by -1 do
    if $h(i) = 0$ and $T_i \subseteq T$ then
        $x_i \leftarrow 1$;
        $T \leftarrow T - T_i$;

- **Time Complexity** = $\mathcal{O}(|V|+|E|)$.

7.2 The Weighted Domination Problem on Strongly Chordal Graphs

- **Ex:**



$$h(i) = w_i - \sum_{j \sim i} y_j, \ \forall\, i.$$
$$T_i = \{k : k \sim i \text{ and } y_k > 0\}.$$

$T = \{1, 2, 3, 4, 5\}$

$x_1 = 0$	$x_2 = 0$	$x_3 = 0$	$x_4 = 0$	$x_5 = 0$
$y_1 = 0$	$y_2 = 0$	$y_3 = 0$	$y_4 = 0$	$y_5 = 0$
$h_1 = 2$	$h_2 = 6$	$h_3 = 3$	$h_4 = 4$	$h_5 = 4$
$T_1 = \{ \quad \}$	$T_2 = \{ \quad \}$	$T_3 = \{ \quad \}$	$T_4 = \{ \quad \}$	$T_5 = \{ \quad \}$

- **Note:** It will follow from the algorithm that if G is strongly chordal, the $P(G)$ has 0-1 optimal solution, i.e. $value(P(G)) = value(P_1(G))$.

Stage One: for $j = 1$ to n do
$y_j \leftarrow \min\{h(k): k \sim j\}$;
Modify $h(i)$, T_i;

2,

Stage Two: for $i = n$ to 1 by –1 do
if $h(i) = 0$ and $T_i \subseteq T$ then
$x_i \leftarrow 1$;
$T \leftarrow T - T_i$;

7.2 The Weighted Domination Problem on Strongly Chordal Graphs

- **Thm: The final values of $x^* = \langle x_1, x_2, \ldots, x_n \rangle$ and $y^* = \langle y_1, y_2, \ldots, y_n \rangle$ of algorithmI are optimal solutions to $P(G)$ and $D(G)$.**

  i.e. (1) x^* is a feasible solution of $P(G)$.

  (2) y^* is a feasible solution of $D(G)$.

  (3) x^*, y^* satisfy (CS1) and (CS2).

$$y_j \geq 0, \forall j$$
$$\sum_{j \sim i} y_j \leq w_i, \forall i$$

Proof. (1/5) (2) - (1) - (3)

(2) By stage one, $y_j \geq 0$ and $h(j) \geq 0$, $\forall\, j$ at any time.

If not,

① If first become negative is $h(i')$, then $\exists\, j'$ s.t. $i'j' \in E$ &

$h_{j'-1}(i') \geq 0$, but $h_{j'}(i') < 0$. ($h_k(i) \triangleq h(i)$; $y_i^{(k)} \triangleq y_i$ when $j = k$.)

$$\because\ h_{j'}(i') = w_{i'} - \sum_{j \sim i'} y_j = h_{j'-1}(i') + y_{j'}^{(j'-1)} (=0) - y_{j'}^{(j')} < 0.$$

$$\therefore\ y_{j'}^{(j')} > h_{j'-1}(i') \rightarrow\leftarrow$$

7.2 The Weighted Domination Problem on Strongly Chordal Graphs

- **Thm**: The final values of $x^* = \langle x_1, x_2, \ldots, x_n \rangle$ and $y^* = \langle y_1, y_2, \ldots, y_n \rangle$ of algorithmI are optimal solutions to $P(G)$ and $D(G)$.
  i.e. (1) x^* is a feasible solution of $P(G)$.
  (2) y^* is a feasible solution of $D(G)$.
  (3) x^*, y^* satisfy (CS1) and (CS2).

$$y_j \geq 0, \forall j$$
$$\sum_{j \sim i} y_j \leq w_i, \forall i$$

Proof. (2/5)

(2) By stage one, $y_j \geq 0$ and $h(j) \geq 0$, $\forall j$ at any time.

If not,

② If first become negative is $y_{j'}$, when $j = j'$.

$\because y_{j'} = \min\{h_{j'-1}(k'): k \sim j\}$ and

$h_{j'-1}(k) \geq 0, \forall k$.

$\therefore y_{j'} \geq 0 \rightarrow\leftarrow$

7.2 The Weighted Domination Problem on Strongly Chordal Graphs

- **Thm**: The final values of $x^* = <x_1, x_2, \ldots, x_n>$ and $y^* = <y_1, y_2, \ldots, y_n>$ of algorithmI are optimal solutions to $P(G)$ and $D(G)$.
  i.e. (1) x^* is a feasible solution of $P(G)$.

$$\sum_{i\sim j} x_i \geq 1, \forall j$$
$$x_i \geq 0, \forall i$$

Proof. (3/5)

(1) $x_i \geq 0$ is easily to see. (x_i is either 0 or 1, $\forall i$)

Only need to see if $\forall j, \sum_{i\sim j} x_i \geq 1$, i.e. $\forall\, y_j, \exists\, x_{i^*}$ s.t. $i^* \sim j$ and $x_{i^*} = 1$.

By the choice of y_j, $\exists\, k \sim j$ such that $h_j(k) = 0$ and $\max T_k \leq j$.

Note: $\because\ T_k = \{p: p \sim k, y_p > 0\}$.

In interation 1, 2, …, j： $T_k \subseteq \{1, 2, \ldots, j\}$;

In interation p, $\forall\, p > j$ ：

① $p \sim k$: $\because\ 0 \leq h_p(k) \leq h_j(k) = 0$, $\therefore\ h_p(k) = 0 \Rightarrow y_p = 0 \Rightarrow T_k \subseteq \{1, 2, \ldots, j\}$

② If $p \nsim k$: T_k不會變 $\therefore\ T_k \subseteq \{1, 2, \ldots, j\}$

7.2 The Weighted Domination Problem on Strongly Chordal Graphs

- **Thm: The final values of $x^* = \langle x_1, x_2, \ldots, x_n \rangle$ and $y^* = \langle y_1, y_2, \ldots, y_n \rangle$ of algorithmI are optimal solutions to $P(G)$ and $D(G)$.**
  i.e. (1) x^* is a feasible solution of $P(G)$.

$$\sum_{i \sim j} x_i \geq 1, \forall j$$
$$x_i \geq 0, \forall i$$

Proof. (4/5)

(1) Case 1: If $x_k = 1$, let $i^* = k$, it's done!!

Case 2: If $x_k = 0$, $\because h(k) = 0 \therefore T_k \not\subseteq T$ when $i = k$ in stage Two.

i.e. $\exists\, l > k$ s.t. $x_l = 1$ and $T_k \cap T_l \neq \phi$.

Let $a \in T_k \cap T_l$

$$\begin{cases} a \leq j \ (\because a \in T_k \therefore \max T_k \leq j) \\ k < l \ (\because \text{the choice of } l) \\ a \sim k \ (\because a \in T_k) \\ a \sim l \ (\because a \in T_l) \\ j \sim k \ (\because \text{the choice of } k) \end{cases}$$

$\therefore$ By (SEO2), $j \sim l$, let $i^* = l$, it's done!!

7.2 The Weighted Domination Problem on Strongly Chordal Graphs

- **Thm: The final values of $x^* = \langle x_1, x_2, \ldots, x_n \rangle$ and $y^* = \langle y_1, y_2, \ldots, y_n \rangle$ of algorithmI are optimal solutions to $P(G)$ and $D(G)$.**
  i.e. (3) x^*, y^* satisfy (CS1) and (CS2).

Proof. (5/5)

(3) (CS2) $\forall i$, if $x_i > 0$, then $x_i = 1$ and by stage two, $h(i)=0$. i.e. $\sum_{j \sim i} y_i = w_i$

(CS1) If $\sum_{i \sim j} x_i > 1$ for some j, then

$$\left.\begin{array}{r} \exists\, i_1 \neq i_2 \text{ s.t. } i_1 \sim j,\ x_{i_1} = 1 \\ i_2 \sim j,\ x_{i_2} = 1 \end{array}\right\} \Rightarrow T_{i_1} \cap T_{i_2} = \phi.$$

If $y_j > 0$, then $j \in T_{i_1} \cap T_{i_2} \rightarrow\leftarrow$

$\therefore y_j = 0$

i.e. $\forall j, \sum_{i \sim j} x_i \neq 1 \Rightarrow y_j \not> 0$

$\therefore \forall j, y_j > 0 \Rightarrow \sum_{i \sim j} x_i = 1$

7.2 The Weighted Domination Problem on Strongly Chordal Graphs

- **Def: $\eta(G, w)$ = the maximum number of vertices of G (with repetition allowed) such that each vertex v_i of G is equal or adjacent to at most w_i of them.**

- **Corollary: If G is a strongly chordal graph with non-negative integer weight, then $\gamma(G, w) = \eta(G, w)$.**

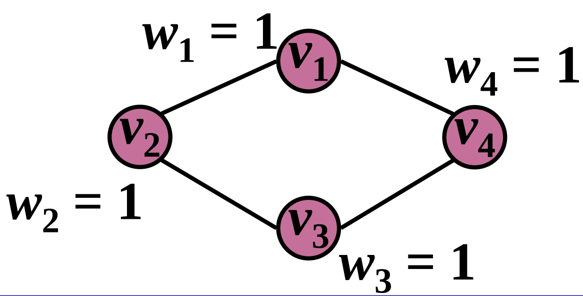
- **Note: For an arbitrary graph G or chordal graph G, solve $P(G)$ can not find a minimum weight dominating set.**

  $(\because value(P(G)) \neq value(P_1(G)))$

7.2 The Weighted Domination Problem on Strongly Chordal Graphs

- **Ex:**

①

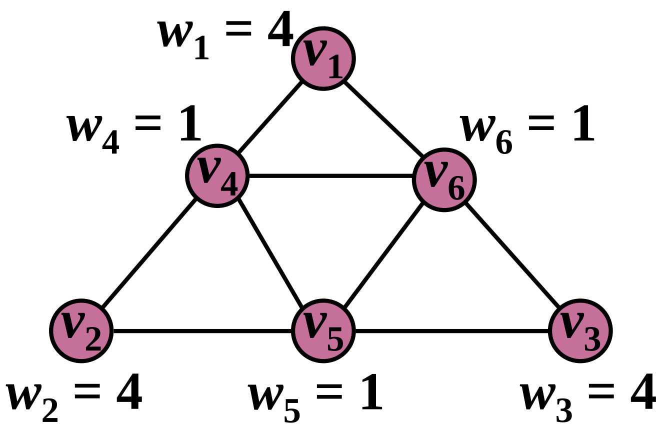


$x_1 = x_2 = x_3 = x_4 = 1/3 = y_1 = y_2 = y_3 = y_4$

value(P(G)) = 4/3

value(D(G)) = 4/3

②



$x_4 = x_5 = x_6 = 1/2 = y_1 = y_2 = y_3$

$x_1 = x_2 = x_3 = 0 = y_4 = y_5 = y_6$

value(P(G)) = 3/2

value(D(G)) = 3/2

∴ **neither *P*(*G*) nor *D*(*G*) has integer optimal solution!!**

- **Remark: The weighted independent domination problem is linearly solvable for strongly chordal graphs in a similar way.**