Computer Science and Information Engineering
National Chi Nan University

# Combinatorial Optimization

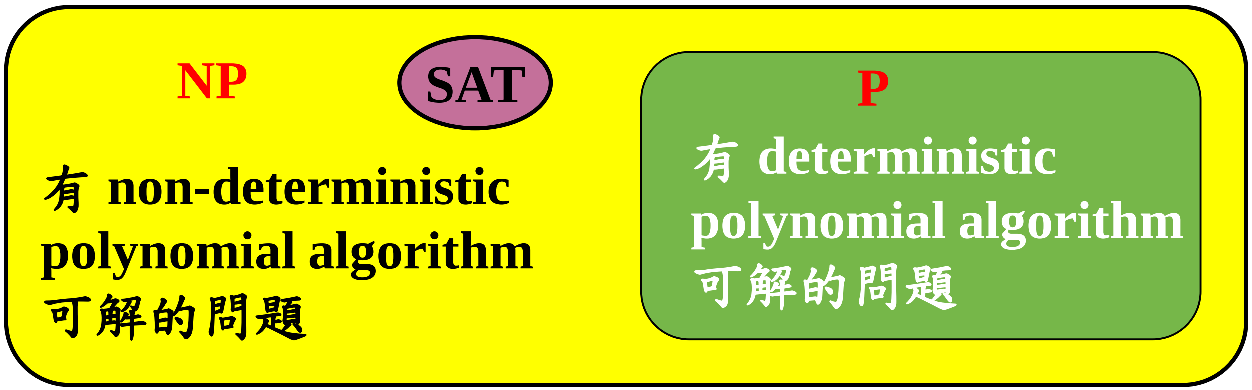
Dr.  Justie Su-Tzu Juan

# Lecture 6. NP-Completeness

## §6.1 Dom is NP-complete

# 6.1 Dom is NP-complete

- **Def:**




- **Question: Domination problem 是否 ∈ P ?**

想法: $\forall D \subseteq V$, Step 1: check $D$是否為dominating set

Step 2: 若是，算出$|D|$

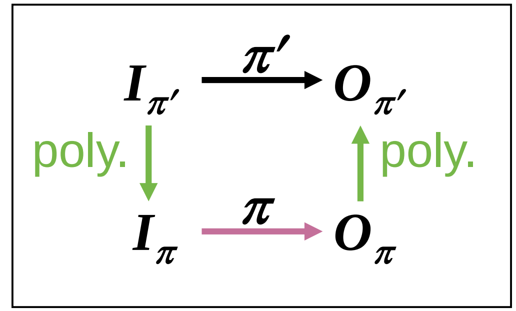
Step 3: 求其中最小$|D|$

有$2^n$種可能 ∴此法不可得証 $\in P$

但仍不曉得 $\in P$ 或 $\notin P$ !

# 6.1 Dom is NP-complete

- **Def:** $\pi$ is called **NP-complete** if
  ① $\pi \in$ NP
  ② $\forall\ \pi' \in$ NP, $\pi' \propto \pi$
  其中 $\propto$ 為 polynomial reduction.




- **Note:** ① $\propto$ 具有遞移性, i.e. $\pi_1 \propto \pi_2$ and $\pi_2 \propto \pi_3 \Rightarrow \pi_1 \propto \pi_3$.
  ② If $\pi^*$ is NP-complete and $\pi^* \in$ P, then NP = P.
  ③ If $\pi^*$ is NP-complete and $\pi^* \propto \pi$ where $\pi \in$ NP, then $\pi$ is NP-complete.

- **Def:** **Optimization** problem v.s. **Decision** problem

- **Ex:** ① Given $G$, 回答 $\gamma(G)$ (=min{$|D|$: $D$ is a dominating set of $G$.})
  ② Given $k$, $G$, 回答 $\begin{cases} \text{yes, if } \gamma(G) \le k \text{ (if } \exists \text{ dom. set of size} \le k\text{);} \\ \text{no, otherwise.} \end{cases}$

# 6.1 Dom is NP-complete

- **Def: SAT problem:**

  **Input:** $f = f(x_1, x_2, \ldots, x_n) = \prod_{1 \le i \le m} (x_{i,1} + x_{i,2} + \ldots + x_{i,a_i})$ **(+: or, ·: and)**

  **where $x_1, x_2, \ldots, x_n$: logical varibles,**

  **and $x_{i,j}$ is $x_k$ or $\neg x_k$ for some $k \in \{1, 2, \ldots, n\}$**

  **Output:** $\begin{cases} \text{Yes, if we can assign } x_1, x_2, \ldots, x_n \text{ such that } f \text{ is true;} \\ \text{No, otherwise.} \end{cases}$

- **Ex:** $f = x \cdot y \cdot z$ $(m = 3, a_1 = a_2 = a_3 = 1)$

  **Solve: assign $x \leftarrow 1, y \leftarrow 1, z \leftarrow 1$**

  $\Rightarrow f = 1$ $\therefore$ **Yes!**

- **Ex:** $f = (x + \neg x)(x)(\neg x)$ $(m = 3, a_1 = 2, a_2 = 1, a_3 = 1)$

  **Solve: No!**

# 6.1 Dom is NP-complete

- **Thm: (Cook) SAT is NP-complete.**

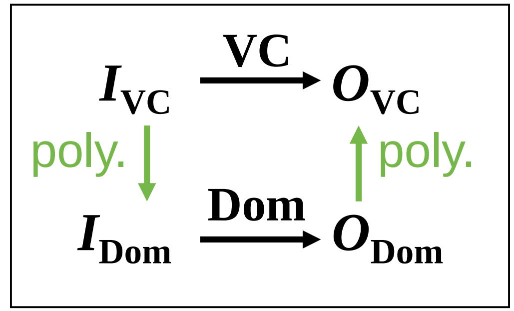
- **Remark: R. Karp 列出許多Combinatorial problem証其為NP-complete.**

- **Def: VC problem:**
  **Input:** $G = (V, E)$ **and** $k \leq |V|$.
  **Output:** $\begin{cases} \text{Yes, if } \exists \text{ vertex covering of size} \leq k; \\ \text{No, otherwise.} \end{cases}$

- **Def: Dom problem:**
  **Input:** $G = (V, E)$ **and** $k \leq |V|$.
  **Output:** $\begin{cases} \text{Yes, if } \exists \text{ dominating set of size} \leq k; \\ \text{No, otherwise.} \end{cases}$

# 6.1 Dom is NP-complete

- **Thm: VC is NP-complete.**
- **Thm: Dom is NP-complete.**




① Construct ↓↑

② Prove 正確性

**Proof. (1/2)**

**欲証: VC ∝ Dom**

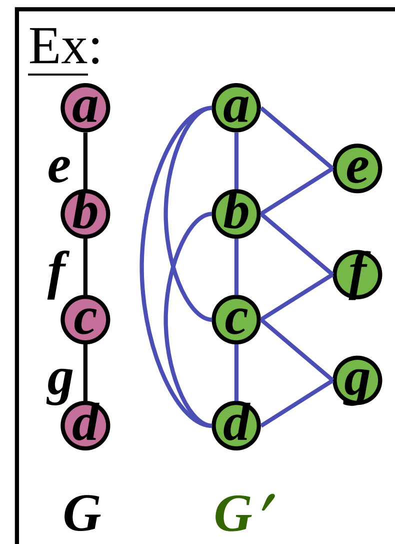
$\forall\, G = (V, E)$ **and** $k \le |V|$

**Construct** $G' = (V', E')$ **and** $k' \le |V'|$ **such that**

$$\begin{cases} V' = V \cup E \\ E' = \{xy : x \neq y \text{ in } V\} \cup \{xe : x \in V, e \in E, x \in e\} \\ k' = k \end{cases}$$

**Claim:** $G$ **has a vertex cover** $C$ **of size at most** $k \Leftrightarrow$

$G'$ **has a dominating set** $D$ **of size at most** $k'$.



# 6.1 Dom is NP-complete

- **Thm: Dom is NP-complete.**

**Proof. (2/2)** Claim: $G$ has a vertex cover $C$ of size at most $k \Leftrightarrow$ $G'$ has a dominating set $D$ of size at most $k'$.

<pf> ($\Rightarrow$) Suppose $C$ is a vertex cover of $G$, $|C| \leq k$.

Then, by the definition of $G'$,

$D = C$ is also a dominating set of $G'$ with $|D| \leq k = k'$.

($\Leftarrow$) Suppose $D$ is a dominating set of $G'$, $|D| \leq k' = k$.

If $\exists$ some $e = xy \in D \cap E$, where $x, y \in V$,

then $D' = (D \setminus \{e\}) \cup \{x\}$ is also a dominating set of $G'$ with

$|D'| \leq |D| \leq k$, since $N_{G'}[e] \subseteq N_{G'}[x]$.

In this way, we may assume that $D \subseteq V$.

Then $C = D$ is a vertex cover of $G$ with $|C| \leq k$.

Therefore, VC is NP-complete implies that Dom is NP-complete by Note ③.

# 6.1 Dom is NP-complete

- **Def: $G = (V, E)$ is called a split graph if $V = C \uplus S$, where $G_C$ is a clique and $G_S$ is a stable set.**

- **Corollary: The domination (total domination, connected domination) problem is NP-complete in split graphs.**

- **Def: $G = (V, E)$ is called a bipartite graph if $V = A \uplus B$, where $G_A$ and $G_B$ both are stable sets.**

- **Thm: The domination problem is NP-complete in bipartite graphs.**

# 6.1 Dom is NP-complete

- **Thm: The domination problem is NP-complete in bipartite graphs.**

**Proof. (1/2)**

$\forall\, G = (V, E)$ and $k \le |V|$

construct $G' = (V', E')$ and $k' \le |V'|$ such that

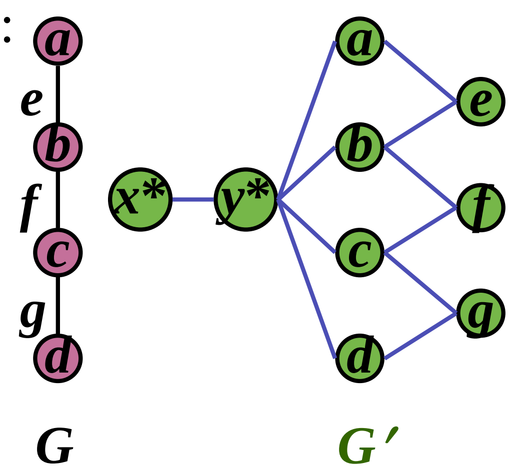
$$\begin{cases} V' = V \cup E \cup \{x^*, y^*\} \\ E' = \{xe: x \in V, e \in E, x \in e\} \cup \{x^*y^*\} \cup \{y^*x: x \in V\} \\ k' = k+1 \end{cases}$$

**Claim:**

$G$ has a vertex cover $C$ of size at most $k \Leftrightarrow$

$G'$ has a dominating set $D$ of size at most $k'$.

Ex:



# 6.1 Dom is NP-complete

- **Thm: The domination problem is NP-complete in bipartite graphs.**

**Proof. (2/2) Claim: $G$ has a vertex cover $C$ of size at most $k \Leftrightarrow$**
**$G'$ has a dominating set $D$ of size at most $k'$.**

**<pf> ($\Rightarrow$) Suppose $C$ is a vertex cover of $G$ and $|C| \leq k$.**

**Then, by the definition of $G'$,**

**$D = C \cup \{y^*\}$ is a dominating set of $G'$ with $|D| \leq k+1 = k'$.**

**($\Leftarrow$) Suppose $D$ is a dominating set of $G'$ and $|D| \leq k'$.**

**Since $N_{G'}[x^*] \subseteq N_{G'}[y^*]$, So we may assume $x^* \notin D$ and $y^* \in D$**

**(Otherwise, replace $D$ by $(D \backslash \{x^*\} \cup \{y^*\})$**

**If $\exists$ $e = xy \in D$, then $D' = (D \backslash \{e\}) \cup \{x\}$ is also a dominating set of size $\leq k'$, since $N_{G'}[e] \subseteq (N_{G'}[x] \cup N_{G'}[y^*])$.**

**In this way, we may assume $D = C \cup \{y^*\}$ where $C \subseteq V$.**

**Then $C$ is a vertex cover of size at most $k' - 1 = k$.**

# 6.1 Dom is NP-complete

- **Exercise 4 (4/26): Prove that the independent domination problem is NP-complete for bipartite graphs.**

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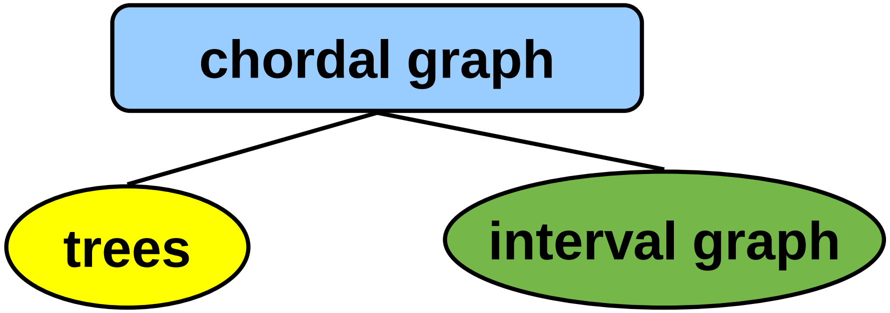
# Lecture 6. NP-Completeness

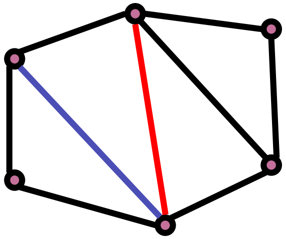
## §6.2 Chordal Graphs

# 6.2 Chordal Graphs




- **<u>Def</u>: A graph *G* is called a chordal graph if every cycle of length greater than 3 has a chord.**

- **<u>Ex</u>:**

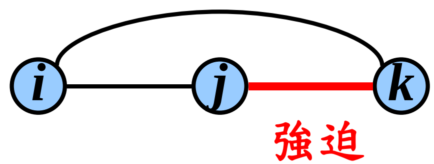


**no** **no** **yes**

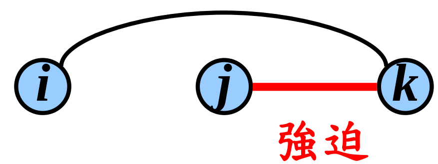
# 6.2 Chordal Graphs

- Def: An ordering of $V$, $\sigma = [v_1, v_2, \ldots, v_n]$ in graph $G = (V, E)$, is called a perfect elimination ordering (PEO) if $\forall\ 1 \le i < j < k \le n$, $v_i v_j \in E$ and $v_i v_k \in E \Rightarrow v_j v_k \in E$.
- (Thm: $G = (V, E)$ is chordal graph iff we can order $V$ into (PEO).)

- Note: ① 其中 $1 \le i < j < k \le n$ 可改成 $1 \le i < j \le n$, $1 \le i < k \le n$, $j \ne k$.
  ② 圖示:



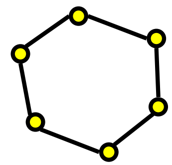

- Recall: Interval ordering (IO):

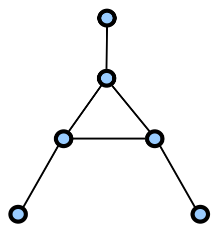



- Note: A vertex ordering $\sigma$ of $G = (V, E)$ is (IO) $\Rightarrow$ $\sigma$ is (PEO).

- Remark: ① Interval graphs are chordal graphs.
  ② $\exists$ example that are chordal graph but not interval graph.

# 6.2 Chordal Graphs

- **<u>Ex</u>:** ①  $C_n$, $n \geq 4$ 不是interval graph.

  ②  是chordal graph (by definition)

  不是interval graph

  (∵畫不出interval intersection representation)

- **<u>Thm</u>: $G = (V, E)$ is chordal graph iff we can order $V$ into (PEO).**

**Proof. (1/5)**

(⇐) Suppose $G$ has a (PEO) $\sigma = [v_1, v_2, \ldots, v_n]$

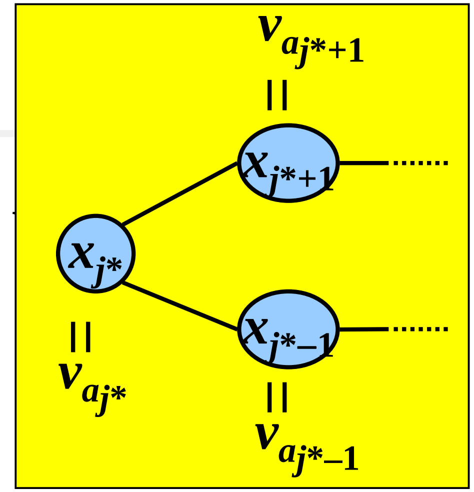
$\forall$ $k$-cycle ($k \geq 4$), say $x_1x_2\ldots x_{k-1}x_k(=x_0)x_1(=x_{k+1})$ in $G$

Let $x_j = v_{a_j}$, $\forall\ 1 \leq j \leq k$.

Let $a_{j^*} = \min\limits_{1 \leq j \leq k} a_j$ ($a_{j^*} < a_{j^*+1}$, $a_{j^*} < a_{j^*-1}$)

# 6.2 Chordal Graphs

- **Thm**: $G = (V, E)$ is chordal graph iff we can order




**Proof. (2/5)**

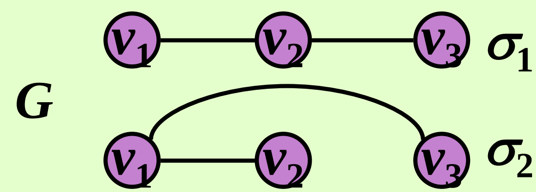
($\Leftarrow$) $\because x_{j^*}x_{j^*-1}, x_{j^*}x_{j^*+1} \in E$, $\therefore x_{j^*-1}x_{j^*+1} \in E$ by (PEO)

So the cycle has a chord. ($\because k \geq 4$)

($\Rightarrow$) For any ordering $\sigma = [v_1, v_2, \ldots, v_n]$

Let $d(\sigma) = [d_n, d_{n-1}, \ldots, d_1]$ where $d_i = |\{j: j > i \text{ and } v_j \in N[v_i]\}|$.

**Ex:**




$\sigma_1 \rightarrow d(\sigma_1) = [0, 1, 1]$

$\sigma_2 \rightarrow d(\sigma_2) = [0, 0, 2]$

$d(\sigma_1) \succcurlyeq d(\sigma_2)$

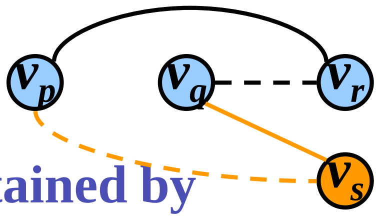
**Def**: For two sequences $s_1 = [a_1, a_2, \ldots, a_n]$, $s_2 = [b_1, b_2, \ldots, b_n]$, we say $s_1 \succcurlyeq s_2$ if $\exists\ 1 \leq k \leq n$ s.t. $a_i = b_i\ \forall\ 1 \leq i < k$ and $a_k > b_k$.

Choose an ordering $\sigma^*$ such that $d(\sigma^*)$ is **lexicographically** largest.

# 6.2 Chordal Graphs

- **Thm:** $G = (V, E)$ is chordal graph iff we can order $V$ into (PEO).

**Proof. (3/5)**




(⇒) Claim ①: for $p < q < r$ and $v_p v_r \in E$ and $v_q v_r \notin E$

$\Rightarrow \exists\, s > q$ s.t. $v_q v_s \in E$ and $v_p v_s \notin E$

<pf>Let $\sigma' = [v_1', v_2', \ldots, v_n']$ be an ordering of $V$ obtained by interchanging $v_p$ and $v_q$, i.e. $\begin{cases} v_i' = v_i, \text{ for all } 1 \le i \le n, i \ne p, i \ne q. \\ v_p' = v_q \\ v_q' = v_p \end{cases}$

Let $d_i' = d_i$ in $\sigma'$

$\forall\, s > q$, $d_s' = d_s$ since $\forall\, s > q$, $v_s' = v_s$

$d_q = |\{t: t > q \text{ and } v_t \in N(v_q)\}|$

$d'_q = |\{t: t > q \text{ and } v_{t'} (= v_t) \in N(v'_q)\}| = |\{t: t > q \text{ and } v_t \in N(v_p)\}|$

Since $d(\sigma) \not\geq d(\sigma')$, we have $d_q \ge d_q'$.

But $r > q$, $v_r \in N(v_p) \backslash N(v_q)$. Hence $\exists\, s > q$ such that $v_s \in N(v_q) \backslash N(v_p)$.

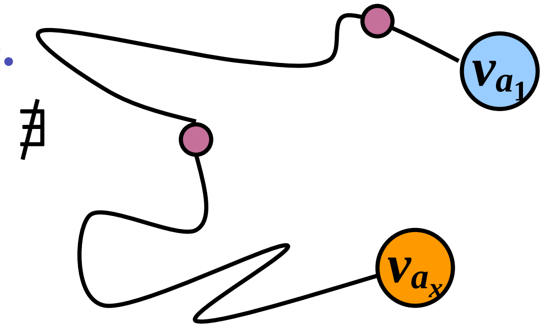
# 6.2 Chordal Graphs

- **<u>Thm</u>: $G = (V, E)$ is chordal graph iff we can order $V$ into (PEO).**

**Proof. (4/5)**

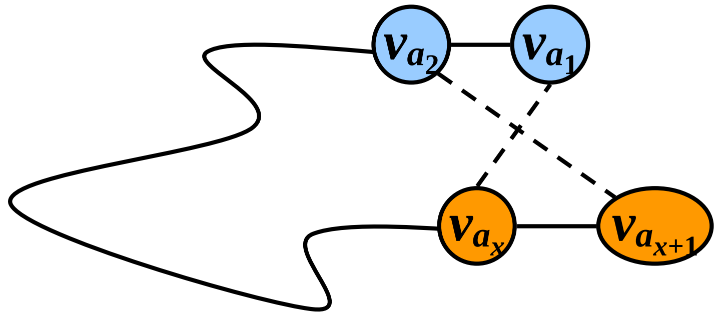
**(⇒) <u>Claim ②</u>: ∄ chordless path $P$: $v_{a_1}, v_{a_2}, \ldots, v_{a_x}$ with $x \geq 3$ and**

**$a_y < a_x < a_1$ for all $y \in \{2, 3, \ldots, x-1\}$.**




**<pf> Suppose such a path exists,**

**choose one such $P$ with max. $a_x$**

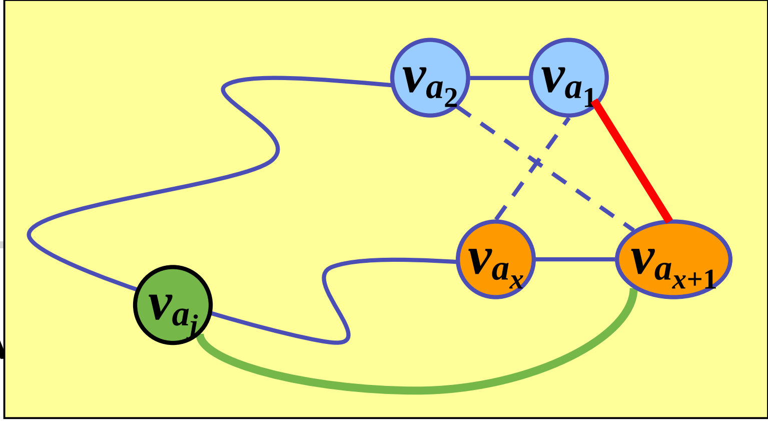



**By ①, ∃ $v_{a_{x+1}}$ s.t. $a_x < a_{x+1}$**

**and $v_{a_{x+1}}v_{a_x} \in E$, $v_{a_{x+1}}v_{a_2} \notin E$**

**(∵ $a_2 < a_x < a_1$ and $v_{a_x}v_{a_1} \notin E$, $v_{a_2}v_{a_1} \in E$)**

# 6.2 Chordal Graphs




- **Thm**: $G = (V, E)$ is chordal graph iff w

**Proof. (5/5)**

(⇒) Claim ②: ∃ chordless path $P$: $v_{a_1}, v_{a_2}, \ldots, v_{a_x}$ with $x \geq 3$ and $a_y < a_x < a_1$ for all $y \in \{2, 3, \ldots, x-1\}$.

<pf> Choose a min $j \geq 3$ such that $v_{a_{x+1}}v_{a_j} \in E$

(∵ $a_x$ is max., ∴ ∃ such $j$)

Consider $P'$: $v_{a_1}, v_{a_2}, \ldots, v_{a_j}, v_{a_{x+1}}$, $|P'| \geq 4$

Case 1: $v_{a_1}v_{a_{x+1}} \notin E \Rightarrow (P')^{-1}$, the inverse of $P'$, is a chordless path with larger second max. vertex then $P$. →←

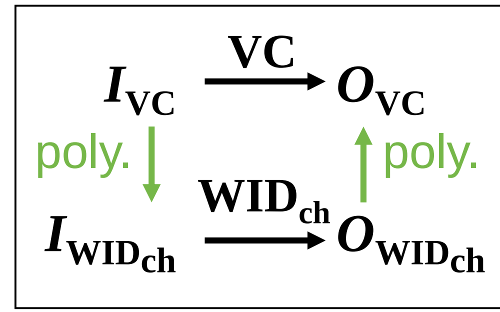
Case 2: $v_{a_1}v_{a_{x+1}} \in E \Rightarrow$ ∃ chordless cycle of length ≥ 4 (∵ $G$ is a chordal graph) →←

If $i < j < k$, $v_iv_j \in E$ and $v_iv_k \in E$, but $v_jv_k \notin E$ then ∃ chordless path $P$: $v_k, v_i, v_j$ →← (∵ ②)

Hence, the ordering $\sigma^*$ is a (PEO).

# 6.2 Chordal Graphs

- **Thm: The weighted independent domination problem is NP-complete for chordal graphs.**




**Proof. (1/2)**

**欲証: VC $\propto$ $WID_{chordal}$**

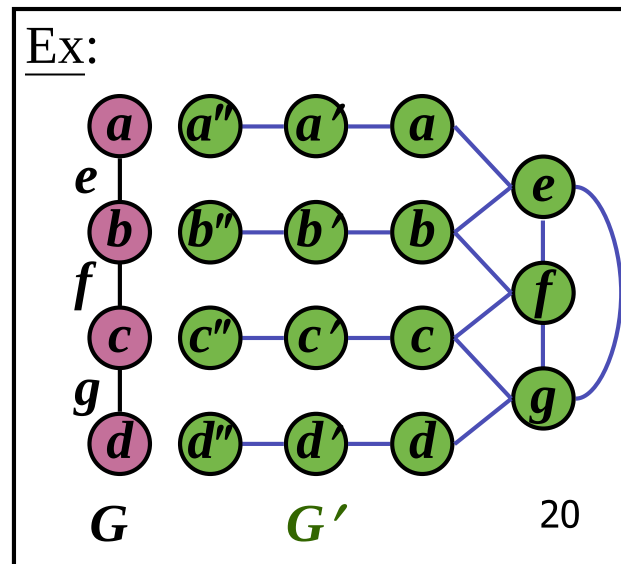
**$\forall\, G = (V, E)$ and $k \leq |V|$**

**Construct $G' = (V', E')$ and $k'$**

by
$$\begin{cases} V' = \{x'', x', x \mid x \in V\} \cup E \\ E' = \{x''x', x'x \mid x \in V\} \cup \{xe \mid x \in V, e \in E, x \in e\} \cup \{e_1e_2 \mid e_1 \neq e_2 \text{ in } E\} \\ k' = k + |V| \end{cases}$$

and
$$\begin{cases} w(e) = 3|V|, \forall\, e \in E \\ w(x) = w(x') = w(x'') = 1, \forall\, x \in V. \end{cases}$$

**Note that $G'$ is a chordal graph.**



# 6.2 Chordal Graphs

- **Thm: The weighted independent domination problem is NP-complete for chordal graphs.**

**Proof. (2/2)**

**Claim: $G$ has a vertex cover of size at most $k \Leftrightarrow$**

**$G'$ has an independent dominating set of weight at most $k'$.**

**<pf>($\Rightarrow$) Suppose $C$ is a vertex cover of $G$, $|C| \le k$.**

**Then, $D = \{x, x'' \mid x \in C\} \cup \{x' \mid x \in V \setminus C\}$ is a independent dominating set of weight at most**

**$|D| = 2|C| + (|V| - |C|) = |C| + |V| \le k + |V| = k'$.**

**($\Leftarrow$) Suppose $D$ is a independent dominating set of $G'$ and $w(D) \le k'$.**

**$\because \forall e \in E$, $w(e) = 3|V|$ and the definition of $G'$**

**$\therefore D \cap E = \phi$**

**$\Rightarrow D = \{x, x'' \mid x \in C\} \cup \{x' \mid x \in V \setminus C\}$ for some $C \subseteq V$**

**then $C$ is a vertex cover of $G$ and**

**$|C| = 2|C| + (|V| - |C|) - |V| = |D| - |V| \le k' - |V| = k$.**

# 6.2 Chordal Graphs

- **Thm: (M. Farber) The independent domination problem is polynomially solvable for chordal graphs.**