Computer Science and Information Engineering
National Chi Nan University

Combinatorial Optimization

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Lecture 5. The Domatic Number Problem on Interval Graphs

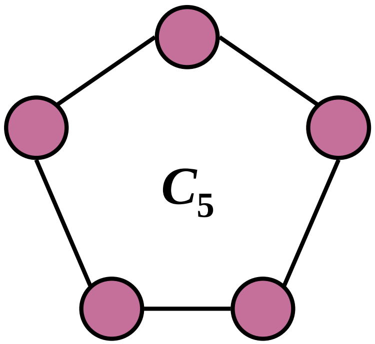
§5.1 Method 1: maximum flow

Slides for a Course Based on the Paper
Alan A. Bertossi, *"On the Domatic Number of Interval Graphs"*, Inform. Process. Lett. 28(6) (1988), 275-280.

5.1 Method 1: maximum flow

- **Def: Given a graph G, the domatic number, denoted by $d(G)$ = maximum number k such that $V = \dot{\bigcup}_{1 \le i \le k} D_i$, where D_i is a dominating set for all $1 \le i \le k$.**

- **Ex:**



$d(C_5) = 2$

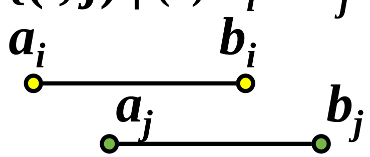
- **Note 1: $d(G)$ = maximum number k such that $\exists$ k disjoint dominating sets in G.**

5.1 Method 1: maximum flow

- **Def**: Given an interval graph $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$ and $\{I_i \mid I_i = [a_i, b_i]\}$ is an interval representation of G with $b_1 < b_2 < \ldots < b_n$. (Hence $\forall\, i, j, k \in V$, $i < j < k$ and $ik \in E \Rightarrow jk \in E$)
  - ① Define a graph $G'= (V', E)$ where $V'= \{0, 1, \ldots, n, n+1\}$, and

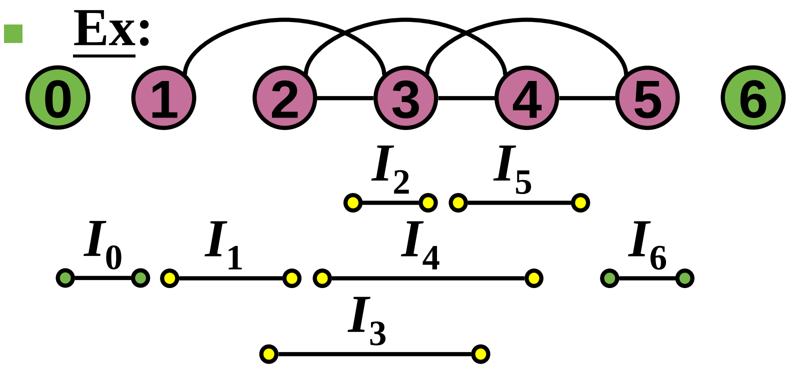
    $I_0 = [a_0, b_0]$, $b_0 < a_i\ \forall\, i$; $I_{n+1} = [a_{n+1}, b_{n+1}]$, $b_n < a_{n+1}$.
  - ② Define a digraph $H'= (V', A')$ where

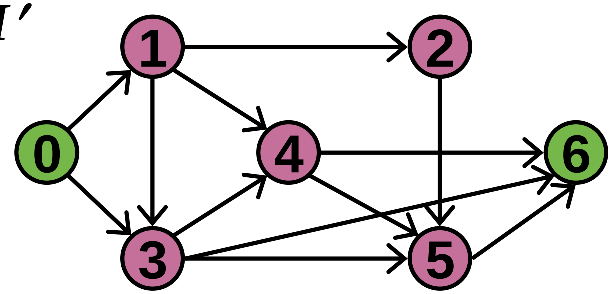
    $A'= \{(i, j) \mid$ (1) $a_i < a_j \leq b_i < b_j$ or (2) $\nexists\, h$ s.t. $b_i < a_h \leq b_h < a_j\}$.

(1)


(2)


- **Ex**:

G'


H'


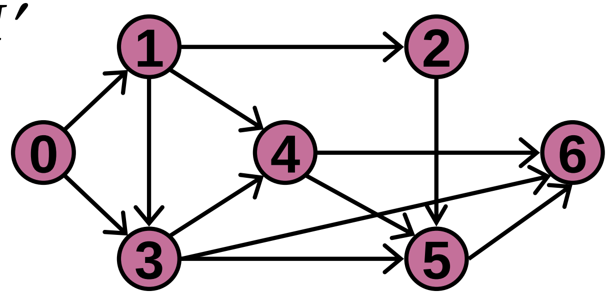
5.1 Method 1: maximum flow

- **<u>Note 2</u>: $\forall\ (i, j) \in A'$, $i < j$ and $i < h < j \Rightarrow (ih \in E$ or $hj \in E)$.**

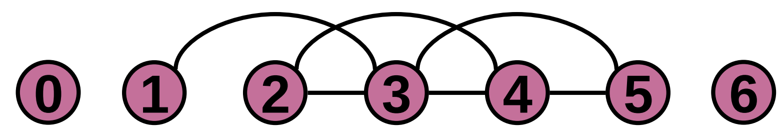
- **<u>Lemma 5.1</u>: $\forall$ 0-(n+1) dipath P in $H' \Rightarrow D = \{v \mid v \in V(P)\}$ is a dominating set in G'.**

- **<u>Ex</u>:**

H′



G′



5.1 Method 1: maximum flow

- **Lemma 5.1: $\forall$ 0-(n+1) dipath P in H' $\Rightarrow$ $D = \{v \mid v \in V(P)\}$ is a dominating set in G'.**

Proof.

Let $P = i_0\ (=0) \rightarrow i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_r \rightarrow i_{r+1}\ (= n+1)$

then $D = \{i_0, i_1, i_2, \ldots, i_r, i_{r+1}\}$ where $i_0 < i_1 < i_2 < \ldots < i_r < i_{r+1}$.

$\forall\ h \notin D$:

$\because h \neq 0, n+1$ $\therefore$ we can find $i_j, i_{j+1} \in D$ s.t. $i_j < h < i_{j+1}$

By Note 2, $\because (i_j, i_{j+1}) \in A'$, $\therefore$ $i_jh \in E$ or $i_{j+1}h \in E$

$\Rightarrow$ either i_j or i_{j+1} dominate h.

$\therefore$ D is a dominating set in G'.

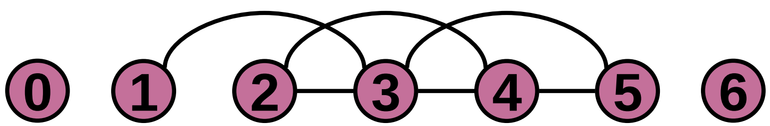
5.1 Method 1: maximum flow

Lemma 5.2: For any minimal dominating set D' of G'

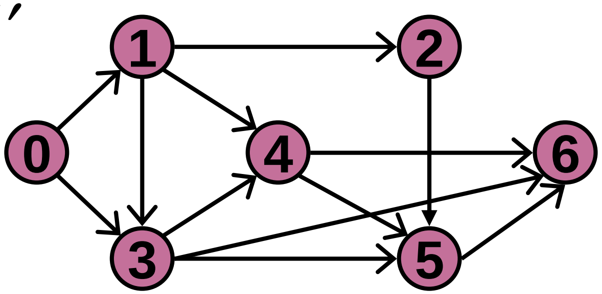
$\Rightarrow \{ x \mid x \in D' \}$ forms a 0-(n+1) dipath in H'.

- **Ex**:

G′



H′



5.1 Method 1: maximum flow

Lemma 5.2: For any minimal (proper) dominating set D' of G'

$\Rightarrow \{ x \mid x \in D' \}$ forms a 0-(n+1) dipath in H'.

Proof. (1/2)

Let $D' = \{0=k_0, k_1, k_2, \ldots, k_s, k_{s+1}=n+1\}$ where $k_0 < k_1 < k_2 < \ldots < k_s < k_{s+1}$.

Claim: $(k_j, k_{j+1}) \in A'$

<pf> for any $0 \le j \le s$:

Case 1: $I_{k_j} \cap I_{k_{j+1}} \neq \phi$:

$\because b_{k_j} < b_{k_{j+1}} \therefore a_{k_{j+1}} \le b_{k_j}$

If $a_{k_{j+1}} \le a_{k_j}$, then $I_{k_j} \subseteq I_{k_{j+1}}$

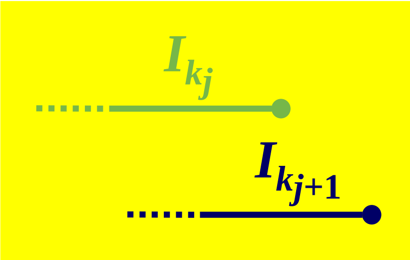
$\Rightarrow N[k_j] \subseteq N[k_{j+1}]$

$\Rightarrow D \backslash \{k_j\}$ is still a dominating set $\rightarrow\leftarrow$

($\because$ D is a "minimal" dominating set of G')

$\therefore a_{k_j} < a_{k_{j+1}} < b_{k_j} < b_{k_{j+1}}$

Hence $(k_j, k_{j+1}) \in A'$.



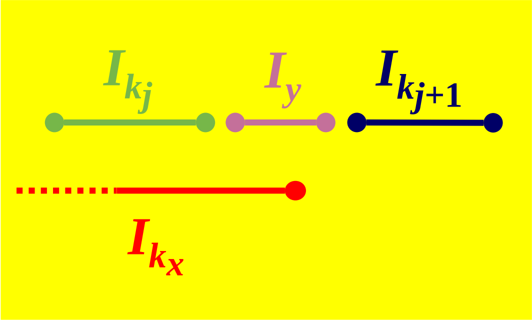
5.1 Method 1: maximum flow

Lemma 5.2: For any minimal dominating set D' of G'

$\Rightarrow \{ x \mid x \in D' \}$ forms a 0-(n+1) dipath in H'.

Proof. (2/2)

Case 2: $I_{k_j} \cap I_{k_{j+1}} = \phi$:



Suppose $\exists\, y \in V$ such that $b_{k_j} < a_y \le b_y < a_{k_{j+1}}$

$\because$ D is a dominating set of G'

$\therefore \exists\, I_{k_x} \in D$ such that $I_{k_x} \cap I_y \neq \phi$

$\because \nexists\, I_{k_x} \in D$ s.t. $j < x < j+1$

$\therefore k_x > y \Rightarrow x > j+1 \Rightarrow I_{k_{j+1}} \subseteq I_{k_x}$

$\Rightarrow D \backslash \{k_{j+1}\}$ is still a dominating set $\rightarrow\leftarrow$

($\because$ D is a "minimal" dominating set of G')

$\therefore \nexists\, y$ s.t. $b_{k_j} < a_y \le b_y < a_{k_{j+1}}$

Hence $(k_j, k_{j+1}) \in A'$.

5.1 Method 1: maximum flow

- **Thm 5.1: $d(G)$ = maximum number of internally vertex-disjoint 0-(n+1) path in H'.**

Proof.

By Note 1 $d(G)$ = maximum number of disjoint dominating set in G.

= maximum number of minimal dominating set in G' such that $\forall$ proper dominating set $D_i' \neq D_j'$, $D_i' \cap D_j' = \{0, n+1\}$.

By lemma 5.1 $\geq$ maximum number of internally disjoint 0-(n+1) path in H'.

By lemma 5.2 $\geq$ maximum number of dominating set in G' such that $\forall$ dominating set $D_i' \neq D_j'$, $D_i' \cap D_j' = \{0, n+1\}$.

= maximum number of disjoint dominating set in G.

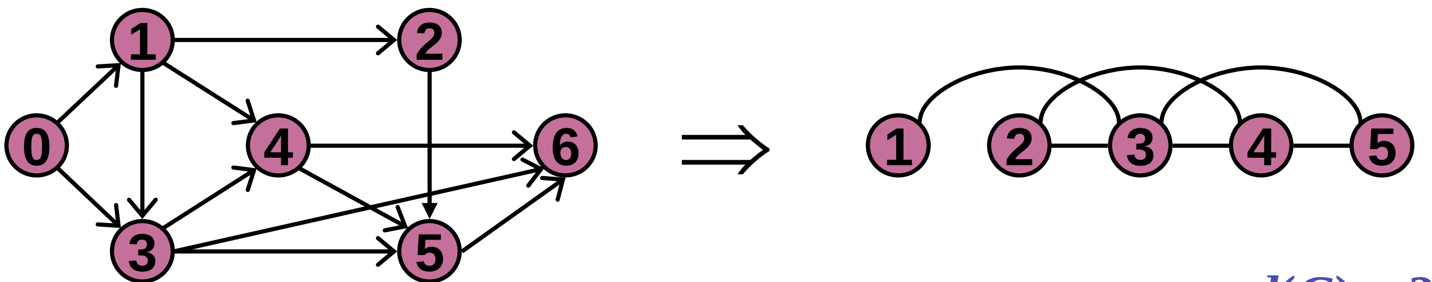
By Note 1 = $d(G)$.

Hence $d(G)$ = maximum number of internally vertex-disjoint 0-(n+1) path in H'.

5.1 Method 1: maximum flow

- **Remark: Since maximum flow = minimum cut, the best method to find the maximum flow need $\mathcal{O}(n^{2.5})$.** (?)

- **Ex:**



$\therefore d(G) = 2$

H' G

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Lecture 5. The Domatic Number Problem on Interval Graphs

§5.2 Method 2: Dynamic

Slides for a Course Based on the Paper
T. L. Lu, P. H. Ho, and G. J. Chang, *"The domatic number problem in interval graphs"*, SIAM J. Disc. Math. 3 (1990), 531-536.

5.2 Method 2: Dynamic

- **<u>Thm 5.2</u>: (Weakly duality inequality)**

  $d(G) \le \delta(G)+1$, where $\delta(G) = \min\{deg(x) \mid x \in V(G)\}$.

Proof.

Suppose $D_1, D_2, \ldots, D_{d(G)}$ are disjoint dominating set.

Choose $x \in V(G)$ such that $deg(x) = \delta(G)$

$\because$ D_i is a dominating set, $\forall\ 1 \le i \le d(G)$

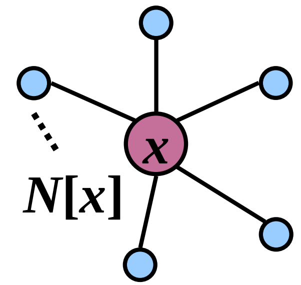
$\therefore$ $D_i \cap N[x] \ne \phi$, $\forall\ 1 \le i \le d(G)$

$\forall 1 \le i < j \le d(G)$:

$\because$ D_i, D_j are disjoint

$\therefore$ $(D_i \cap N[x]) \cap (D_j \cap N[x]) = \phi$

$\Rightarrow d(G) \le |N[x]| = \delta(G)+1$



5.2 Method 2: Dynamic

- **Algorithm:**

Given an interval graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$ in interval ordering.

for $0 \le i \le \delta(G)$
    $D_i \leftarrow \phi$;

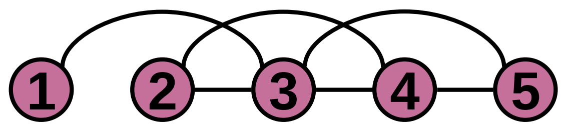
for $j = n$ to 1 step – 1
    choose a maximum $k \in N[j]$ that is not dominated by all D_i;
    say k is not dominated by D_{i^*}, If $\nexists$ such k then choose any D_i as D_{i^*}
    $D_{i^*} \leftarrow D_{i^*} \cup \{j\}$.

- **Time Complexity** = $\mathcal{O}(|V|+|E|)$

for $j = n$ to 1 step – 1
choose a maximum $k \in N[j]$ that is not dominated by all D_i;
say k is not dominated by D_{i*}, If $\nexists$ such k then choose any D_i as D_{i*}
$D_{i*} \leftarrow D_{i*} \cup \{j\}$.

5.

- **Ex:**



k	1	ϕ	2	5	5
i^*	{1}	ϕ	{0}	{1}	{0} (or {1})

D_0 = { 5, 3, 2 }

D_1 = { 4, 1 }

- **Goal**: When the algorithm stops, if $D_0, \ldots, D_{\delta(G)}$ are dominating sets, then $\delta(G)+1 \le d(G) \le \delta(G)+1$. $(\Rightarrow d(G) = \delta(G)+1)$

  need prove   w.d.i.

- **Notation:**

  ① $ndom_j(h)$ = number of D_i s.t. h is not dominated by D_i at iteration j.

  ② $R_j(h)$ = number of vertices in $N[h]$ not yet assigned at iteration j.

5.2 Method 2: Dynamic

- **Note:**
  ① $ndom_n(h) = \delta(G)+1, \forall\ 1 \le h \le n$
  ② $R_n(h) = deg(h)+1 \ge \delta(G)+1, \forall\ 1 \le h \le n$
  ③ $R_0(h) = 0, \forall\ 1 \le h \le n$
  ④ $ndom_i(h) \ge ndom_j(h), \forall\ 1 \le h \le n, n \ge i > j \ge 0.$

- **Lemma 5.3**: $\forall\ 1 \le h \le n$, $R_j(h) \ge ndom_j(h)$ in any iteration j.

- **Thm 5.3**: $D_0, \ldots, D_{\delta(G)}$ are dominating sets.

Proof.

By Lemma 5.3, when $j = 0$: $0 = R_0(h) \ge ndom_0(h)$
$\Rightarrow ndom_0(h) = 0$ for all h
i.e. $\forall\ h$, h is dominated by D_i, $\forall\ 1 \le i \le \delta(G)$
$\Rightarrow D_0, \ldots, D_{\delta(G)}$ are dominating sets.

5.2 Method 2: Dynamic

- **<u>Lemma 5.3</u>: $\forall\ 1 \le h \le n$, $R_j(h) \ge ndom_j(h)$ in any iteration j.**

Proof. (1/3)

Prove by induction on j (back induction)

When $j = n$: $R_n(h) = deg(h)+1 \ge \delta(G)+1 = ndom_n(h)$, $\forall\ 1 \le h \le n$

Suppose $R_{j+1}(h) \ge ndom_{j+1}(h)$ for some $2 \le j \le n$, $\forall\ 1 \le h \le n$

For the case of j, $\forall\ 1 \le h \le n$:

① $h \notin N[j+1]$: $R_j(h) = R_{j+1}(h) \ge ndom_{j+1}(h) = ndom_j(h)$

I.H.

② $h \in N[j+1]$: $R_j(h) = R_{j+1}(h) - 1 \ge ndom_{j+1}(h) - 1$

I.H.

依演算法中所選出k之大小分別討論:

<u>Case 1</u>: $k < h$ or $\nexists$ such k

<u>Case 2</u>: $h \le k$: (k未被D_{i*} dominate)

5.2 Method 2: Dynamic

- **Lemma 5.3**: $\forall\ 1 \le h \le n$, $R_j(h) \ge ndom_j(h)$ in any iteration j.

Proof. (2/3)

② $h \in N[j+1]$: $R_j(h) = R_{j+1}(h) - 1 \ge ndom_{j+1}(h) - 1$

Case 1: $k < h$ or $\nexists$ such k

By the algorithm, $ndom_j(h) = 0 = ndom_{j+1}(h)$

$\Rightarrow R_j(h) \ge 0 = ndom_j(h)$

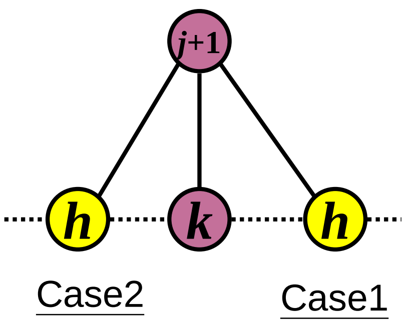
Case 2: $h \le k$: (k未被D_{i^*} dominate)

Claim: 未將j+1放入D_{i^*}之前, D_{i^*} does not dominate h.

$\because$ j+1放入D_{i^*}之後， D_{i^*}就dominate h

$\therefore$ $ndom_j(h) = ndom_{j+1}(h) - 1$.

$\Rightarrow R_j(h) \ge ndom_{j+1}(h) - 1 = ndom_j(h)$.

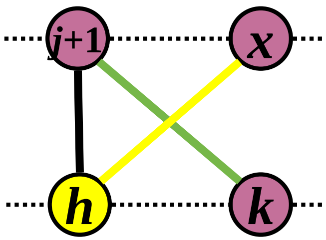


$i < j < k$ and $v_i v_k \in E \Rightarrow v_j v_k \in E$ (∗)

5.2 Method 2: Dynamic

- **Lemma 5.3**: $\forall\ 1 \le h \le n$, $R_j(h) \ge ndom_j(h)$ in any iteration j.

Proof. (3/3)



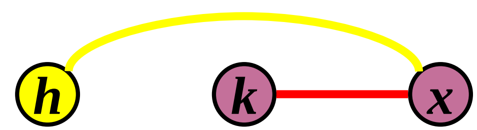
Case 2: $h \le k$: (k未被D_{i^*} dominate)

Claim: 未將j+1放入D_{i^*}之前, D_{i^*} does not dominate h.

<pf> Suppose not,

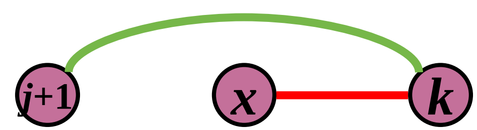
i.e. $\exists\ x \in D_{i^*}$, x dominate h where $x \in \{j+2, \ldots, n\}$

考慮 k 與 x 之大小關係:



(1) If $h \le k < x$: ($\because h \le k$)

By (∗), $hx \in E \Rightarrow kx \in E$



(2) else $j+1 < x \le k$: ($\because j+1 < x$)

By (∗), $(j+1)k \in E \Rightarrow xk \in E$

$\therefore$ In any case, k was dominated by D_{i^*} →←

5.2 Method 2: Dynamic

- **Reference**

- Glenn Manacher, Terrance Mankus, "*Finding a Domatic Partition of an Interval Graph in Time O*(*n*)," SIAM J. Discrete Math. 9 (1996) Issue 2, pages 167 – 172.
- Haim Kaplan, Ron Shamir, "*The domatic number problem on some perfect graph families*," Information Processing Letters, Volume 49, Issue 1 (January 1994) Pages: 51 – 56.
- G. J. Chang (1994), "*The domatic number problem*," Disc. Math. 125, 115-122.
- A. S. Rao and C. P. RanganLinear, "*algorithm for domatic number problem on interval graphs*," Information Processing Letters, Volume 33 , Issue 1 (October 1989), Pages: 29 - 33
- Shen-Lung Peng and Maw-Shang Chang, "*A simple linear time algorithm for the domatic partition problem on strongly chordal graphs*," Information Processing Letters, Volume 43 , Issue 6 (October 1992), Pages: 297 - 300

5.2 Method 2: Dynamic

- **Exercise 3 (4/12): Develop a primal-dual algorithm for the vertex cover problem in tree. Or find an counterexample. (The dual problem is the maximum matching problem which is to find a matching of maximum size. A matching of a graph is a subset of edge set in which no two distinct edges have a common vertex.)**