Computer Science and Information Engineering
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Combinatorial Optimization

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Lecture 4. The Domination Problems on Interval Graphs

§4.3 Domination Problems

Slides for a Course Based on the Paper
G. Ramalingam and C. Pandu Rangan, *"A unified approach to domination problems on interval graphs"*, Inform. Process. Lett. 27 (1988), 271-274.

4.3 Domination Problems

1. Introduction and notations

- **Def**: Let $G = (V, E)$ be a graph, let $D \subseteq V$

① D is a **independent dominating set** of G iff

$$\begin{cases} \text{(1) } D \text{ is a dominating set of } G. \\ \text{(2) } D \text{ is a independent set. } (\forall\, x \neq y \text{ in } D \Rightarrow xy \notin E) \end{cases}$$

$\gamma_i(G) = \min\{|D|: D$ is a independent dominating set of $G\}$ $(\geq \gamma(G))$

② D is a **connected dominating set** of G iff

$$\begin{cases} \text{(1) } D \text{ is a dominating set of } G. \\ \text{(2) } G[D] \text{, the subgraph induced by } D \text{ is connected.} \end{cases}$$

$\gamma_c(G) = \min\{|D|: D$ is a connected dominating set of $G\}$

③ D is a **total dominating set** of G iff

$$\begin{cases} \text{(1) } D \text{ is a dominating set of } G. \\ \text{(2) } deg_{G[D]}(x) \neq 0,\ \forall\, x \in D. \end{cases} \equiv \bigcup_{x \in D} N(x) = V$$

$(= \forall\, x \in D \Rightarrow \exists\, y \neq x$ and $y \in D$ such that $xy \in E)$

$\gamma_t(G) = \min\{|D|: D$ is a total dominating set of $G\}$

4.3 Domination Problems

- **Def: Given graph $G = (V, E)$, and $\forall$ vertex v, $\exists$ real number $w(v)$**

  ① $\gamma(G, w) = \min \{w(D) = \sum_{v \in D} w(v)$: D is a dominating set of $G\}$.

  ② $\gamma_i(G, w) = \min \{w(D) = \sum_{v \in D} w(v)$: D is an independent dom. set of $G\}$.

  ③ $\gamma_c(G, w) = \min \{w(D) = \sum_{v \in D} w(v)$: D is a connected dom. set of $G\}$.

  ④ $\gamma_t(G, w) = \min \{w(D) = \sum_{v \in D} w(v)$: D is a total dom. set of $G\}$.

- **Remark: The problems of finding a minimum-cardinality (independent, connected, total) dominating set and finding a minimum-weight (independent, connected, total) dominating set of a graph are NP-complete. [Garey and Johnson, 1979]**

4.3 Domination Problems

- **Def: Given an interval graph $G = (V, E)$ and the interval ordering of G is $\{1, 2, \ldots, n\}$. Define**

  ① $V_i = \{1, 2, \ldots, i\}$, $V_0 = \phi$.

  ② G_i = the subgraph of G induced by V_i

  ③ $low(i) = \min\{x \mid x \in N[i]\}$

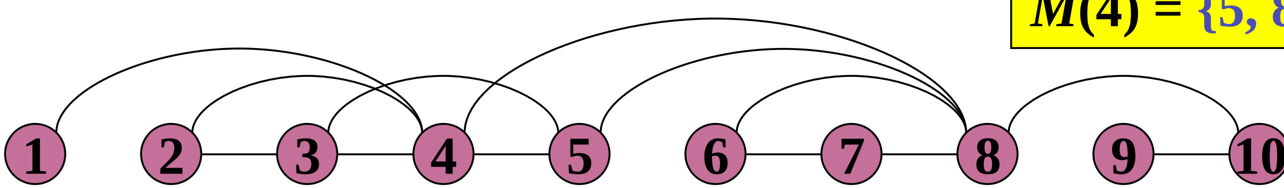
  ④ $maxlow(i) = \max\{low(j) \mid low(i) \leq j \leq i\}$

  ⑤ $L(i) = \{maxlow(i), maxlow(i)+1, \ldots, i\}$

  ⑥ $M(i) = \{j \mid j > i$ and j is adjacent to $i\}$

$L(4) = \{2, 3, 4\}$
$M(4) = \{5, 8\}$

- **Ex:**



i	1	2	3	4	5	6	7	8	9	10
$low(i)$	1	2	2	1	3	6	6	4	9	8
$maxlow(i)$	1	2	2	2	3	6	6	6	9	9

$L(3) = \{2, 3\}$
$M(3) = \{4, 5\}$

$L(8) = \{6, 7, 8\}$
$M(8) = \{10\}$

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- **Note**: ① $N_{G_i}(i) = \{low(i), low(i)+1, \ldots, i-1\}$
  ② $G[L(i)]$ form a maximal clique in G_i.
  ③ Let $j \in V$ s.t. $low(i) \le j \le i$ and $maxlow(i) = low(j)$
  Then, (1) $low(i) \le low(j)$
  (2) $j \le i$
  (3) $N[j] \subseteq L(i) \cup M(i)$
  (4) $N_{G_i}(j) \subseteq L(i)$

- **Def**: **Min X** = the element D^* in X, such that
  $$w(D^*) = \min\{w(D) \mid D \in X\}.$$

4.3 Domination Problems

2. Domination

- **Def:** ① Let D_i = subset of $V(G)$ s.t. $\forall\, x \in V_i$, $\exists\, y \in D_i$ with $x \sim y$.
  ② Let MD_i = the minimum-weight D_i.

- **Thm:** (a) $MD_0 = \phi$
  (b) For $1 \le i \le n$, $MD_i = \min\{\{j\} \cup MD_{low(j)-1} \mid j \in L(i) \cup M(i)\}$

Proof. (b) (1/2)

① $\because\ \exists\, j \in L(i)$ s.t. $N(j) \subseteq L(i) \cup M(i)$
$\therefore\ \exists\, k \in D_i \cap (L(i) \cup M(i))$ s.t. $kj \in E$
$\Rightarrow D_i = \{k\} \cup D$ for some $k \in L(i) \cup M(i)$,
where $D = D_i - \{k\}$, for any D_i.

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- **<u>Thm</u>: (b) For $1 \le i \le n$, $MD_i = \min\{\{j\} \cup MID_{low(j)-1} \mid j \in L(i) \cup M(i)\}$**

Proof. (b) (2/2)

② ∵ The definition of *low* function,

∴ k dominates no vertex in $V_{low(k)-1}$

⇒ D is a dominating set of $G_{low(k)-1}$

By ①, ②, a set is a $D_i \Leftrightarrow D_i = \{k\} \cup D_{low(k)-1}$ for some $k \in L(i) \cup M(i)$

∴ $MD_i = \mathrm{Min}\{D_i \mid \text{for all possible } D_i\}$

$= \mathrm{Min}\{\{k\} \cup D_{low(k)-1} \mid k \in L(i) \cup M(i)\}$

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- **Algorithm 4.3:**

Given the interval ordering {1, 2, …, n} and *low*, *maxlow* functions for every vertex of a interval graph G.

$MD_0 := \phi$;

$w(MD_0) := 0$;

for $i = 1$ **to** n **do**

  $min := \infty$;

  for $j \in L[i] \cup M[i]$ **do**

    if $(j \geq maxlow(i)$ **and** $w(j) + w(MD_{low(j)-1}) < min)$ **then**

      $min := w(j) + w(MD_{low(j)-1})$;

      $minj := j$

  $MD_i := \{minj\} \cup MD_{low(minj)-1}$;

  $w(MD_i) := min$

- **Time Complexity = $\mathcal{O}(|V|+|E|)$**

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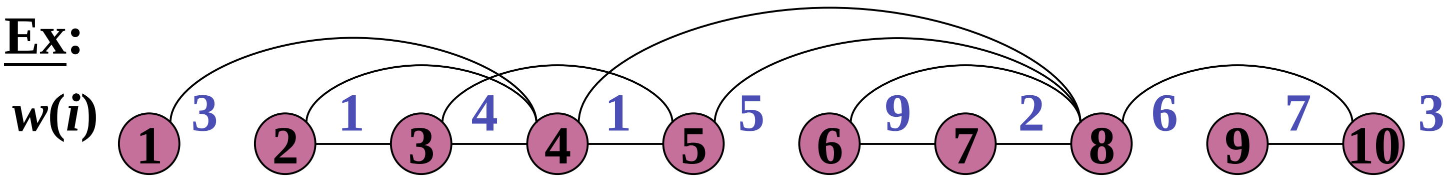
- **Ex:**



***low*(*i*)**	1	2	2	1	3	6	6	4	9	8
***maxlow*(*i*)**	1	2	2	2	3	6	6	6	9	9
***L*(*i*)**	{1}	{2}	{2,3}	{2,3,4}	{3,4,5}	{6}	{6,7}	{6,7,8}	{9}	{9,10}
***M*(*i*)**	{4}	{3,4}	{4,5}	{5,8}	{8}	{7,8}	{8}	{10}	{10}	ϕ
MD_i										
$w(MD_i)$										
min										
minj										

```
min := ∞;
for j ∈ L[i]∪M[i] do
   if (j ≥ maxlow(i) and w(j) + w(MD_{low(j)–1}) < min) then
      min := w(j) + w(MD_{low(j)–1});
      minj := j
MD_i := {minj} ∪ MD_{low(minj)–1};
w(MD_i) := min
```

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3. Independent domination

- **Def**: ① Let ID_i = an independent dominating set of G_i.
  ② Let MID_i = the minimum-weight ID_i.

- **Thm**: (a) $MID_0 = \phi$
  (b) For $1 \le i \le n$, $MID_i = \text{Min}\{\{j\} \cup MID_{low(j)-1} \mid j \in L(i)\}$

Proof. (b) (1/3) (≥)

Let $MID_i = D \cup \{j\}$, where $j = \max\{x \mid x \in MID_i\}$, $D = MID_i \setminus \{j\}$.

① D is an independent set.

② D is a dominating set of $G_{low(j)-1}$:

$\forall x \in V_{low(j)-1}$, $\exists y \in MID_i$ s.t. $x \sim y$

But $\because x < low(j)$ $\therefore$ By the definition of *low* function, we have $y \neq j$.

Then $y \in D$. $\therefore$ D is a dominating set of $G_{low(j)-1}$.

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- **Thm: (b) For $1 \le i \le n$, $MID_i = \text{Min}\{\{j\} \cup MID_{low(j)-1} \mid j \in L(i)\}$**

Proof. (b) (2/3) (≥)

③ $j \in L(i)$:

Suppose $k^* \in V$ s.t. $low(i) \le k^* \le i$ and $maxlow(i) = low(k^*)$

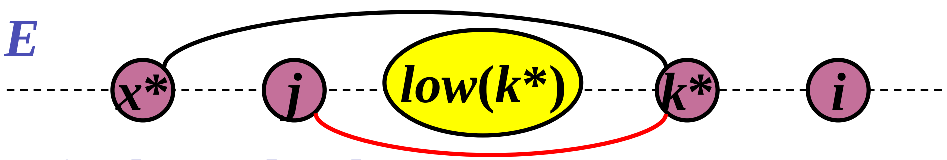
$\Rightarrow maxlow(i) = \max\{low(k) \mid low(i) \le k \le i\} \ge low(j)$

($\because N_{G_i}(i) = \{low(i), low(i)+1, \ldots, i-1\}$, $\therefore low(i) \le j \le i$)

If $j \notin L(i)$, then $j < maxlow(i) = low(k^*) \le k^*$

By definition of independent dominating set,

$\exists\, x^* \in MID_i$ s.t. $x^*k^* \in E$



$\because x^* < \max\{x \mid x \in MID_i\} = j < k^*$ and $x^*k^* \in E$

$\therefore$ By (∗), $jk^* \in E \rightarrow\leftarrow$ ($\because j < low(k^*)$)

By ①, ②, ③ $\Rightarrow w(D) \ge w(j) + w(MID_{low(j)-1})$ for some $j \in L(i)$.

$\Rightarrow MID_i \ge \text{Min}\{\{j\} \cup MID_{low(j)-1} \mid j \in L(i)\}$.

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- **Thm: (b) For $1 \le i \le n$, $MID_i = \text{Min}\{\{j\} \cup MID_{low(j)-1} \mid j \in L(i)\}$**

Proof. (b) (3/3) (≤) $\forall j \in L(i)$:

① $\because N_{G_i}(j) = \{x \in V \mid x \ge low(j)\}$

$\therefore$ j is not adjacent to any vertex in $G_{low(j)-1}$

$\Rightarrow \{j\} \cup MID_{low(j)-1}$ is an independent set of G_i.

② (1) $\forall x \in \{low(j), low(j)+1, \ldots, j\}$:

$\because low(j) \le x \le j$

$\therefore$ By the definition of *low* function and (∗), $x \sim j$.

(2) $\forall x \in \{j+1, j+2, \ldots, i\}$:

$\because j \in L(i)$ and $low(i) \le maxlow(i) \le j < x < i$

$\therefore low(x) \le maxlow(i) \le j < x$

$\therefore$ By the definition of *low* function and (∗), $jx \in E$

or say

$\because L(i)$ is a clique

$\therefore \forall x \in L(i), x \sim j$.

$\Rightarrow \{j\} \cup MID_{low(j)-1}$ is an indp. dom. set of G_i

By ①, ②, $MID_i \le \text{Min}\{\{j\} \cup MID_{low(j)-1} \mid j \in L(i)\}$.

4.3 Domination Problems

- **Algorithm 4.2:**

Given the interval ordering {1, 2, …, n} and *low*, *maxlow* functions for every vertex of a interval graph G.

$MID_0 := \phi$;
$w(MID_0) := 0$;
for $i = 1$ **to** n **do**
  $min := \infty$;
  for $maxlow(i) \le j \le i$ **do**
    if $(w(j) + w(MID_{low(j)-1}) < min)$ **then**
      $min := w(j) + w(MID_{low(j)-1})$;
      $minj := j$
  $MID_i := \{minj\} \cup MID_{low(minj)-1}$;
  $w(MID_i) := min$

- **Time Complexity = $\mathcal{O}(|V|+|E|)$**

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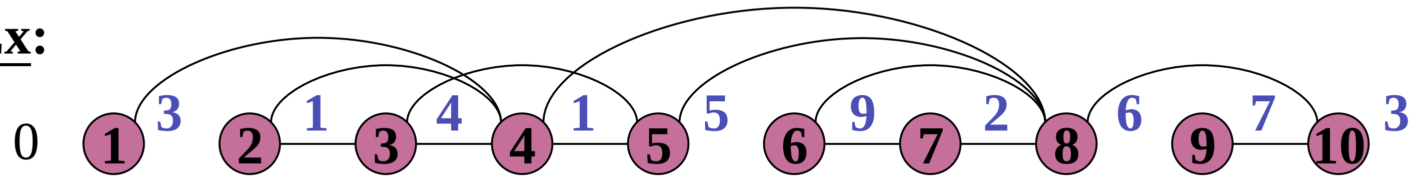
- **Ex:**



***w*(*i*)**	3	1	4	1	5	9	2	6	7	3
***low*(*i*)**	1	2	2	1	3	6	6	4	9	8
***maxlow*(*i*)**	1	2	2	2	3	6	6	6	9	9
***L*(*i*)**	{1}	{2}	{2,3}	{2,3,4}	{3,4,5}	{6}	{6,7}	{6,7,8}	{9}	{9,10}
MID_i										
$w(MID_i)$										
min										
minj										

$min := \infty$;
for $maxlow(i) \le j \le i$ **do**
  if $(w(j) + w(MID_{low(j)-1}) < min)$ **then**
    $min := w(j) + w(MID_{low(j)-1})$;
    $minj := j$
$MID_i := \{minj\} \cup MID_{low(minj)-1}$;
$w(MID_i) := min$

4.3 Domination Problems

- **Note:**
  - ① **The method can be used to solve the "maximum-weight" version.**
  - ② **For the other 3 domination problems, we can consider nonnegative weights only.**

4.3 Domination Problems

4. Total domination and connected domination

- **Exercise 加分 (4/13): Develop a dynamic programming algorithm for total domination or connected domination; or show there exist an counterexample.**

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5. Related works

Interval Graphs				Circular-arc Graphs			
TD	WTD	WD, WID	WCD	D	WD	WTD	WID, WCD
Open[84]				$\mathcal{O}(nm)$[85]			
$\mathcal{O}(n^2)$[86]					$\mathcal{O}(n^3)$[89]		
$\mathcal{O}(n+m)$ [88]				$\mathcal{O}(n)$[91]			
	$\mathcal{O}(n\log n)$ [88]				$\mathcal{O}(n^2\log n)$ [91]		
	$\mathcal{O}(n+m)$[88]						
	$\mathcal{O}(n+m)$[98]		$\mathcal{O}(n\log\log n)$ [98]		$\mathcal{O}(n+m)$[98]		

Where $n = |V|$ and $m = |E|$