Computer Science and Information Engineering
National Chi Nan University

Chapter 10
NP-completeness

§ 10.3 Some NP-Complete Problems (2)

§ 10.3 Some NP-Complete Problems

- **Def:**

  Clique Problem:

  input: Graph G and $k \in N$.

  output: $\begin{cases} \text{“yes”, if } G \text{ has a clique of size } k; \\ \text{“no”, o.w..} \end{cases}$

- **Def: C is a clique of $G = (V, E)$**

  if $C \subseteq V$ and for any $x \in C$ and $y \in C$, $xy \in E$.

- **Theorem: SAT $\propto$ Clique (i.e. Clique is *NP*-complete)**

§ 10.3 Some NP-Complete Problems

- **Def:**

  Vertex-cover Problem:

  input: Graph G' and $k' \in N$.

  output: $\begin{cases} \text{"yes", if } G' \text{ has a vertex cover of size } k'; \\ \text{"no", o.w..} \end{cases}$

Recall

- **Def: C is a vertex-cover of $G = (V, E)$**

  if $C \subseteq V$ and every edge $xy \in E$ either $x \in C$ or $y \in C$.

§ 10.3 Some NP-Complete Problems

- **Theorem: Clique ∝ Vertex-cover**

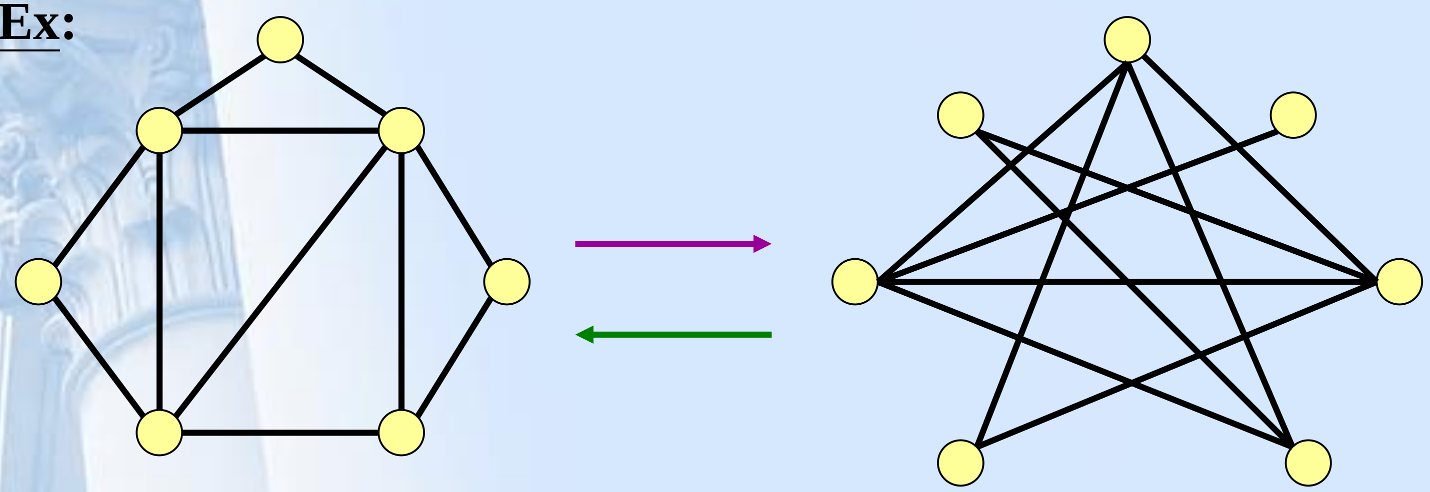
  Proof. (1/3)

  Given G and k

  Let $G' = G^C$ and $k' = |V(G)| - k$

  Claim: G has a clique of size k ⇔ G' has a vertex-cover size k'.

  Ex:



$k = 3$ **G** **$k' = 4$** **$G' = G^C$**

§ 10.3 Some NP-Complete Problems

- **<u>Theorem</u>: Clique ∝ Vertex-cover**

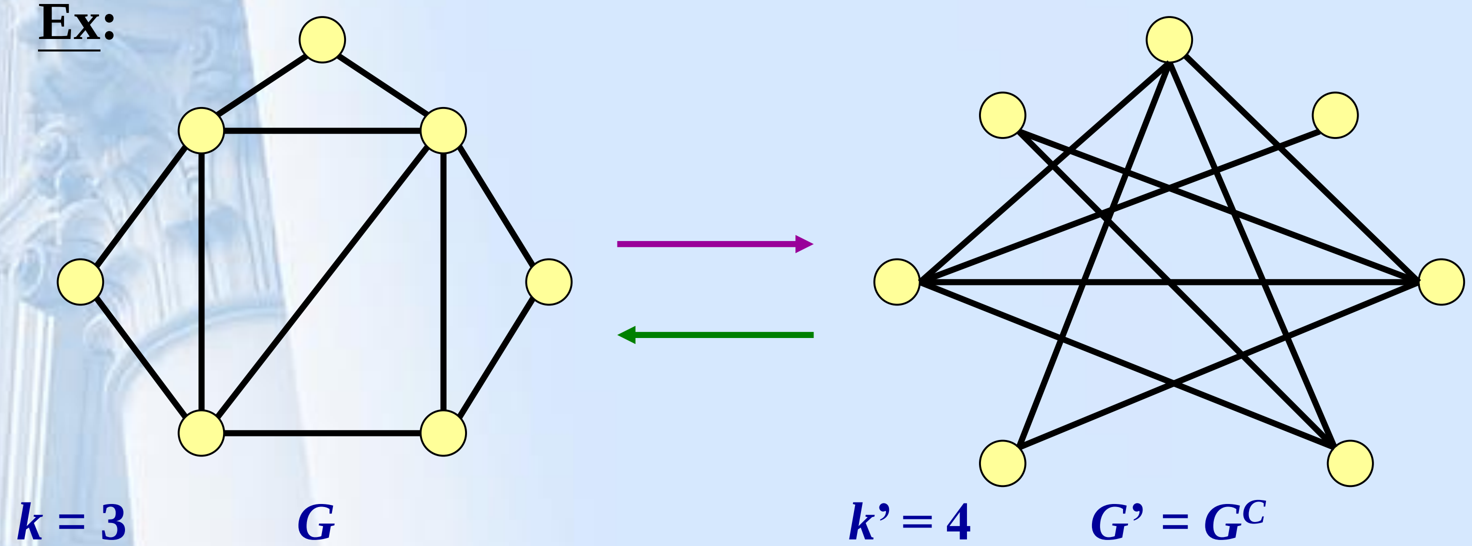
  Proof. (1/3)

  Given G and k

  Let $G' = G^C$ and $k' = |V(G)| - k$

  <u>Claim</u>: G has a clique of size $k \Leftrightarrow G'$ has a vertex-cover size k'.

  <u>Ex</u>:



§ 10.3 Some NP-Complete Problems

- **Theorem: Clique ∝ Vertex-cover**

  Proof. (2/3)

  Claim: G has a clique of size $k \Leftrightarrow G'$ has a vertex-cover size k'.

  Proof.

  ($\Rightarrow$) Suppose C is a clique of G with $|C| = k$.

  Let $C' = V(G) - C$

  Then $|C'| = k'$.

  $\forall\, xy \in E(G') = E(G^C)$

  Assume $x \notin C'$, $y \notin C'$, then $x \in C$, $y \in C$.

  $\Rightarrow xy \in E(G)$ (since C is a clique) $\rightarrow\leftarrow$ ($xy \in E(G^C)$)

  Hence $x \in C'$ or $y \in C'$

  $\therefore$ C' is a vertex cover.

§ 10.3 Some NP-Complete Problems

- Theorem: Clique $\propto$ Vertex-cover

  Proof. (3/3)

  Claim: G has a clique of size $k \Leftrightarrow G'$ has a vertex-cover size k'.

  Proof.

  ($\Leftarrow$) Suppose C' is a vertex cover of G' with $|C'| = k'$.

  Let $C = V(G') - C' = V(G) - C'$.

  Then $|C| = |V(G)| - k' = k$.

  $\forall\ x \in C, y \in C$, then $x \notin C', y \notin C'$.

  Assume $xy \notin E(G)$, then $xy \in E(G^C) = E(G')$.

  $\rightarrow\leftarrow$ $xy \in E(G')$, but $x \notin C', y \notin C'$.

  Hence $xy \in E(G)$

  Hence C is a clique of size k.

§ 10.3 Some NP-Complete Problems

- **Def:**

  Domination Problem:

  input: Graph G and $k \in N$.

  output: $\begin{cases} \text{"yes", if } \exists\, D \subseteq V(G),\ |D| = k \\ \qquad \text{and } N[x] \cap D \neq \emptyset,\ \forall\, x \in V(G); \\ \text{"no", o.w. .} \end{cases}$

Recall

- **Def: A domination set of a graph $G = (V, E)$ is a subset D of V s.t. $\forall\, x \in V \setminus D$, $\exists\, y \in D$ with $xy \in E$.**

§ 10.3 Some NP-Complete Problems

- **Theorem: 3 SAT ∝ Domination.**

  Proof. (1/5)

  Given $f = \prod_{i=1}^{r}(x_{i1} + x_{i2} + x_{i3})$ over $\{x_1, x_2, \ldots, x_n\}$.

  Consider $G = (V, E)$ as

  $V = \{x_j, \neg x_j, y_j: 1 \le j \le n\} \cup \{(i, j): 1 \le i \le r, 1 \le j \le 4\}$

  $E = \{x_j(\neg x_j), (\neg x_j)y_j, x_j y_j: 1 \le j \le n\} \cup$

  $\{(i, 1)(i, 2), (i, 2)(i, 4), (i, 3)(i, 4), (i, 3)(i, 1): 1 \le i \le r\} \cup$

  $\{x_j(i, k): x_j = x_{ik}\} \cup \{\neg x_j(i, k): (\neg x_j) = x_{ik}\}.$

  Let $k = n + r$.

  claim: f is satisfiable ⇔ G has a dominating set of size k.

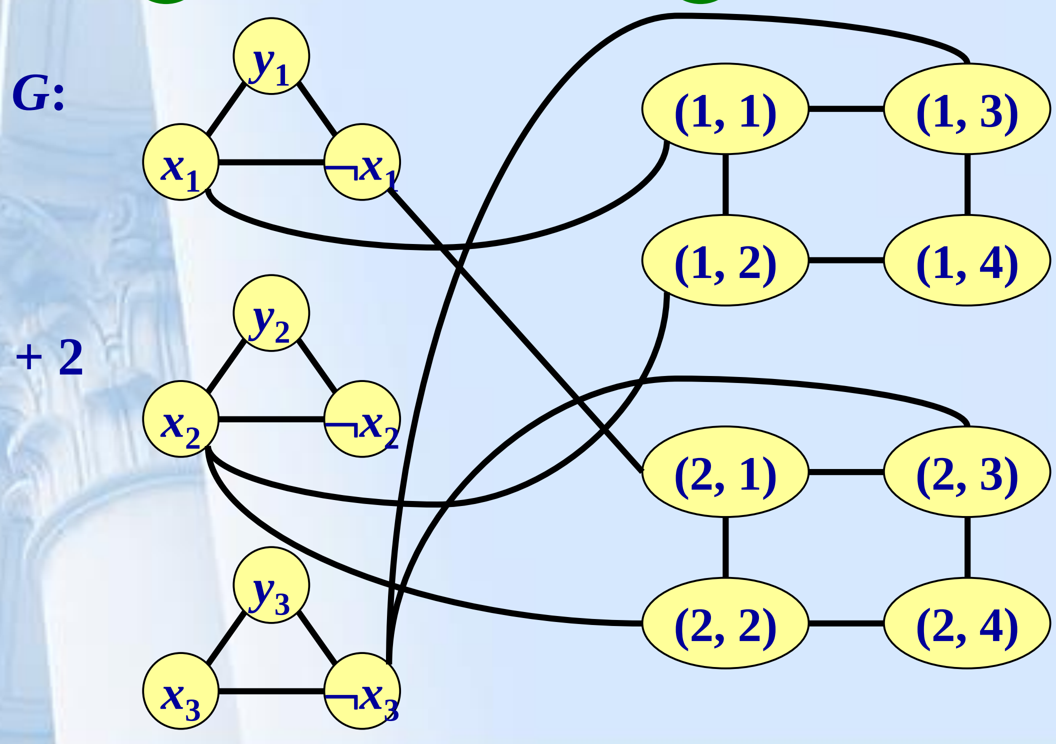
§ 10.3 Some NP-Complete Problems

- **Theorem: 3 SAT ∝ Domination.**

  Proof. (2/5)

  Ex: $f = (x_1 + x_2 + (\neg x_3))((\neg x_1) + x_2 + (\neg x_3))$

$x_1 = 1$,
$x_2 = 1$,
$x_3 = 1$



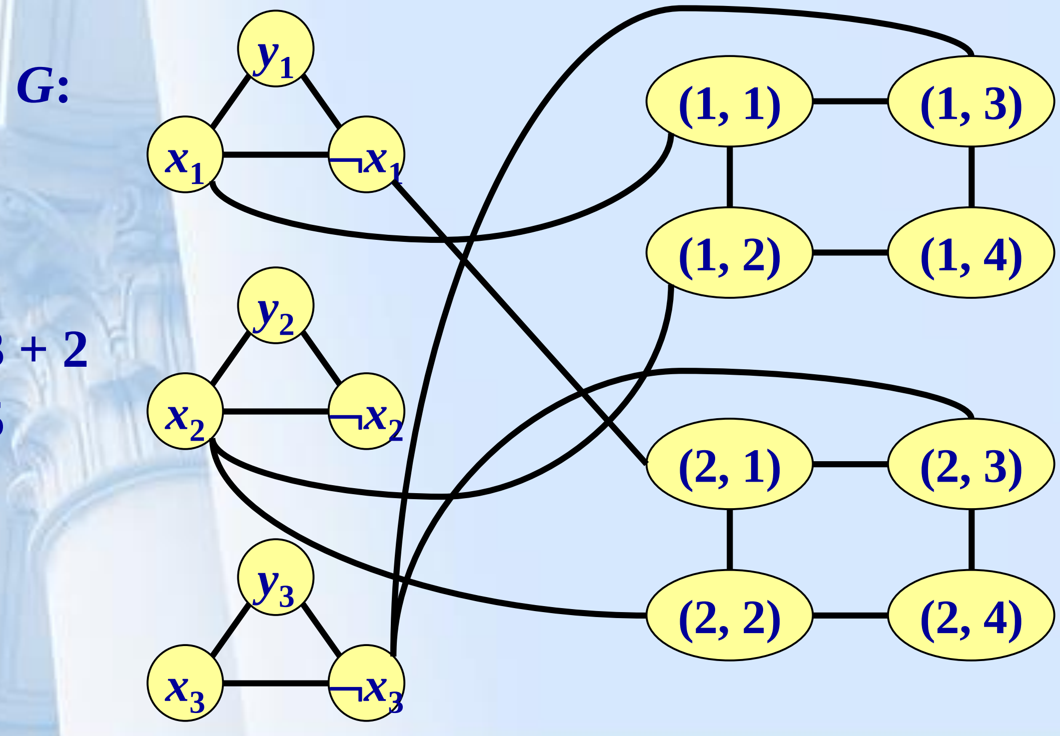
$k = 3 + 2$
$= 5$

§ 10.3 Some NP-Complete Problems

- **Theorem: 3 SAT ∝ Domination.**

  Proof. (2/5)

  Ex: $f = (x_1 + x_2 + (\neg x_3))((\neg x_1) + x_2 + (\neg x_3))$



$x_1 = 1$, $x_2 = 1$, $x_3 = 1$

$k = 3 + 2 = 5$

$x_1 = 1$, $x_2 = 1$, $x_3 = 0$

§ 10.3 Some NP-Complete Problems

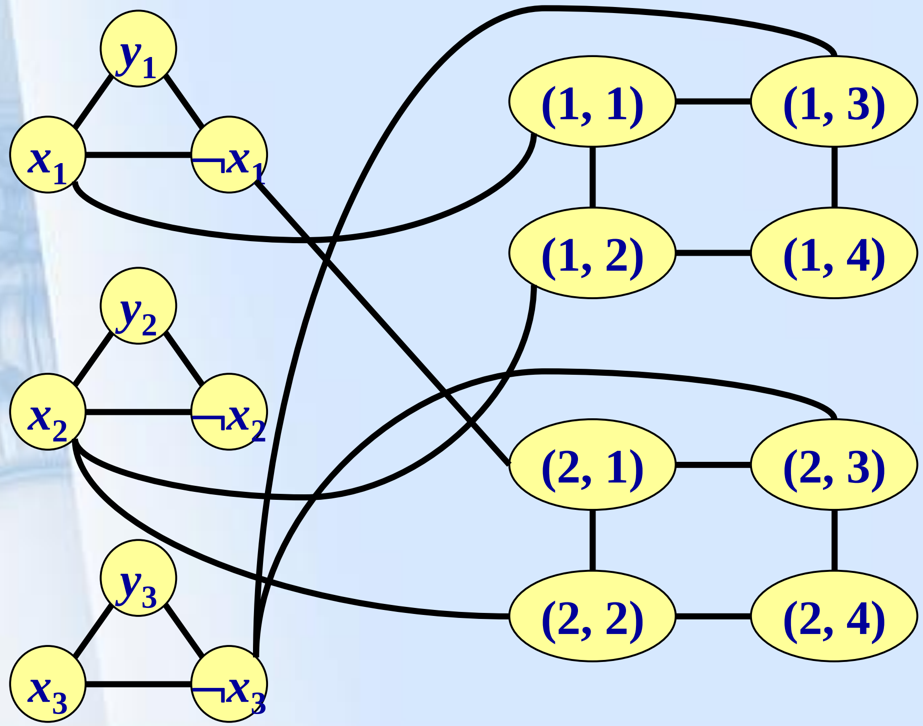
- **Theorem: 3 SAT ∝ Domination.**

  Proof. (2/5)

  Ex: $f = (x_1 + x_2 + (\neg x_3))((\neg x_1) + x_2 + (\neg x_3))$

$x_1 = 1,$
$x_2 = 1,$
$x_3 = 1$

G:



$k = 3 + 2$
$= 5$

$x_1 = 1,$
$x_2 = 1,$
$x_3 = 0$

§ 10.3 Some NP-Complete Problems

- **Theorem: 3 SAT ∝ Domination.**

  Proof. (3/5)

  Claim: f is satisfiable ⇔ G has a dominating set of size k.

  Proof.

  (⇒) If we can assign $x_1, \ldots, x_n$ s.t. f is true,

  ($\because$ f is true, $\therefore$ $\forall\ 1 \le i \le r, \exists\ k_i \in \{1, 2, 3\} \ni x_{ik_i}$ is true.)

  then let $D = \{x_j : x_j$ is assign true$\} \cup$

  $\{\neg x_j : x_j$ is assign false$\} \cup$

  $\{(i, 5 - k_i) : 1 \le i \le r\}$.

  Then it's easy to see that D is a dominating set of G

  and $|D| = n + r = k$.

§ 10.3 Some NP-Complete Problems

- **Theorem: 3 SAT ∝ Domination.**

  Proof. (4/5)

  claim: f is satisfiable ⇔ G has a dominating set of size k.

  Proof.

  (⇐) Suppose G has a dominating set D of size $n + r$.

  Then $|D \cap \{y_j, x_j, (\neg x_j)\}| = 1\ \forall\ 1 \le j \le n$ and

  $|D \cap \{(i, k)\}: 2 \le k \le 4| = 1\ \forall\ 1 \le i \le r$

  ($\because\ \exists\ y_j, \exists\ (i, 4)\ \forall\ 1 \le j \le n, 1 \le i \le r$)

  Assign $x_j = \begin{cases} \text{true, if } x_j \in D; \\ \text{false, if } \neg x_j \in D \text{ or } y_j \in D. \end{cases}$

  check: $x_{i1} + x_{i2} + x_{i3}$ is true, $\forall\ 1 \le i \le r$.

§ 10.3 Some NP-Complete Problems

- **Theorem: 3 SAT ∝ Domination.**

  Proof. (5/5)

  claim: f is satisfiable ⇔ G has a dominating set of size k.

  Proof.

  (⇐) check: $x_{i1} + x_{i2} + x_{i3}$ is true, $\forall\ 1 \le i \le r$.

  let $D \cap \{(i, k): 2 \le k \le 4\} = (i, k_i)$

  Note that (i, k_i) can't dominate $(i, 5 - k_i)$ with $1 \le 5 - k_i \le 3$.

  i.e. $(i, 5 - k_i)$ adjacent to some x_j or $(\neg x_j)$ in D

  $\Rightarrow x_{i5-k_i}$ is true for all $1 \le i \le r$, for some $1 \le 5 - k_i \le 3$

  $\Rightarrow x_{i1} + x_{i2} + x_{i3}$ is true, $\forall\ 1 \le i \le r$.

§ 10.3 Some NP-Complete Problems

- **<u>Corollary</u>:**

① **Clique Problem is *NP*-complete.**

② **Vertex-cover Problem is *NP*-complete.**

③ **Domination Problem is *NP*-complete.**

- **<u>Note</u>: Need to prove**

① **Clique Problem $\in$ *NP*.**

② **Vertex-cover Problem $\in$ *NP*.**

③ **Domination Problem $\in$ *NP*.**

§ 10.3 Some NP-Complete Problems

- **Thm: vertex-cover problem $\propto$ domination problem on bipartite graph.**

  Proof. (1/5)

  Given graph $G = (V, E)$ and $k \in N$ for vertex-cover problem.

  Consider bipartite graph $G' = (X, Y, E')$ as:

  $X = \{\alpha\} \cup V$

  $Y = \{\beta\} \cup E$

  $E' = \{\alpha\beta, \beta x: x \in V\} \cup \{xe: x \in V, e \in E, x \in e\}$

  Let $k' = k + 1$

  Claim: G has a vertex-cover C of size k, $k \leq |V|$

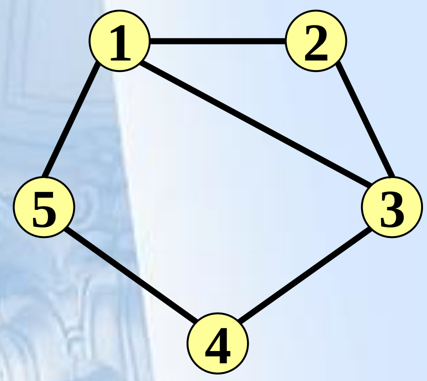
  $\Leftrightarrow$ **G' has a dominating set D of size $k' = k + 1$, $k' \leq |X|$.**

§ 10.3 Some NP-Complete Problems

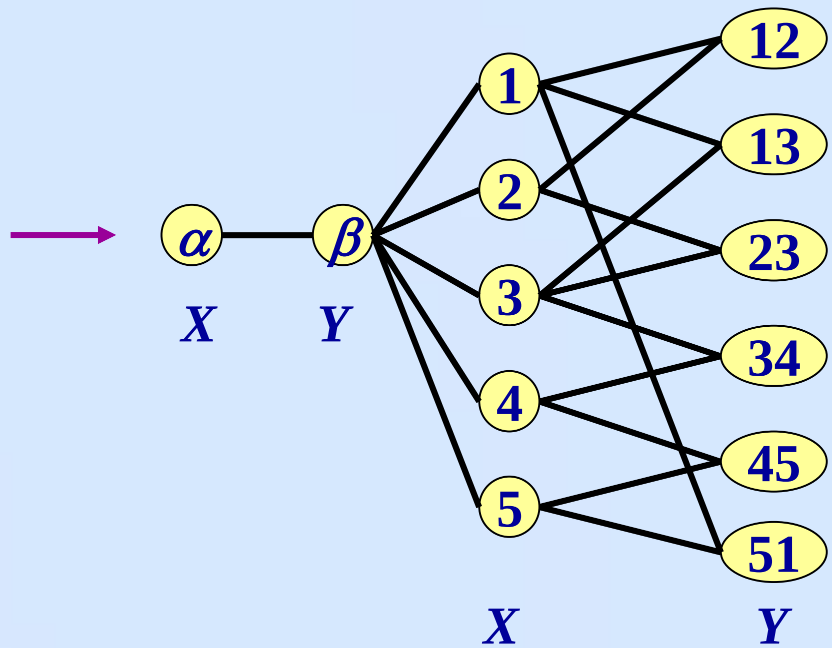
- **<u>Thm</u>: vertex-cover problem ∝ domination problem on bipartite graph.**

  Proof. (2/5)

  <u>Ex</u>:



k* = 3 *G



k'* = *k* + 1 *G'

= 4

§ 10.3 Some NP-Complete Problems

- **Thm: vertex-cover problem $\propto$ domination problem on bipartite graph.**

  Proof. (3/5)

  Claim: G has a vertex-cover C of size k, $k \leq |V|$

  $\Leftrightarrow$ G' has a dominating set D of size $k' = k + 1$, $k' \leq |X|$.

  Proof.

  ($\Rightarrow$) Given C, let $D = \{\beta\} \cup C$

  It's easy to see that D is a dominating set of G' with size k'.

  ($\Leftarrow$) Suppose $|D| = k' = k + 1$, D is a dominating set of G'.

  ① If $\beta \notin D$, then $\alpha \in D$.

  Let $D' = (D - \{\alpha\}) \cup \{\beta\}$ be also a dominating set of G' with size k'.

  $\therefore$ We may assume $\beta \in D$.

§ 10.3 Some NP-Complete Problems

- **Thm: vertex-cover problem $\propto$ domination problem on bipartite graph.**

  Proof. (4/5)

  Claim: G has a vertex-cover C of size k, $k \leq |V|$

  $\Leftrightarrow$ G' has a dominating set D of size $k' = k + 1$, $k' \leq |X|$.

  Proof.

  ($\Leftarrow$) Suppose $|D| = k' = k + 1$, D is a dominating set of G'.

  ② If $\exists\ e = xy \in D$, $x \in X$, $y \in X$,

  then let $D' = (D - \{e\}) \cup \{x\}$ is also a dominating set of

  G' with size k' (or $k' - 1$, if $x \in D$. If so, since $k' \leq |X|$,

  then let $D' = (D - \{e\}) \cup \{z\}$, for some $z \in V - D$.)

  $\therefore$ We may assume $D \cap E = \varnothing$.

§ 10.3 Some NP-Complete Problems

- <u>Thm</u>: vertex-cover problem $\propto$ domination problem on bipartite graph.

  Proof. (5/5)

  <u>Claim</u>: G has a vertex-cover C of size k, $k \leq |V|$

  $\Leftrightarrow$ G' has a dominating set D of size $k' = k + 1$, $k' \leq |X|$.

  Proof.

  ($\Leftarrow$) Suppose $|D| = k' = k + 1$, D is a dominating set of G'.

  $\therefore$ Set $C = D \cap V$.

  Then C is clearly a vertex cover of G with $|C| = k$.

- <u>Corollary</u>: Domination Problem for bipartite graph is *NP*-complete. (Need to prove Domination Problem for bipartite graph $\in NP$.)