Computer Science and Information Engineering
National Chi Nan University

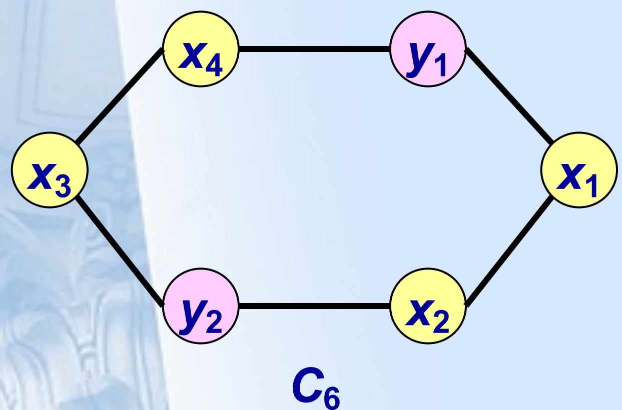
Chapter 8
Other Problems on Trees

§ 8.1 Domination on Tree

8.1 Domination on Tree

- **Def**: A **domination set** of a graph $G = (V, E)$ is a subset D of V s.t. $\forall\, x \in V \setminus D, \exists\, y \in D$ with $xy \in E$.

- **Ex**:



Let $D_1 = \{y_1, y_2\}$,
$D_2 = \{x_1, y_2, x_4\}$,
then D_1, D_2 are domination set of C_6.

- **Notation**: $\gamma(G) = \min_{\forall \text{ domination set } D} |D|$

E. J. Cockayne, S. E. Goodman and S. T. Hedetniemi, *"A linear algorithm for the domination number of a tree"*, Inform. Process. Lett. 4 (1975), 41-44.

8.1 Domination on Tree

- **Linear-time algorithm for trees:**

  Cockayne, Goodman, Hedetniemi, IPL, 1975

  想法: (1/2)

  Suppose D is an optimal domination set,

  $\forall$ x: a leaf adjacent to y:

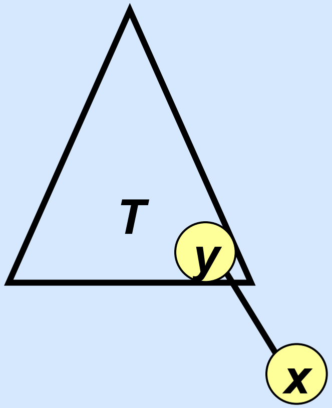
  ① $y \in D \Rightarrow x \notin D$.

  (o.w. $D \setminus \{x\}$ is also a dominating set with $|D \setminus \{x\}| < |D|$)

  ② $y \notin D \Rightarrow x \in D$

  $\Rightarrow$ let $D' = (D \setminus \{x\}) \cup \{y\}$

  be also a dominating set as $N[x] \subseteq N[y]$.



8.1 Domination on Tree

- **Linear-time algorithm for trees:**

  Cockayne, Goodmen, Hedeteriemi, IPL, 1975

  想法: (2/2)

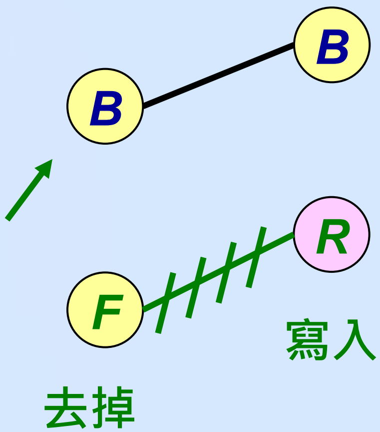
  So, always ∃ an optimal dominating set *D*

  s.t. $y \in D$ and $x \notin D$ for any leaf x adj. to y.

  ⇒ Step 1: 配合"bound" vertex $N[y]$ *B*

  Step 2: 產生"free" vertex x *F*

  Step 3: 再產生"required" vertex y *R*



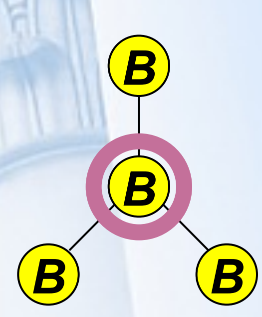
8.1 Domination on Tree

- **Def:** $G = (V, E)$ is a graph, $L : V \to \{B, R, F\}$, (i.e., each vertex x has a label $L(x) \in \{B, R, F\}$.)
  An ***L*-dominating (mixed dominating) set** of G is a subset $D \subseteq V(G)$
  s.t. ▯ $L(x) = R \Rightarrow x \in D$
  ▯ $L(x) = B \Rightarrow N[x] \cap D \neq \phi$

- **Notation:** $\gamma(G, L) = \min\{|D| : D \text{ is a } L\text{-dominating set of } G\}$.

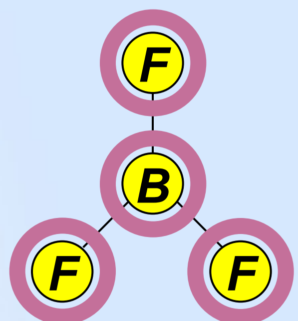
- **Ex:**

G_1



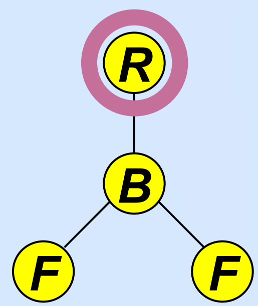
$\gamma(G_1, L_1) = 1$

G_2



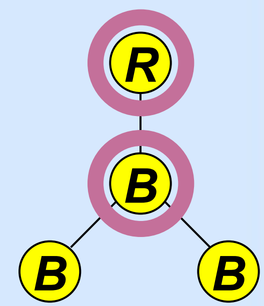
$\gamma(G_2, L_2) = 1$

G_3



$\gamma(G_3, L_3) = 1$

G_4

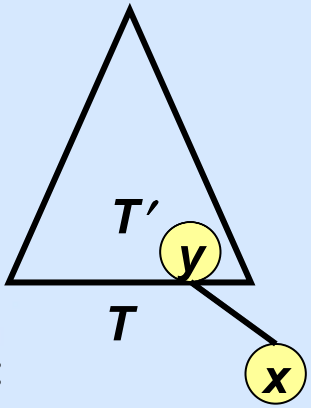


$\gamma(G_4, L_4) =$ 2

8.1 Domination on Tree

- **Note**: If $L(v) = B$, $\forall\ v \in V(G)$,
  an L-dominating set = dominating set and $\gamma(G) = \gamma(G, L)$.

- **Theorem**: T has a leaf x adjacent to y, $T' \leftarrow T - x$ then
  (1) $L(x) = F \Rightarrow \gamma(T, L) = \gamma(T', L)$
  (2) $L(x) = B \Rightarrow \gamma(T, L) = \gamma(T', L')$:
  $$L'(v) = \begin{cases} L(v), & \forall\ v \neq y \\ R, & v = y \end{cases}$$
  (3) $L(x) = L(y) = R \Rightarrow \gamma(T, L) = \gamma(T', L) + 1$
  (4) $L(x) = R$ but $L(y) \neq R \Rightarrow \gamma(T, L) = \gamma(T', L') + 1$:
  $$L'(v) = \begin{cases} L(v), & \forall\ v \neq y \\ F, & v = y \end{cases}$$



8.1 Domination on Tree

- **Thm: A tree T has a leaf x adjacent to vertex y, let $T' = T - x$ then**

  (1) $L(x) = F \Rightarrow \gamma(T, L) = \gamma(T', L)$

Proof. (1/2)

Choose a L-dominating set D of $T - x$ such that $|D| = \gamma(T', L)$,

then D is also a L-dominating set of T.

Hence $\gamma(T, L) \leq |D| = \gamma(T', L)$.

Choose a L-dominating set D of T such that $|D| = \gamma(T, L)$,

Case 1: $x \notin D \Rightarrow D$ is also a L-dominating set of $T - x$

$\Rightarrow \gamma(T, L) = |D| \geq \gamma(T', L)$.

Case 2: $x \in D$, consider $D' = (D - \{x\}) \cup \{y\}$

D' is a L-dominating set of $T - x$

8.1 Domination on Tree

- **Thm**: A tree T has a leaf x adjacent to vertex y, let $T' = T - x$ then
  (1) $L(x) = F \Rightarrow \gamma(T, L) = \gamma(T', L)$

Proof. (2/2)

Case 2: $x \in D$, consider $D' = (D - \{x\}) \cup \{y\}$

D' is a L-dominating set of $T - x$

($\forall v \in V(T - x)$: 1. $v \neq y$, $L(v) = B$: $\exists u \in D - \{x\} \subseteq D'$, s.t. $uv \in E(T)$.

2. $v \neq y$, $L(v) = R$: $\because v \in D \Rightarrow v \in D' = (D - \{x\}) \cup \{y\}$.

3. $v = y$: $\because y \in D'$, $\therefore N[y] \cap D' \neq \phi$.)

Also, $|D'| \leq |D|$, $\gamma(T, L) = |D| \geq |D'| \geq \gamma(T', L)$.

So, $\gamma(T, L) = \gamma(T', L)$.

8.1 Domination on Tree

- **Thm**: A tree T has a leaf x adjacent to vertex y, let $T' = T - x$ then

  (4) $L(x) = R$, but $L(y) \neq R \Rightarrow \gamma(T, L) = \gamma(T', L')+1$ where $L'(v) = \begin{cases} L(v), & \forall\ v \neq y \\ F, & v = y \end{cases}$

Proof.

Let D be a minimum L'-dominating set of $T - x$.

$D \cup \{x\}$ is a L-dominating set of T,

$\therefore \gamma(T, L) \leq |D| + 1 = \gamma(T', L') + 1.$

Let D is a minimum L-dominating set of T. By definition, $x \in D$.

$\because L'(y) = F \Rightarrow \therefore$ no matter $y \in D$ or $y \notin D$,

$D' = D - \{x\}$ is a L'-dominating set of $T - x$.

$\gamma(T', L') \leq |D'| = |D| - 1 = \gamma(T, L) - 1.$

Hence $\gamma(T, L) = \gamma(T', L') + 1.$

8.1 Domination on Tree

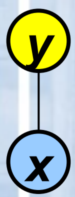
- **Algorithm:**

```
Given tree ordering [x1, x2, …, xn] of T
D ← φ;
for i = 1 to n do L(xi) = B.
for i = 1 to n–1 do
        choose j □ i such that xixj ∈ E;
        if (L(xi) = B) then L(xj) ← R;
        if (L(xi) = R) then
                if (L(xj) = B) then L(xj) ← F;
                D ← D ∪ {xi};
if (L(xn) ≠ F) then D ← D ∪ {xn};
```

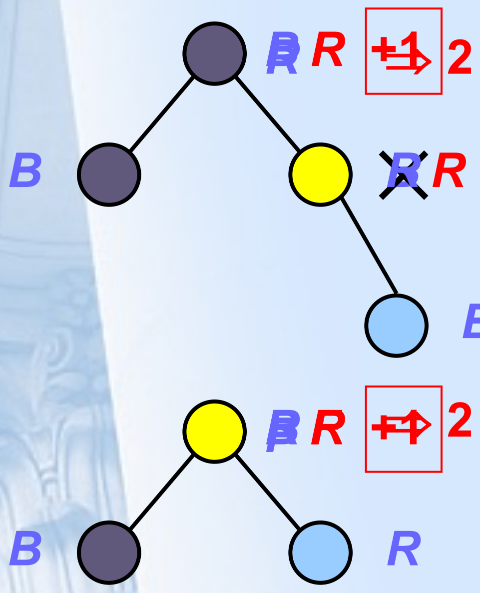
- **Time complexity = O(n).**

8.1 Domination on Tree

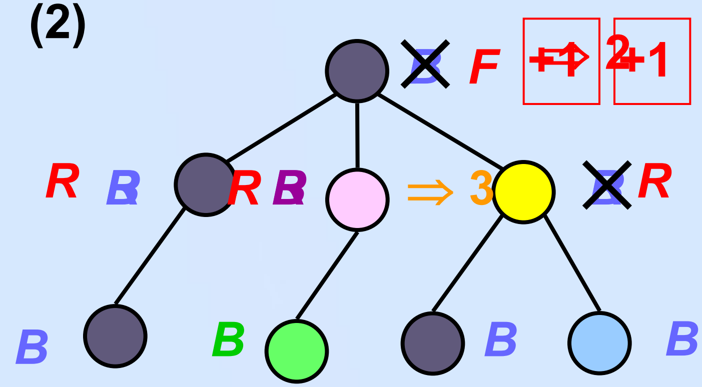
- Ex:



(1)



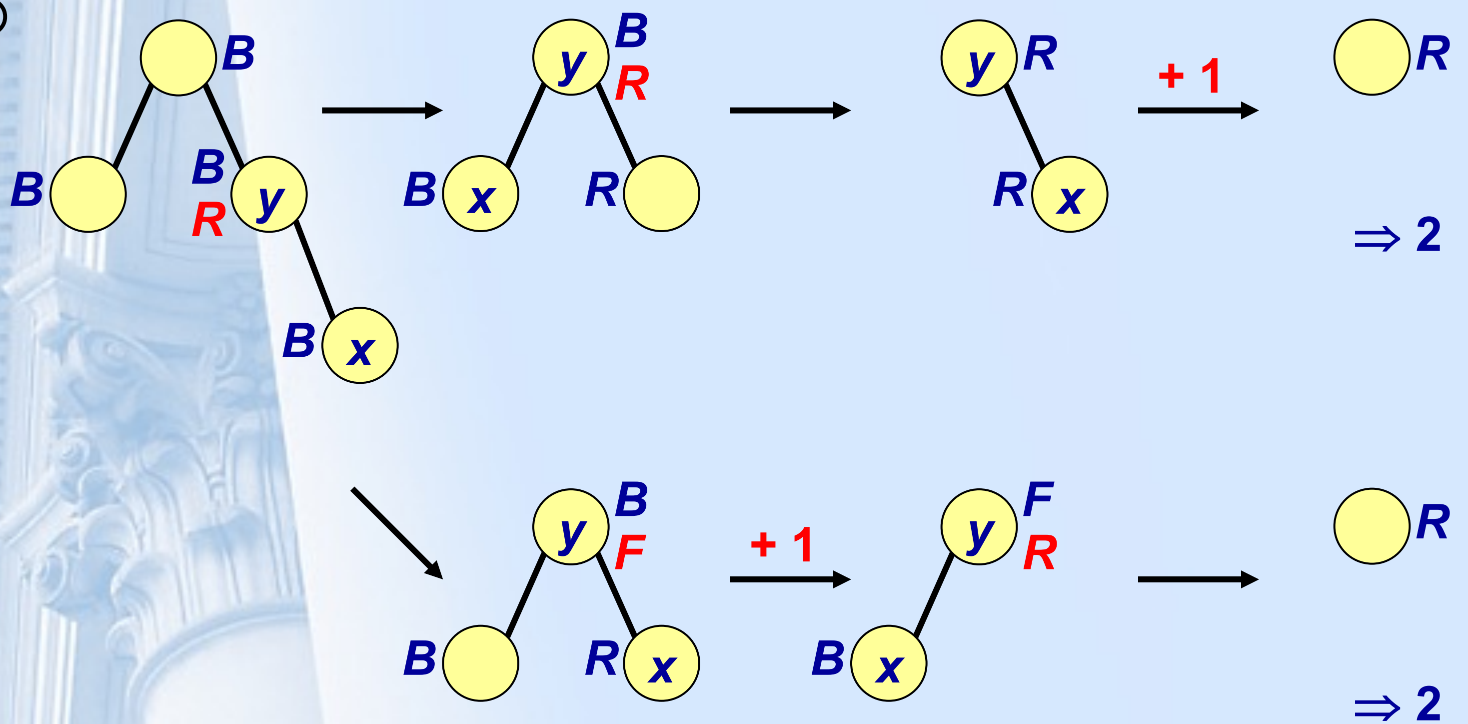
(2)



8.1 Domination on Tree

- **Ex:**

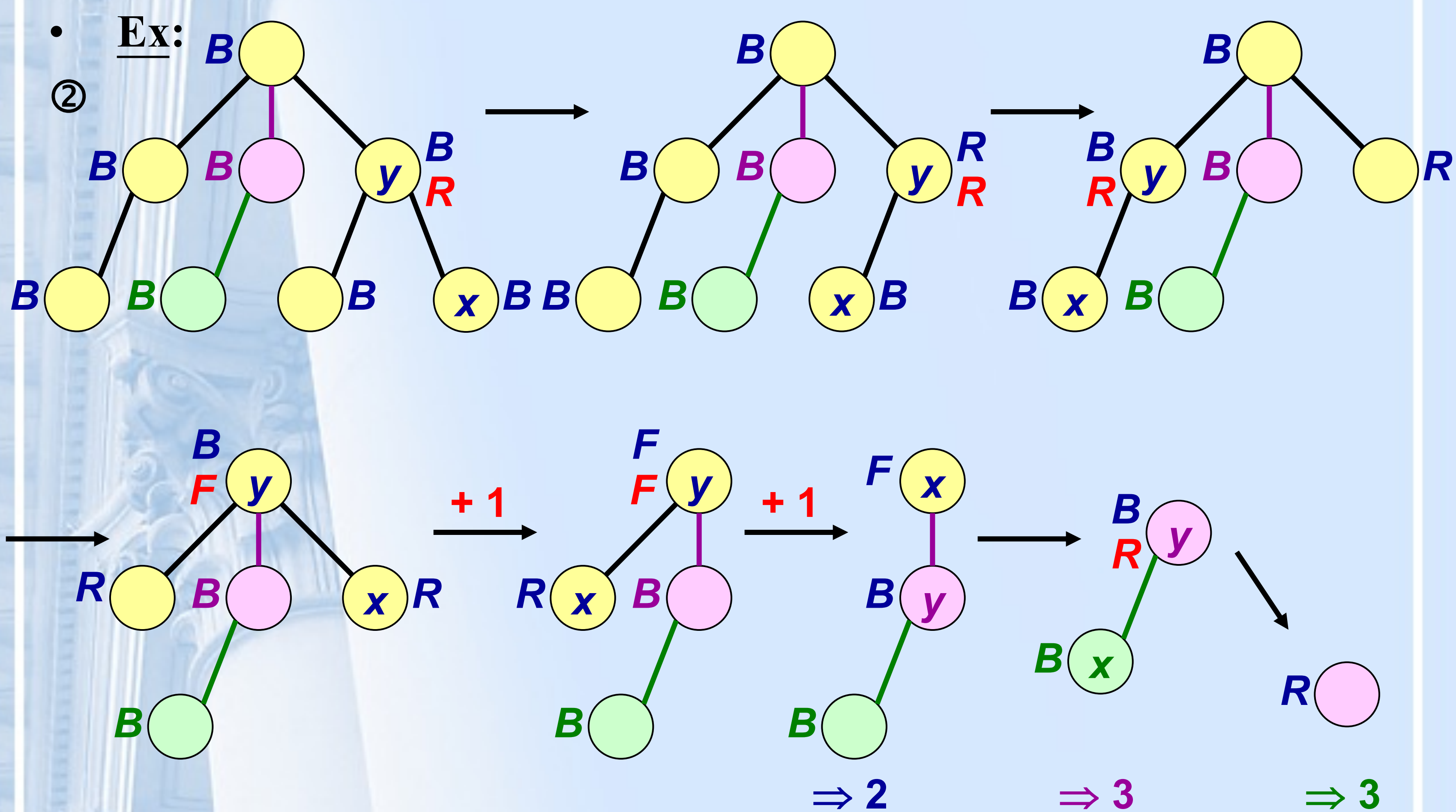
①



8.1 Domination on Tree

- **Ex:**

②



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Chapter 8
Other Problems on Trees

§ 8.2 Total Interval Number

Thomas M. Kratzke and Douglas B. West,
The Total Interval Number of a Graph II: Trees and Complexity
SIAM J. Dis. Math. 9 (1996) p.p. 339-348

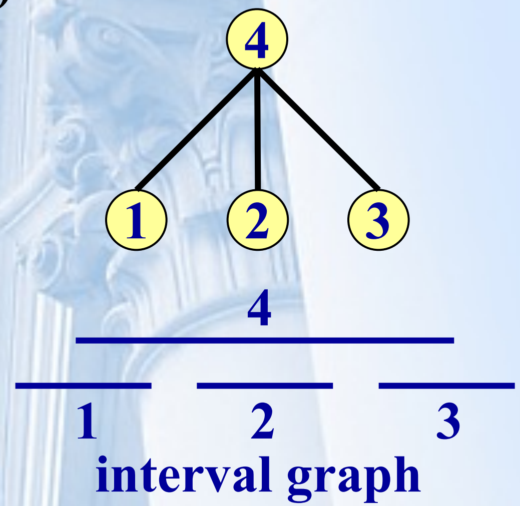
8.2 Total Interval Number

- **Def:**

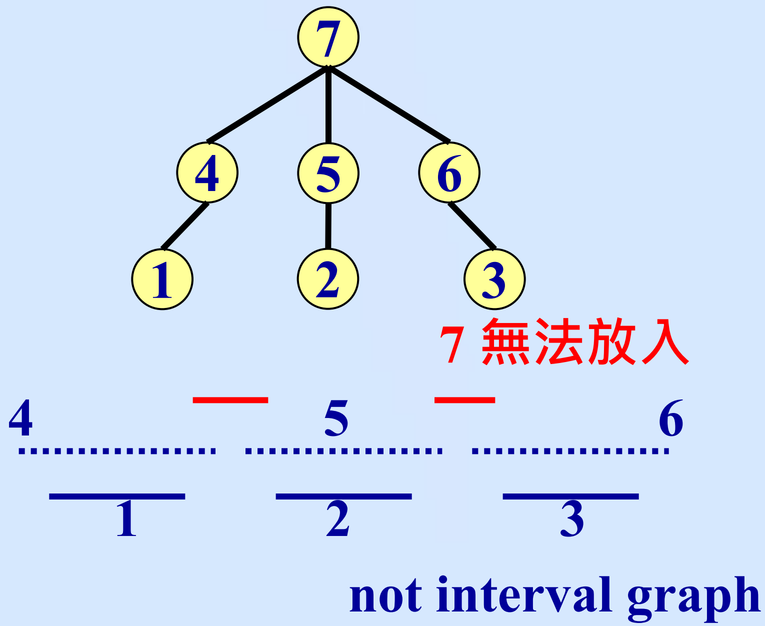
① **A graph $G=(V, E)$ is called an interval graph if $\exists$ intervals $\{I_x: x \in V\}$ s.t. $x \neq y \in V$, $xy \in E \Leftrightarrow I_x \cap I_y \neq \varnothing$.**

- **Ex:**

(a)



(b)



8.2 Total Interval Number

- **Def:**

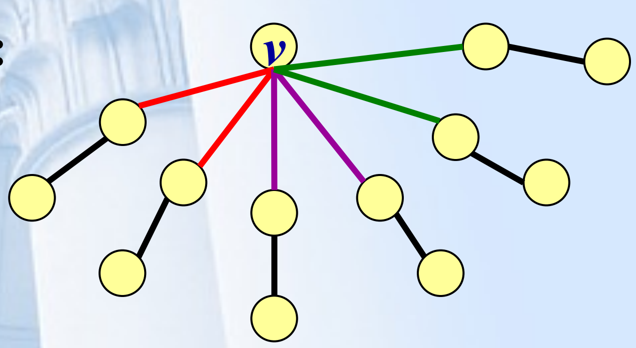
② A **multiple-interval representation** of $G = (V, E)$ is a function $f: V \to \{\cup\text{intervals}\}$ s.t. $x \neq y \in V$, $xy \in E \Leftrightarrow f(x) \cap f(y) \neq \emptyset$.

③ **total interval number** (or **TIN**) of G:

$I(G) = \min_f \sum_{x \in V}$ (# of intervals in $f(x)$) $= \min_f \sum_{x \in V} |f(x)|$.

- **Remark:** G is an interval graph $\Leftrightarrow I(G) = |V|$.

- **Ex:**



$I(G) = |V| + 2$
$= |V| + \lceil \deg(v) / 2 \rceil - 1.$

8.2 Total Interval Number

Note: In a tree G, x is a leaf.

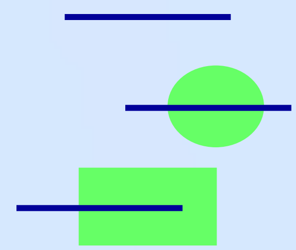
If f is an optimal solution (multiple-interval representation) of $G = (V, E)$ then $|f(x)| = 1$.

Sol.

$\because$ $\deg(x) = 1$.

- **Def: Given a graph $G = (V, E)$. Let L: $V \rightarrow$ {NC, FP, FB} s.t.**

  $\forall$ $x \in V$, $L(x) =$
  - **NC: (non-constrained)**
  - **FP: (free portion)**
  - **FB: (free boundary)**



8.2 Total Interval Number

- **Def:**

① An ***L*-multiple-interval representation** of *G* is a multiple- interval representation *f* with *L*: $V \to$ {NC, FP, FB} s.t.

(i) $L(x)$ = FP $\Rightarrow f(x)$ has an interval having a free portion.

(ii) $L(x)$ = FB $\Rightarrow f(x)$ has an interval having a free boundary.

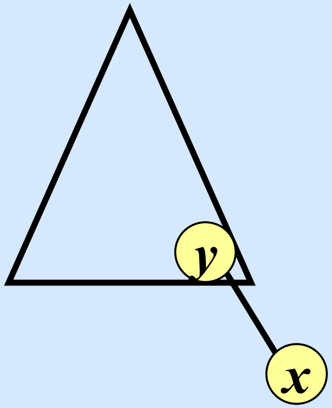
② $I(G, L) = \min_f \sum_{x \in V} |f(x)|$ (***L*-total interval number**)

Note: If $L(x)$ = NC for all vertex *x* of *G*,

(1) an *L*-multiple-interval representation = multiple-interval representation.

(2) $I(G) = I(G, L)$.

8.2 Total Interval Number



- **Theorem**: G has a leaf x adj. to y, $G' \leftarrow G - x$ and define L' on G' by L': $\begin{cases} L(z), \text{ if } z \neq y \\ (\spadesuit), \text{ if } z = y \end{cases}$

  Case 1: $L(x) = L(y) = \text{NC}$:

  $(\spadesuit) = \text{FP}$ and $I(G, L) = I(G', L') + 1$.

  Proof. (1/2)

  (≥) If f is an optimal solution for $I(G, L)$;

  $f(y)$ 有一段只 $\cap f(x)$.

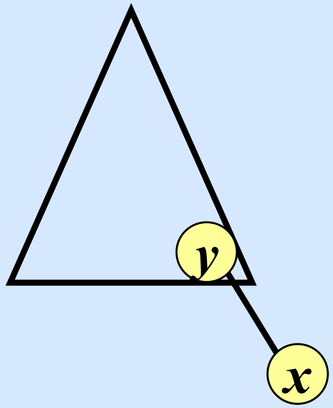
  Let $f'(v) = f(v)$, $\forall\, v \in V(G')$

  $\Rightarrow f'(y)$ has a FP and $I(G, L) = \sum_{v \in V} |f(v)|$

  $= \sum_{v \in V-x} |f(v)| + |f(x)|$

  $\geq I(G', L') + 1.$

8.2 Total Interval Number



- **Theorem**: G has a leaf x adj. to y, $G' \leftarrow G - x$ and define L' on G' by L': $\begin{cases} L(z), \text{ if } z \neq y \\ (\spadesuit), \text{ if } z = y \end{cases}$

  Case 1: $L(x) = L(y) = \text{NC}$:

  $(\spadesuit) = \text{FP}$ and $I(G, L) = I(G', L') + 1$.

  Proof. (2/2)

  (≤) If f' is an optimal solution for $I(G', L')$;

  $f'(y)$ 有一段FP: J_y.

  Let $f(v) = \begin{cases} f'(v), & \text{if } v \neq x \\ \text{an interval} \subseteq J_y, & \text{if } v = x \end{cases}$

  $\Rightarrow I(G, L) \leq I(G', L') + 1$.

8.2 Total Interval Number

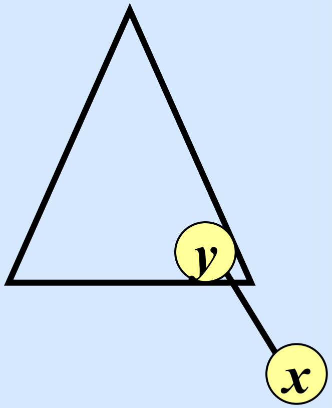
- **Theorem**: G has a leaf x adj. to y, $G' \leftarrow G - x$ and define L' on G' by L': $\begin{cases} L(z), \text{ if } z \neq y \\ (\spadesuit), \text{ if } z = y \end{cases}$

  Case 1: $L(x) = L(y) =$ NC
  $\Leftrightarrow (\spadesuit) =$ FP and $I(G, L) = I(G', L') + 1$.

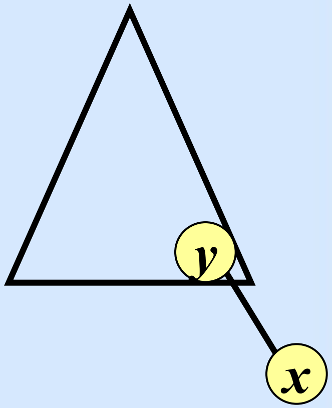
  Case 2: $L(x) =$ FP, $L(y) =$ NC
  $\Leftrightarrow (\spadesuit) =$ FB and $I(G, L) = I(G', L') + 1$.

  Case 3: $L(x) =$ FB, $L(y) =$ NC
  $\Leftrightarrow (\spadesuit) =$ FB and $I(G, L) = I(G', L') + 1$.

  Case 4: $L(x) =$ NC, $L(y) =$ FP
  $\Leftrightarrow (\spadesuit) =$ FP and $I(G, L) = I(G', L') + 1$.



8.2 Total Interval Number



- **Theorem**: G has a leaf x adj. to y, $G' \leftarrow G - x$ and define L' on G' by L': $\begin{cases} L(z), \text{ if } z \neq y \\ (\spadesuit), \text{ if } z = y \end{cases}$

  Case 5: $L(x)$ = FP, $L(y)$ = FP

  $\Leftrightarrow$ $(\spadesuit)$ = FB and $I(G, L) = I(G', L') + 1$.

  Case 6: $L(x)$ = FB, $L(y)$ = FP

  $\Leftrightarrow$ $(\spadesuit)$ = FB and $I(G, L) = I(G', L') + 1$.

  Case 7: $L(x)$ = NC, $L(y)$ = FB

  $\Leftrightarrow$ $(\spadesuit)$ = FB and $I(G, L) = I(G', L') + 1$.

  Case 8: $L(x)$ = FP, $L(y)$ = FB

  $\Leftrightarrow$ $(\spadesuit)$ = NC and $I(G, L) = I(G', L') + 2$.

  Case 9: $L(x)$ = FB, $L(y)$ = FB

  $\Leftrightarrow$ $(\spadesuit)$ = NC and $I(G, L) = I(G', L') + 2$.

8.2 Total Interval Number

- **Algorithm:**

Given tree ordering $[x_1, x_2, \ldots, x_n]$ of T
$I(G, L) \leftarrow n$;
for $i = 1$ to n do $L(x_i) = \text{NC}$;
for $i = 1$ to $n-1$ do
    choose j □ i such that $x_i x_j \in E$;
    if ($L(x_j) = \text{FB}$) then
        if ($L(x_i) \neq \text{NC}$) then
            $L(x_j) \leftarrow \text{NC}$;
            $I(G, L) \leftarrow I(G, L) + 1$;
    else if ($L(x_i) = \text{NC}$) then $L(x_j) \leftarrow \text{FP}$;
    else $L(x_j) \leftarrow \text{FB}$;

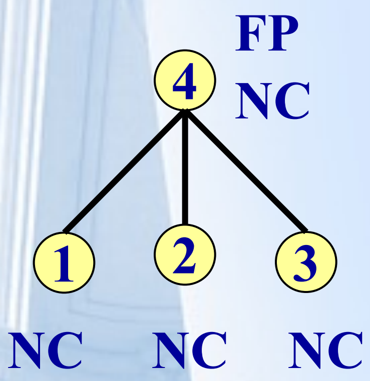
Time complexity = $O(n)$.

$L(x)$	$L(y)$	$L'(y)$	I
NC	NC	FP	
FP	NC	FB	
FB	NC	FB	
NC	FP	FP	
FP	FP	FB	
FB	FP	FB	
NC	FB	FB	
FP	FB	NC	1
FB	FB	NC	1

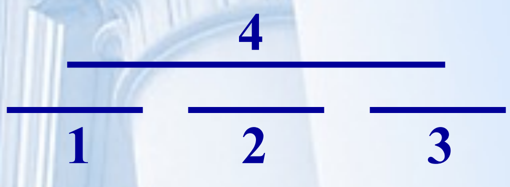
8.2 Total Interval Number

Ex:

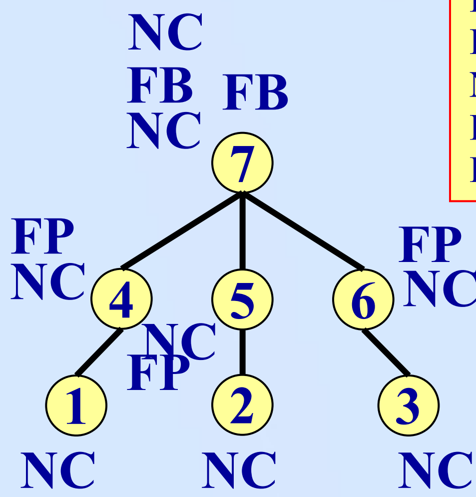
(a)



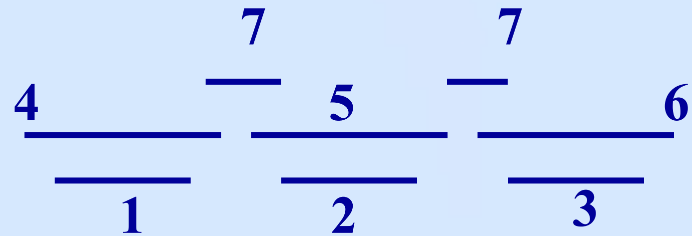
$I(G, L) = 4$



(b)



$I(G, L) =$ ~~7~~ 8

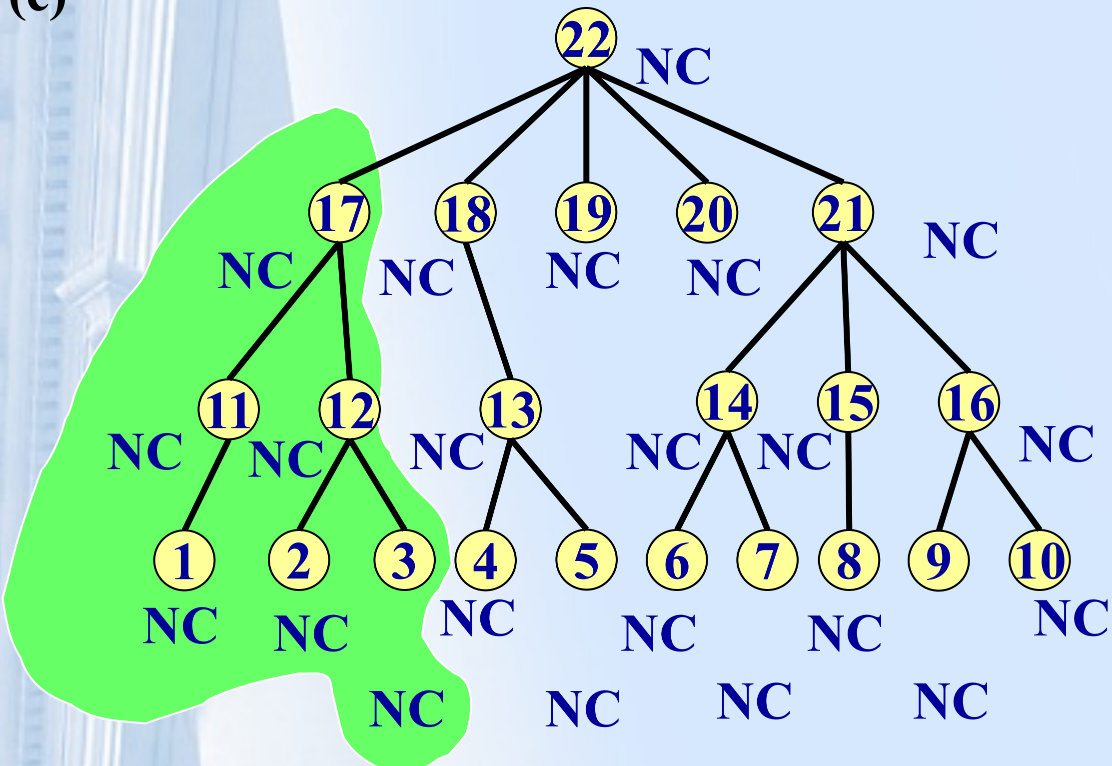


8.2 Total Interval Number

$L(x)$	$L(y)$	$L'(y)$	I
NC	NC	FP	
FP	NC	FB	
FB	NC	FB	
NC	FP	FP	
FP	FP	FB	
FB	FP	FB	
NC	FB	FB	
FP	FB	NC	1
FB	FB	NC	1

Ex:

(c)



$I(G, L) = ?$

Exercise 7 (11/14): 只有綠色區域的 Tree $T + 22$時，為何 $I(T, L) = 7$？該如何修改演算法？