Computer Science and Information Engineering
National Chi Nan University

# Chapter 7
# Coloring Problems

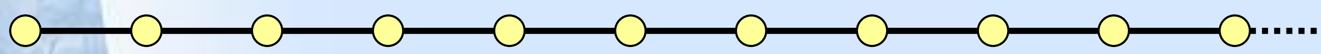
## § 7.4 Vertex Ranking on Trees

# 7.4 Vertex Ranking on Trees

- **Def:**

① $f$ is a **vertex ranking** of $G$ if $f: V(G) \to \{1, 2, \ldots\}$ s.t.
$\forall\ x \neq y,\ f(x) = f(y), \forall\ x\text{-}y$ path $P$, $\exists\ z \in V(P)$ with $f(z) > f(x)$.

② Notation: **value($f$)** $= \max_{x \in V(G)} f(x)$,
**$r(G)$** $= \min\{\text{value}(f): f$ is a vertex ranking of $G\}$.

- **Ex:** 1. 

[(1 2 1) 3 (1 2 1)] 4 [(1 2 1)…]

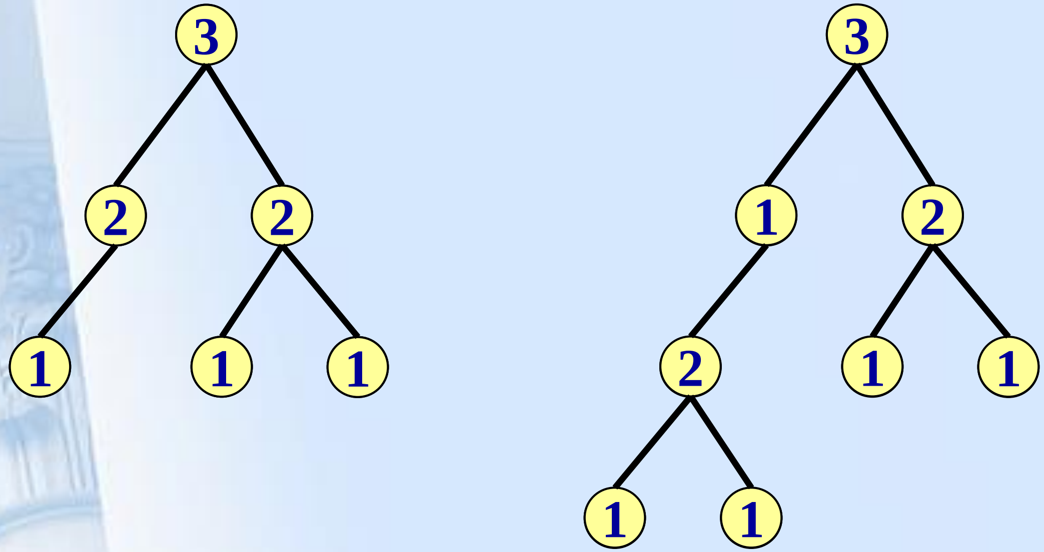
2. $r(K_n) = n$.

- **Exercise 6 (11/7):**

Find $r(P_n)$ for all path $P_n$ (with $n$ vertices).

# 7.4 Vertex Ranking on Trees

- **<u>Note</u>:** $r(G) \leq |V(G)|$
- **<u>Ex</u>:**



# 7.4 Vertex Ranking on Trees

- **Note: In a tree , let $f^*$ be a optimal vertex ranking, then $\forall$ leaf $x$, W.L.O.G., we can say $f^*(x) = 1$.**

  **Proof. (1/2)**

  **Case 1: $f^*(x) = 1$.**

  **Case 2: $f^*(x) \geq 2$: Let $y \in V(G)$ s.t. $xy \in E(G)$.**

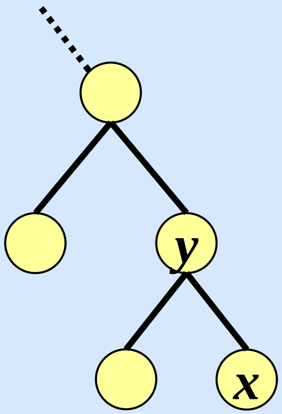
  **Case 2.1: $f^*(y) \geq 2$**

  **$\Rightarrow$ change $f^*(x)$ to 1.**

  **i.e. define $f'$ is a vertex ranking**

  $$\text{s.t. } f'(z) = \begin{cases} f^*(z), & \text{if } z \neq x \\ 1, & \text{if } z = x \end{cases}$$

  **then value($f'$) $\leq$ value($f^*$).**



# 7.4 Vertex Ranking on Trees

- **Note: In a tree , let $f^*$ is a optimal vertex ranking, then $\forall$ leaf $x$, W.L.O.G., we can say $f^*(x) = 1$.**

  **Proof. (2/2)**

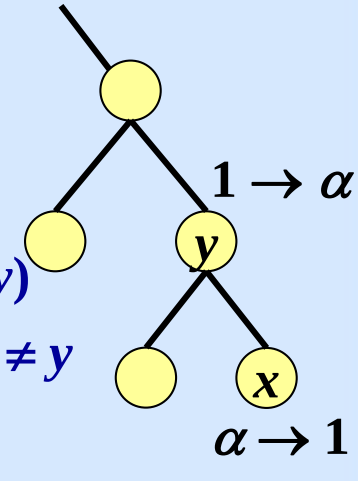
  **Case 2: $f^*(x) \geq 2$: Let $y \in V(G)$ s.t. $xy \in E(G)$**

  **Case 2.2: $f^*(y) = 1$**

  **$\Rightarrow$ interchange value of $f^*(x)$ and $f^*(y)$**

  **i.e. define $f''$: $f''(z) = \begin{cases} f^*(z), \text{ if } z \neq x, z \neq y \\ f^*(x), \text{ if } z = y \\ f^*(y), \text{ if } z = x \end{cases}$**

  **then $f''$ is still a vertex ranking (need be proved) and value($f''$) = value($f^*$).**



# 7.4 Vertex Ranking on Trees

- **New version: For any tree $T$. Given $l: V(G) \to N = \{1, 2, \ldots\}$, $L: V(G) \to 2^N$, and $\forall j \in L(x), l(x) < j, \forall x \in V(G)$.**

- **Def:**

① **An (*L*, *l*)-ranking of $G$ is a function $f$ from $V(G)$ to $N$**

**s.t. (i) $f(x) \geq l(x), \forall x \in V(G)$**

**(ii) $f(x) \notin L(x), \forall x \in V(G)$**

**(iii) $\forall x \neq y, \forall$ $x$-$y$ path $P$:**

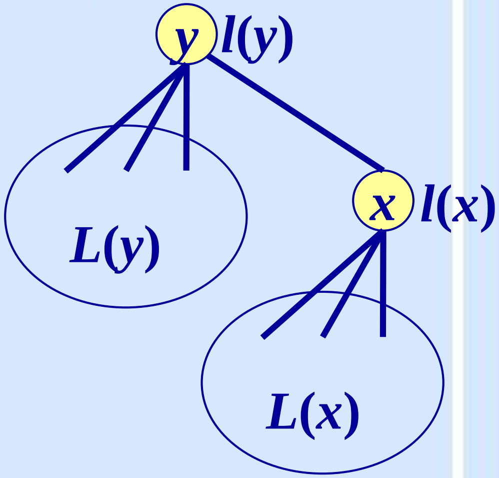
**if $j \in (L(x) \cup \{f(x)\}) \cap (L(y) \cup \{f(y)\})$**

**then $f(z) > j$ for some $z \in P$.**

② **value($L$; $l$; $f$) = $\max_{x \in V(G)}\{\max\{L(x) \cup \{f(x)\}\}\}$.**

③ **$r(G; L; l) = \min\{\text{value}(L; l; f): f \text{ is a } (L, l)\text{-ranking of } G\}$.**

# 7.4 Vertex Ranking on Trees

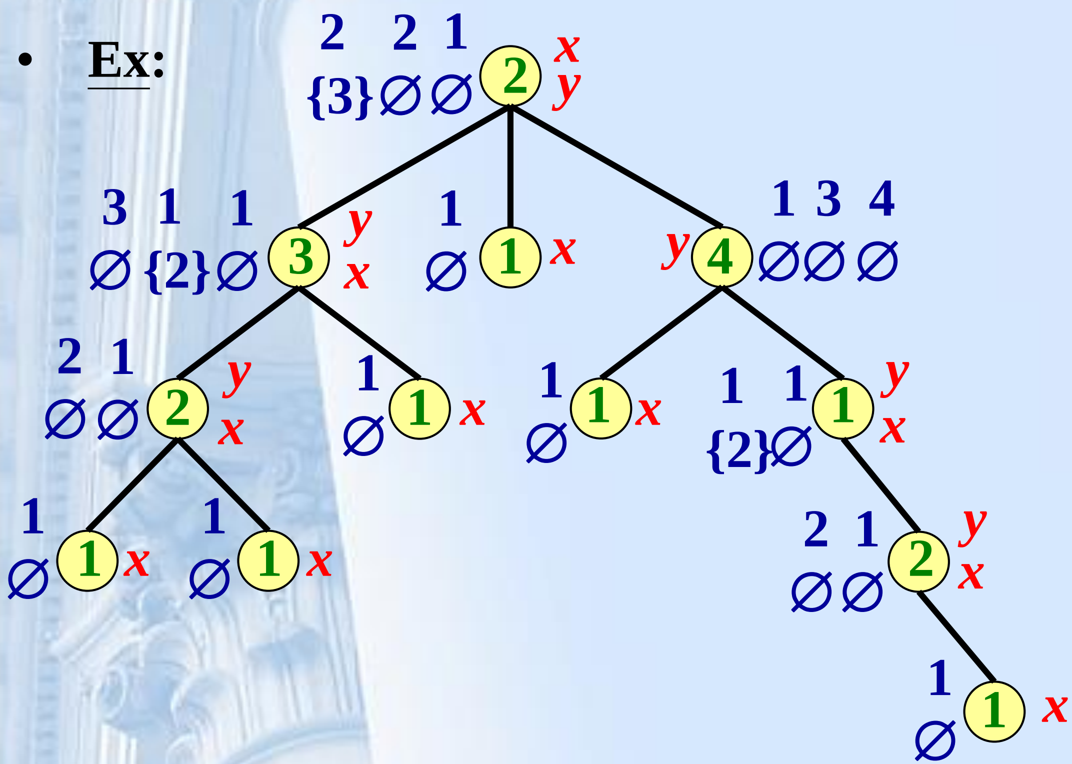



- **Remark**: $\forall x \in V, L(x) = \varnothing, l(x) = 1$:
  $(L, l)$-ranking = vertex ranking;
  value$(L, l, f)$ = value$(f)$; $r(G, L, l) = r(G)$.

- **Theorem**: $G$ has a leaf $x$ adjacent to $y$.
  Let $G' = G - x$,
  $\alpha = \max\{a: a \in ((L(y) \cup \{l(y)\}) \cap (L(x) \cup \{l(x)\})) \cup \{0\}\}$,
  $\beta = \min\{k \in N: k > \alpha, k \notin L(x) \cup L(y) \cup \{l(x)\}\}$,
  $l'(v) = \begin{cases} l(v), \text{ if } v \in V(G') - \{y\}, \\ \max\{\beta, l(y)\}, \text{ if } v = y, \end{cases}$
  $L'(v) = \begin{cases} L(v), \text{ if } v \in V(G') - \{y\}, \\ \{r: r \in L(x) \cup L(y) \cup \{l(x)\} \text{ and } r > l'(y)\}, \text{ if } v = y. \end{cases}$
  Then $r(G; L; l) = r(G'; L'; l')$.

# 7.4 Vertex Ranking on Trees

- Ex:




| $\alpha$ | $\beta$ |
|---|---|
| 1 | 2 |
| 0 | 1 |
| 1 | 3 |
| 0 | 2 |
| 1 | 2 |
| 0 | 2 |
| 0 | 1 |
| 1 | 3 |
| 1 | 2 |
| 0 | 1 |
| 3 | 4 |

$\alpha = \max\{a: a \in ((L(y) \cup \{l(y)\}) \cap (L(x) \cup \{l(x)\})) \cup \{0\}\}$,

$\beta = \min\{k \in N: k > \alpha, k \notin L(x) \cup L(y) \cup \{l(x)\}\}$,

$l'(y) = \max\{\beta, l(y)\}$,

$L'(y) = \{r: r \in L(x) \cup L(y) \cup \{l(x)\}$ and $r > l'(y)\}$.

# 7.4 Vertex Ranking on Trees

- **<u>Another version</u>:**

**Given $G = (V, E)$,**

**define $G^* = (V^*, E^*)$ where $V^* = V \cup \{x^* : x \in V\}$,**

$$E^* = E \cup \{xx^* : x \in V\}.$$

**Given any $(L, l)$-ranking $f$ of $G$,**

**define $L^* : V \to 2^N$ with $L^*(x) = \{l(x)\}, \forall\, x \in V$,**

$$L^*(x^*) = L(x), \forall\, x \in V.$$

**Then $f$ is a $(L, l)$-ranking of $G \Leftrightarrow f: V \to N$ s.t.**

**① $f(x) \geq l(x), \forall\, x \in V(G)$**

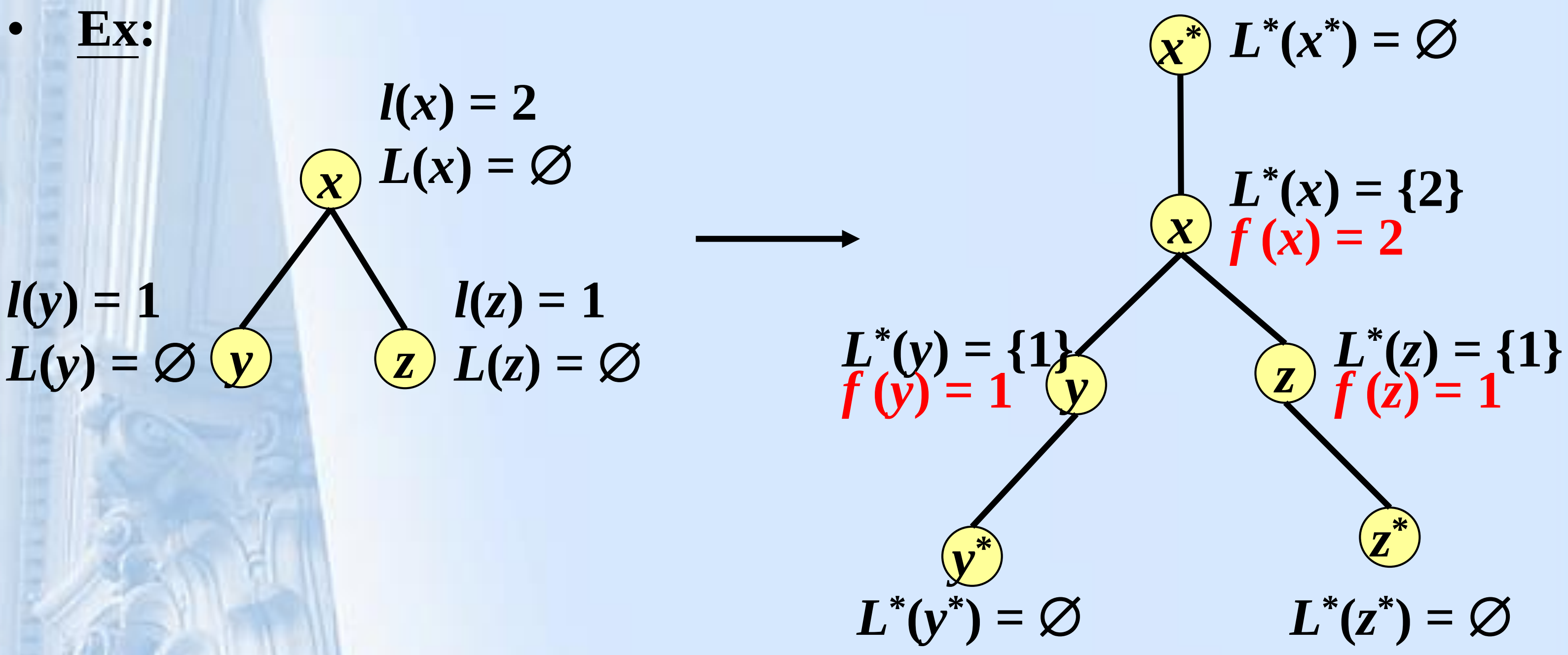
**②③ $\forall\, u \neq v$ in $V^*$, $\forall\, u$-$v$ path $P$,**

**if $j \in L^*(u) \cap L^*(v)$**

**then $\exists\, z \in P \setminus \{u, v\}$ s. t. $f(z) > j$.**

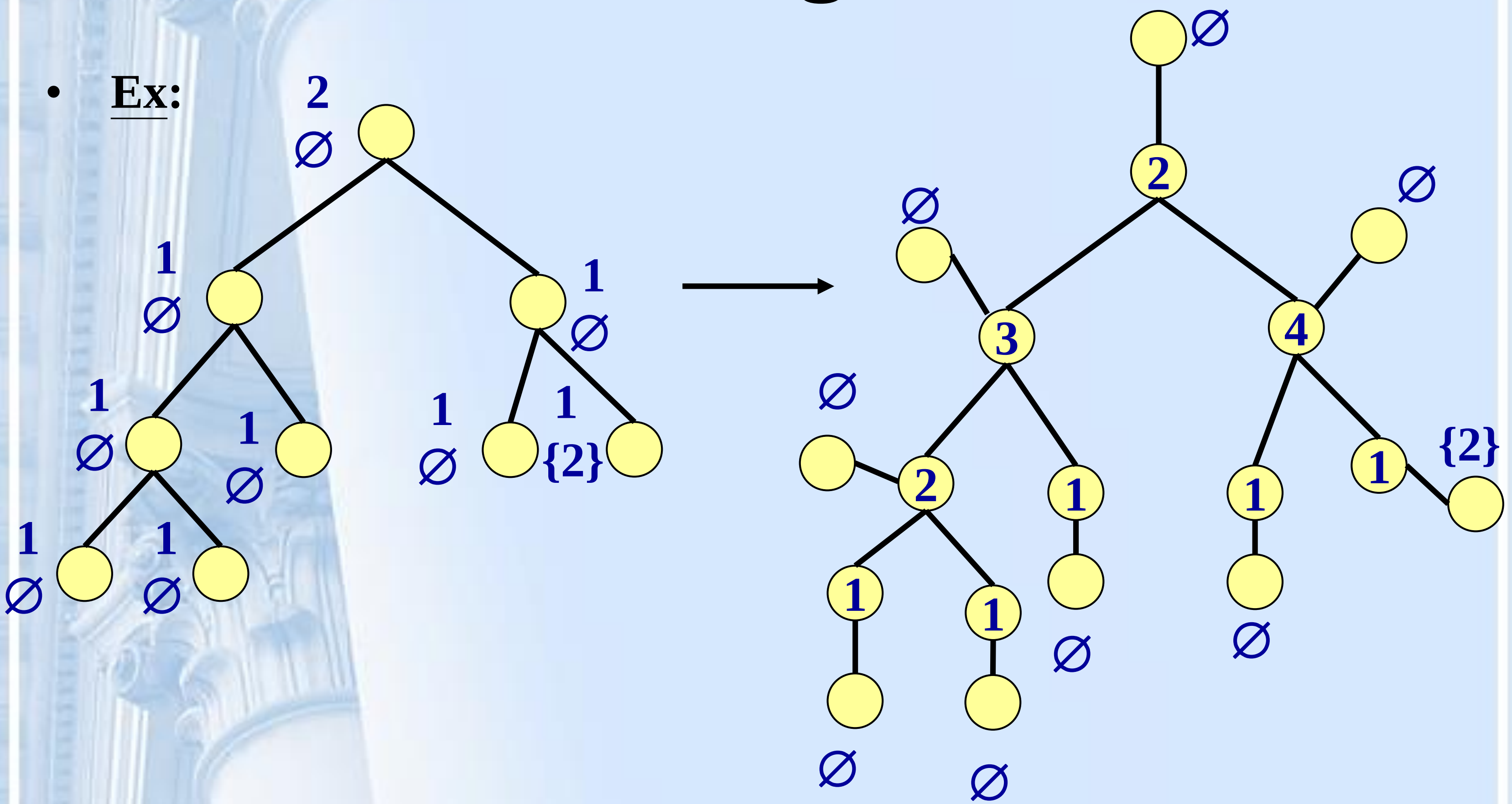
# 7.4 Vertex Ranking on Trees

- **Ex:**



# 7.4 Vertex Ranking on Trees

- **Ex:**



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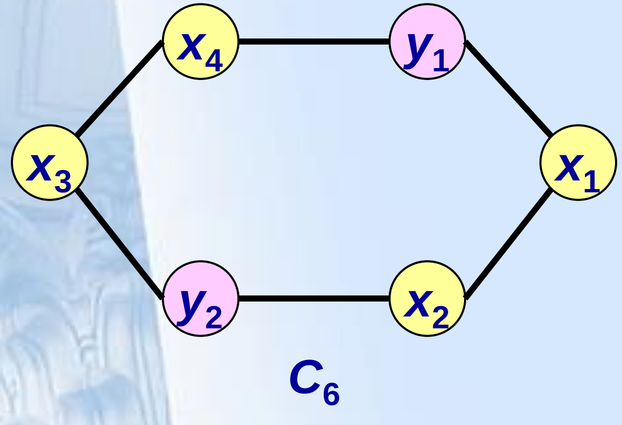
# Chapter 8
# Other Problems on Trees

## § 8.1 Domination on Tree

# 8.1 Domination on Tree

- **Def:** A **domination set** of a graph $G = (V, E)$ is a subset $D$ of $V$ s.t. $\forall\, x \in V \setminus D, \exists\, y \in D$ with $xy \in E$.

- **Ex:**




Let $D_1 = \{y_1, y_2\}$,
$D_2 = \{x_1, y_2, x_4\}$,
then $D_1$, $D_2$ are domination set of $C_6$.

- **Notation:** $\gamma(G) = \min_{\forall \text{ domination set } D} |D|$

E. J. Cockayne, S. E. Goodman and S. T. Hedetniemi, *"A linear algorithm for the domination number of a tree"*, Inform. Process. Lett. 4 (1975), 41-44.

# 8.1 Domination on Tree

- **Linear-time algorithm for trees:**

  **Cockayne, Goodman, Hedetniemi, IPL, 1975**

  **想法: (1/2)**

  **Suppose $D$ is an optimal domination set,**

  **$\forall$ $x$: a leaf adjacent to $y$:**

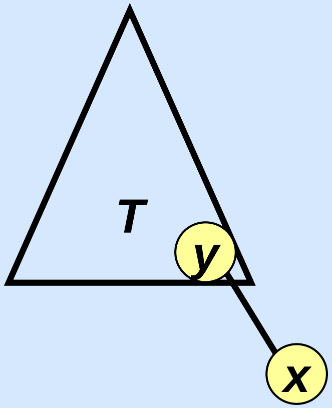
  **① $y \in D \Rightarrow x \notin D$.**

  **(o.w. $D \setminus \{x\}$ is also a dominating set with $|D \setminus \{x\}| < |D|$)**

  **② $y \notin D \Rightarrow x \in D$**

  **$\Rightarrow$ let $D' = (D \setminus \{x\}) \cup \{y\}$**

  **be also a dominating set as $N[x] \subseteq N[y]$.**



# 8.1 Domination on Tree

- **<u>Linear-time algorithm for trees</u>:**

  **Cockayne, Goodmen, Hedeteriemi, IPL, 1975**

  **想法: (2/2)**

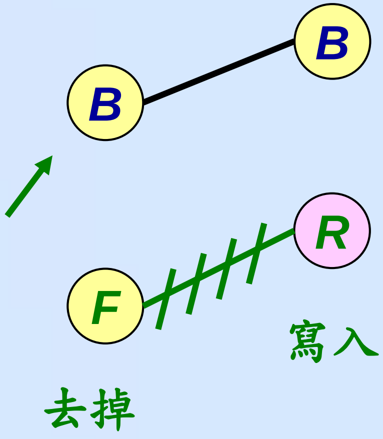
  **So, always ∃ an optimal dominating set *D***

  **s.t. $y \in D$ and $x \notin D$ for any leaf $x$ adj. to $y$.**

  **⇒ <u>Step 1</u>: 配合 "bound" vertex $N[y]$** ***B***

  **<u>Step 2</u>: 產生 "free" vertex $x$** ***F***

  **<u>Step 3</u>: 再產生 "required" vertex $y$** ***R***

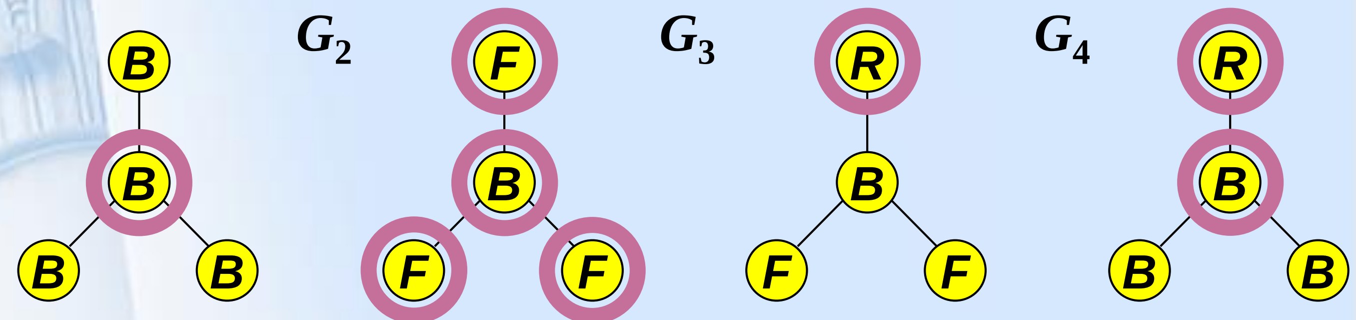


# 8.1 Domination on Tree

- **Def:** $G = (V, E)$ is a graph, $L : V \to \{B, R, F\}$, (i.e., each vertex $x$ has a label $L(x) \in \{B, R, F\}$.)
  An ***L*-dominating** (**mixed dominating**) **set** of $G$ is a subset $D \subseteq V(G)$
  s.t. ① $L(x) = R \Rightarrow x \in D$
  ② $L(x) = B \Rightarrow N[x] \cap D \neq \phi$

- **Notation:** $\gamma(G, L) = \min\{|D| : D \text{ is a } L\text{-dominating set of } G\}$.

- **Ex:** $G_1$ $G_2$ $G_3$ $G_4$




$\gamma(G_1, L_1) = 1$ $\gamma(G_2, L_2) = 1$ $\gamma(G_3, L_3) = 1$ $\gamma(G_4, L_4) = 2$

# 8.1 Domination on Tree

- **Note: If $L(v) = B$, $\forall\, v \in V(G)$,**

  **an $L$-dominating set = dominating set and $\gamma(G) = \gamma(G, L)$.**

- **Theorem: $T$ has a leaf $x$ adjacent to $y$, $T' \leftarrow T - x$ then**

  **(1) $L(x) = F \Rightarrow \gamma(T, L) = \gamma(T', L)$**

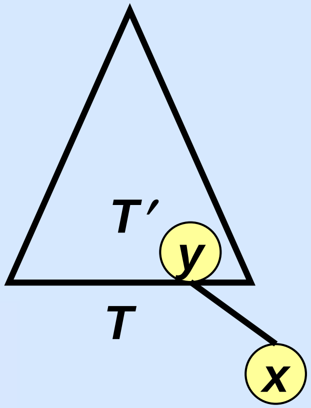
  **(2) $L(x) = B \Rightarrow \gamma(T, L) = \gamma(T', L')$:**

  $$L'(v) = \begin{cases} L(v), & \forall\, v \neq y \\ R, & v = y \end{cases}$$

  **(3) $L(x) = L(y) = R \Rightarrow \gamma(T, L) = \gamma(T', L) + 1$**

  **(4) $L(x) = R$ but $L(y) \neq R \Rightarrow \gamma(T, L) = \gamma(T', L') + 1$:**

  $$L'(v) = \begin{cases} L(v), & \forall\, v \neq y \\ F, & v = y \end{cases}$$



# 8.1 Domination on Tree

- **<u>Thm</u>: A tree $T$ has a leaf $x$ adjacent to vertex $y$, let $T' = T - x$ then**

  **(1) $L(x) = F \Rightarrow \gamma(T, L) = \gamma(T', L)$**

**Proof. (1/2)**

**Choose a $L$-dominating set $D$ of $T - x$ such that $|D| = \gamma(T', L)$,**

**then $D$ is also a $L$-dominating set of $T$.**

**Hence $\gamma(T, L) \le |D| = \gamma(T', L)$.**

**Choose a $L$-dominating set $D$ of $T$ such that $|D| = \gamma(T, L)$,**

**<u>Case 1</u>: $x \notin D \Rightarrow D$ is also a $L$-dominating set of $T - x$**

**$\Rightarrow \gamma(T, L) = |D| \ge \gamma(T', L)$.**

**<u>Case 2</u>: $x \in D$, consider $D' = (D - \{x\}) \cup \{y\}$**

**$D'$ is a $L$-dominating set of $T - x$**

# 8.1 Domination on Tree

- **Thm: A tree $T$ has a leaf $x$ adjacent to vertex $y$, let $T' = T - x$ then**

  **(1) $L(x) = F \Rightarrow \gamma(T, L) = \gamma(T', L)$**

**Proof. (2/2)**

**Case 2: $x \in D$, consider $D' = (D - \{x\}) \cup \{y\}$**

**$D'$ is a $L$-dominating set of $T - x$**

**($\forall v \in V(T - x)$: 1. $v \neq y$, $L(v) = B$ : $\exists u \in D - \{x\} \subseteq D'$, s.t. $uv \in E(T)$.**

**2. $v \neq y$, $L(v) = R$ : $\because v \in D \Rightarrow v \in D' = (D - \{x\}) \cup \{y\}$.**

**3. $v = y$ : $\because y \in D'$, $\therefore N[y] \cap D' \neq \phi$. )**

**Also, $|D'| \leq |D|$, $\gamma(T, L) = |D| \geq |D'| \geq \gamma(T', L)$.**

**So, $\gamma(T, L) = \gamma(T', L)$.**

# 8.1 Domination on Tree

- **Thm:** A tree $T$ has a leaf $x$ adjacent to vertex $y$, let $T' = T - x$ then
  (4) $L(x) = R$, but $L(y) \neq R \Rightarrow \gamma(T, L) = \gamma(T', L')+1$ where $L'(v) = \begin{cases} L(v), & \forall\ v \neq y \\ F, & v = y \end{cases}$

**Proof.**

Let $D$ be a minimum $L'$-dominating set of $T - x$.

$D \cup \{x\}$ is a $L$-dominating set of $T$,

$\therefore \gamma(T, L) \leq |D| + 1 = \gamma(T', L') + 1.$

Let $D$ is a minimum $L$-dominating set of $T$. By definition, $x \in D$.

$\because L'(y) = F \Rightarrow \therefore$ no matter $y \in D$ or $y \notin D$,

$D' = D - \{x\}$ is a $L'$-dominating set of $T - x$.

$\gamma(T', L') \leq |D'| = |D| - 1 = \gamma(T, L) - 1.$

Hence $\gamma(T, L) = \gamma(T', L') + 1.$

# 8.1 Domination on Tree

- **Algorithm:**

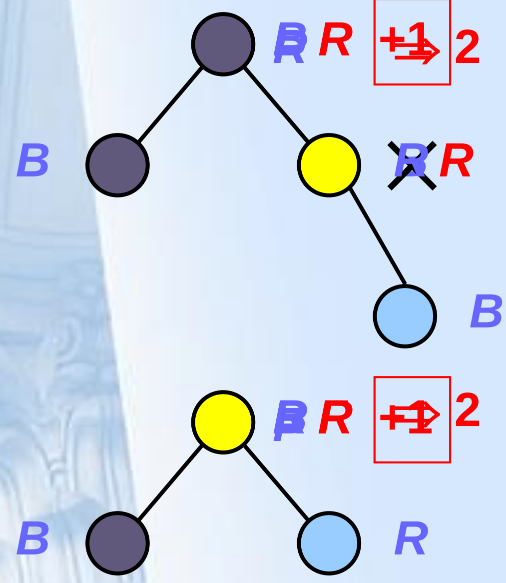
**Given tree ordering $[x_1, x_2, \ldots, x_n]$ of $T$**
**$D \leftarrow \phi$;**
**for $i = 1$ to $n$ do $L(x_i) = B$.**
**for $i = 1$ to $n-1$ do**
  **choose $j > i$ such that $x_i x_j \in E$;**
  **if ($L(x_i) = B$) then $L(x_j) \leftarrow R$;**
  **if ($L(x_i) = R$) then**
    **if ($L(x_j) = B$) then $L(x_j) \leftarrow F$;**
    **$D \leftarrow D \cup \{x_i\}$;**
**if ($L(x_n) \neq F$) then $D \leftarrow D \cup \{x_n\}$;**

- **Time complexity = O($n$).**

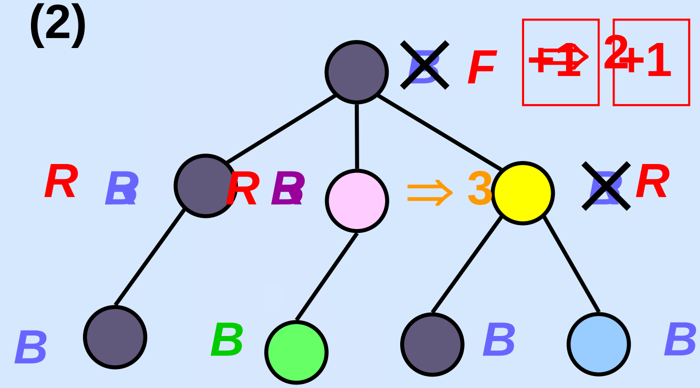
# 8.1 Domination on Tree

- **Ex:**

**(1)**

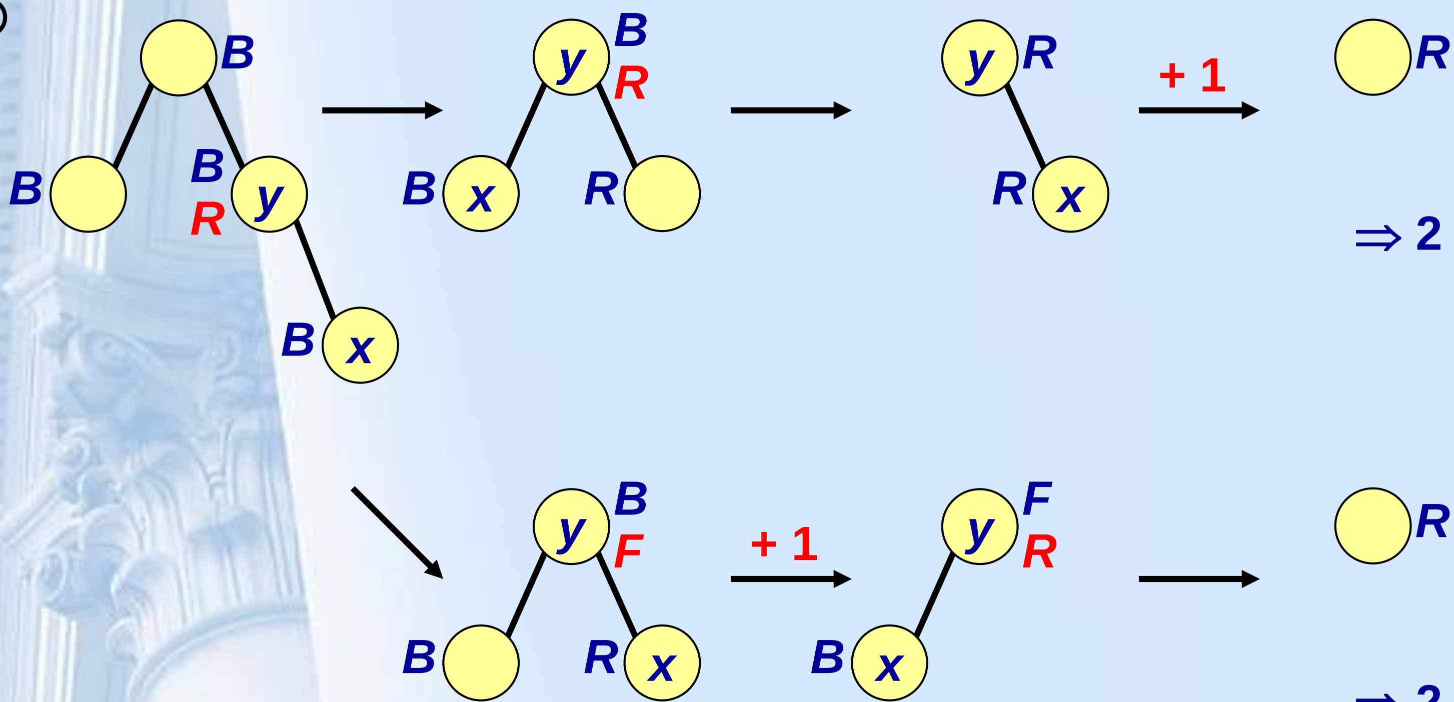

**(2)**



# 8.1 Domination on Tree

- **Ex:**

①



# 8.1 Domination on Tree

- <u>Ex:</u>

②