Computer Science and Information Engineering
National Chi Nan University

Chapter 7
Coloring Problems

§ 7.1 Linear Arboricity of a Tree

7.1 Linear Arboricity of a Tree

- **Def:**

① $la_k(G)$ = min{|n|: $E(G) = \cup_{1 \le i \le n} E_i$ and each E_i induces a subgraph whose components are path of length at most k}.

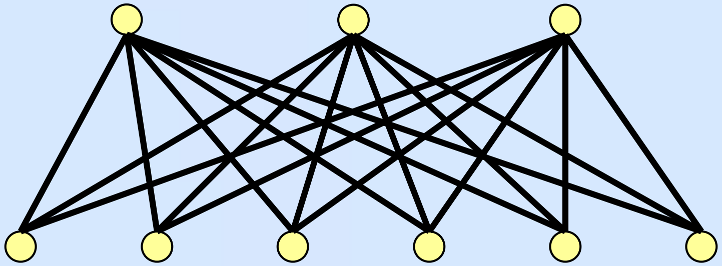
② $la(G)$ = min{|n|: $E(G) = \cup_{1 \le i \le n} E_i$ and each E_i induces a subgraph whose components are path}.

- **Note:** $la_1(G) = \chi'(G)$: min. edge chromatic number.

7.1 Linear Arboricity of a Tree

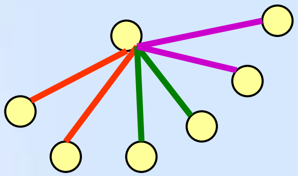
- **Vizing Theorem:** $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$ **if** G **is a simple graph where** $\Delta(G) = \max_{v \in V} \deg(v)$.

- **Ex:** $k = 2$, $G = K_{3,6}$

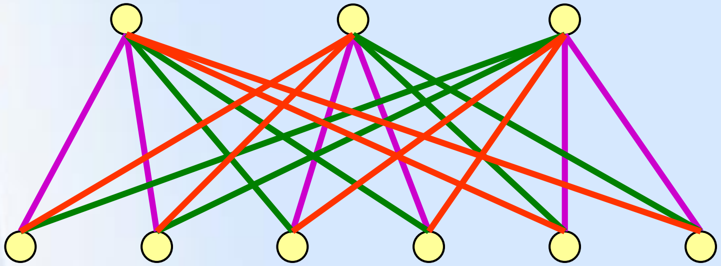


  $\Rightarrow 3 \le \ell a_2(K_{3,6}) \le 3$

  "$3 \le$" $\because \exists$



  "≤ 3"



7.1 Linear Arboricity of a Tree

- **Property:**

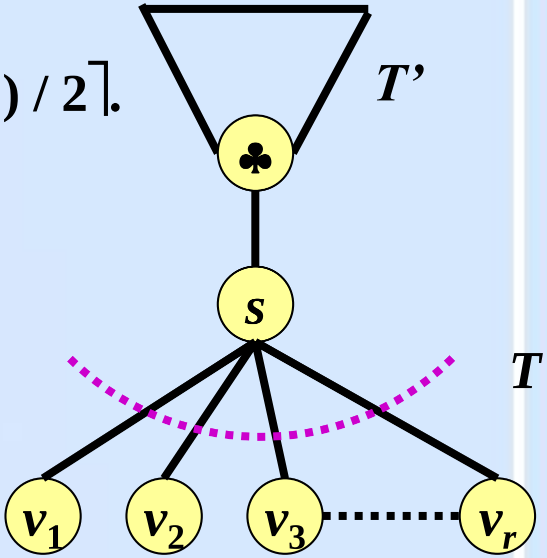
① $la_k \geq \lceil \Delta(G) / 2 \rceil$

② $la_1(G) \geq la_2(G) \geq \ldots \geq la_k(G) \geq la(G)$.

- **Conjecture:** $la(G) \leq \lceil (\Delta(G) + 1) / 2 \rceil$.

7.1 Linear Arboricity of a Tree

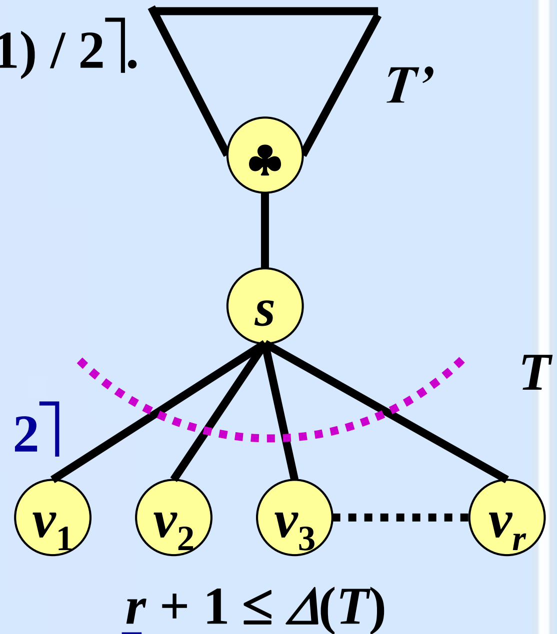
- **<u>Theorem</u>: For any tree T, $la_2(T) \leq \lceil (\Delta(T) + 1) / 2 \rceil$.**
  ($\therefore$ $la(T) \leq \lceil (\Delta(T) + 1) / 2 \rceil$)
  Proof. (1/3)
  Prove by induction on $|V|$:
  ① $|V| = 1$, trivial.
  ② Suppose $|V| < n$, it's true.
  When $|V| = n$:
  Find a "supper leaf" s,
  i.e. all neighbors of s, except at most one,
  are leaves $\{v_1, v_2, \ldots, v_r\}$ s.t. $\deg(s) = r + 1 \leq \Delta(T)$.
  Let $T' \leftarrow T \setminus \{v_1, v_2, \ldots, v_r\}$
  T' is a tree, too.



$r + 1 \leq \Delta(T)$

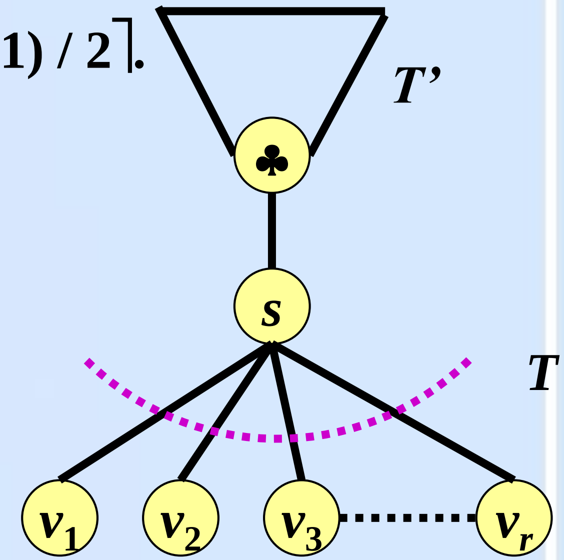
7.1 Linear Arboricity of a Tree

- **Theorem: For any tree T, $\ell a_2(T) \leq \lceil (\Delta(T) + 1) / 2 \rceil$.**
  ($\therefore$ $\ell a(T) \leq \lceil (\Delta(T) + 1) / 2 \rceil$)
  Proof. (2/3)
  ② When $|V| = n$:
  By the strong induction hypothesis,
  $\ell a_2(T') \leq \lceil (\Delta(T')+1) / 2 \rceil \leq \lceil (\Delta(T)+1) / 2 \rceil$
  Since $r + 1 \leq \Delta(T)$
  $\Rightarrow r \leq \Delta(T) - 1$
  $\Rightarrow \lceil r / 2 \rceil \leq \lceil (\Delta(T) - 1) / 2 \rceil = \lceil (\Delta(T) + 1) / 2 \rceil - 1$



7.1 Linear Arboricity of a Tree

- **<u>Theorem</u>: For any tree T, $\ell a_2(T) \le \lceil (\Delta(T) + 1) / 2 \rceil$.**
  $(\therefore\ \ell a(T) \le \lceil (\Delta(T) + 1) / 2 \rceil)$
  Proof. (3/3)
  ② When $|V| = n$:
  Among $\lceil (\Delta(T) + 1) / 2 \rceil - 1$ colors,
  $\exists$ at least $\lceil r / 2 \rceil$ colors can be used
  to color $v_1 s v_2$, $v_3 s v_4$, …
  to get a proper
  linear arboricity coloring of T
  i.e. $\ell a_2(T) \le \lceil (\Delta(T) + 1) / 2 \rceil$.



7.1 Linear Arboricity of a Tree

- **<u>Corollary</u>: T is a tree. If $\Delta(T)$ is odd, then $\ell a_k(T) = \lceil \Delta(T) / 2 \rceil$ for $k \geq 2$.**
  $= (\Delta(T) + 1) / 2$

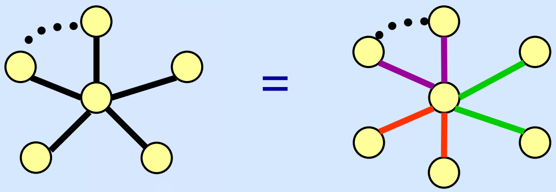
  Proof.

  Since $\Delta(T)$ is odd,
  $\therefore \lceil \Delta(T) / 2 \rceil = \lceil (\Delta(T) + 1) / 2 \rceil$. (upper bound = lower bound)

- **<u>ex</u>: $\lceil (5 - 1) / 2 \rceil = 4 / 2 = 2 = \lceil 3 / 2 \rceil = \lceil (5 - 2) / 2 \rceil$**

7.1 Linear Arboricity of a Tree

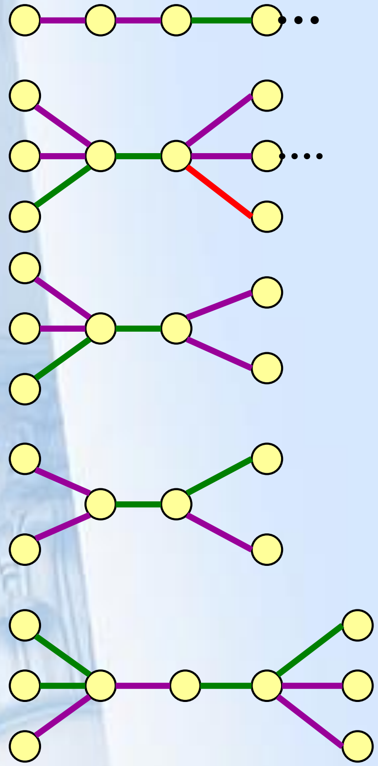
- **Theorem: For any tree T, $\ell a_2(T) = \lceil \Delta(T) / 2 \rceil$**
  if ∃ supper leaf s s.t. deg(s) < Δ in any iteration until $T = K_{1,\Delta}$ (star).
  Proof.
  see the proof of above theorem and



$\therefore\ \ell a_2(K_{1,\Delta}) = \lceil \Delta / 2 \rceil$.

7.1 Linear Arboricity of a Tree

- <u>**ex**</u>:



T_1: $\Delta(T_1) / 2 = 1$, $\ell a_2(T_1) \geq 2$,

T_2: $\Delta(T_2) / 2 = 2$, $\ell a_2(T_2) \geq 3$,

T_3: $\Delta(T_3) / 2 = 2$, $\ell a_2(T_3) = 2$,

T_4: $\lceil \Delta(T_4) / 2 \rceil = 2$, $\ell a_2(T_4) = 2$,

T_5: $\Delta(T_5) / 2 = 2$, $\ell a_2(T_5) = 2$.

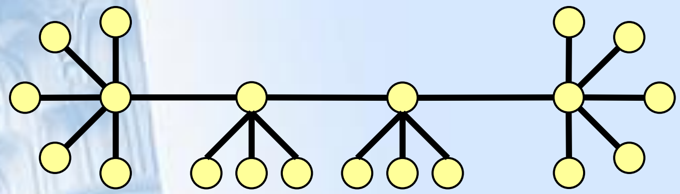
7.1 Linear Arboricity of a Tree

- **Question: For even $\Delta(T)$, decide $\ell a_2(T) = \Delta(T) / 2$ or $(\Delta(T) / 2) + 1$.**

- **Theorem: For any tree T, if $\Delta(T)$ is even, $\exists$ path $x_0, x_1, \ldots, x_k$ in T s.t. $k \geq 1$, $\deg(x_0) = \deg(x_k) = \Delta(T)$, $\deg(x_i) \geq \Delta(T) - 1$ for $1 \leq i \leq k - 1$ $\Rightarrow \ell a_2(T) = (\Delta(T) / 2) + 1$.**

- **Ex:**

① T_1:



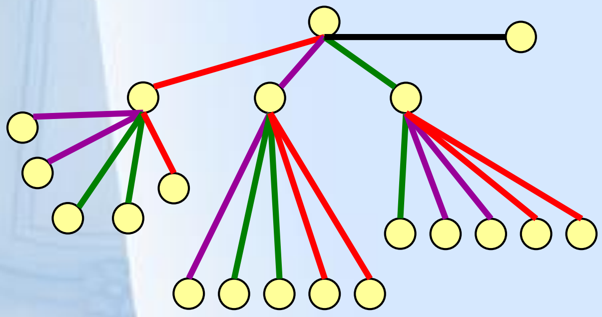
$\ell a_2(T_1) > 3 = \lceil \Delta(T_1) / 2 \rceil$

if exist 3色塗法

7.1 Linear Arboricity of a Tree

- **ex:**

② T_2:



沒有"such" path,

但也不能用3色塗好

i.e. $\ell a_2(T_2) = 4 = \lceil \Delta(T_2) / 2 \rceil + 1$

∴ "⇐" 不能成立

7.1 Linear Arboricity of a Tree

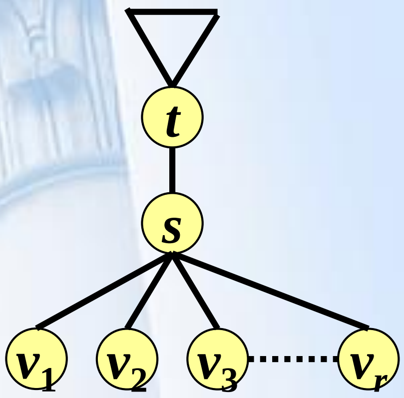
- **Theorem**: T: tree, s: supper leaf adj. to non-leaf t and leaves $v_1, v_2, \ldots, v_r$. $T' = T \setminus \{v_1, \ldots, v_r\}$. Fix α: positive integer,

  $\ell a_2(T) \le \alpha \Leftrightarrow$

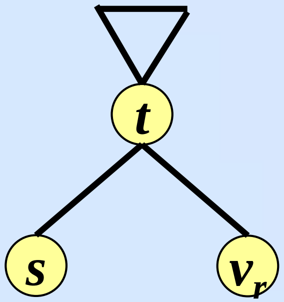
  $$\begin{cases} \text{(i) } r \le 2(\alpha - 1) \text{ and } \ell a_2(T') \le \alpha, \text{ or} \\ \text{(ii) } r = 2\alpha - 1 \text{ and } \ell a_2(T'') \le \alpha \end{cases}$$

  where $T'' = (V, E)$ is a tree s.t. $V(T'') = V(T) \setminus \{v_1, v_2, \ldots, v_{r-1}\}$, $E(T'') = E(T) \setminus \{sv_i: 1 \le i \le r\} \cup \{tv_r\}$.

  i.e. T:



T'':



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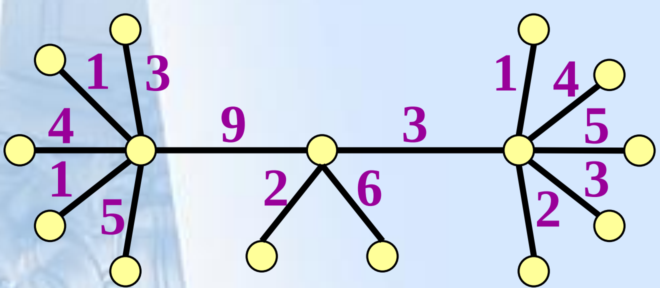
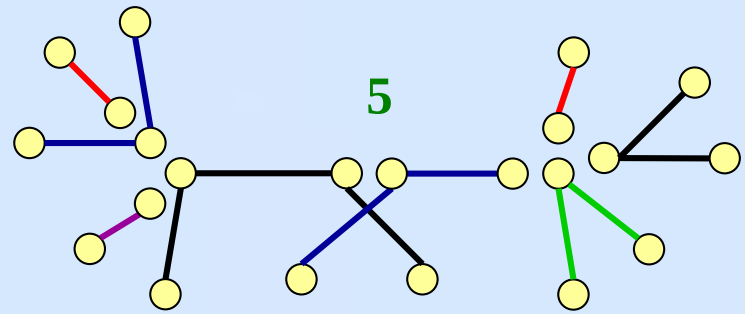
§ 7.2 *L*-Linear Arboricity of a Tree

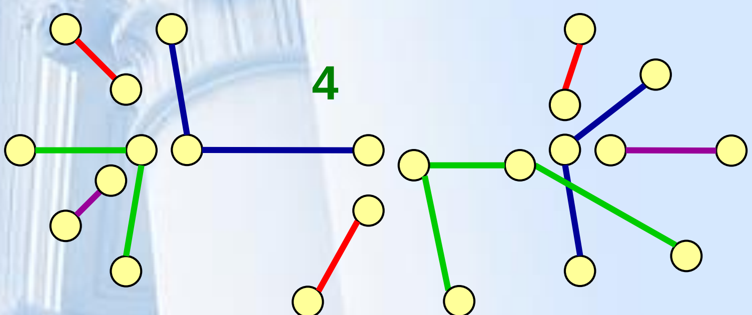
7.2 *L*-Linear Arboricity of a Tree

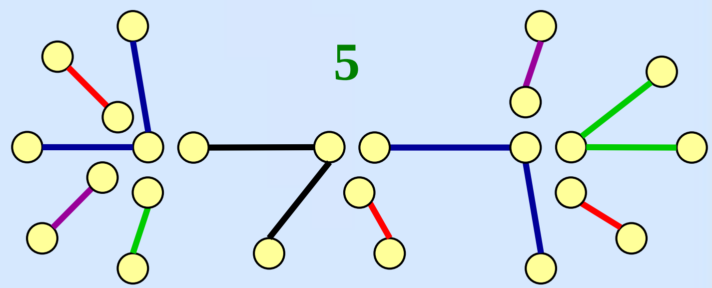
- **<u>Def</u>: T is a tree, L: $E(T) \rightarrow \{1, 2, \ldots\}$,**
  $\ell a(T, L)$ = min # n s.t. $E(T) = \cup_{1 \le i \le n} E_i$ and each component of the subgraph induced by E_i is a path P of length at most $\min_{e \in E_i} L(e)$.

- **<u>ex</u>:**





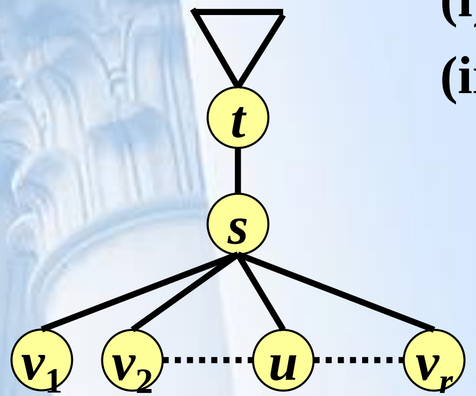




7.2 *L*-Linear Arboricity of a Tree

- **Remark**: $L(e) = k, \forall e \in E(T) \Rightarrow \ell a(T, L) = \ell a_k(T)$.

- **Theorem**:
  T: tree, s: supper leaf adj. to leaves $v_1, v_2, \ldots, v_r$ and non-leaf t.
  p = # of neighbor u of s s.t. $L(su) = 1$. Then fix $\alpha \in N$,
  $\ell a(T, L) \le \alpha \Leftrightarrow r + 1 + p \le 2\alpha$, and (i) or (ii) holds.



(i) $r \le 2(\alpha - 1)$ and $\ell a_2(T', L) \le \alpha$, or

(ii) $r = 2\alpha - 1$ and $\ell a_2(T'', L) \le \alpha$

where $T'' = (V, E)$ is a tree s.t.

$V(T'') = V(T) \setminus \{v_1, v_2, \ldots, v_{r-1}\}$,
$E(T'') = E(T) \setminus \{sv_i: 1 \le i \le r\} \cup \{tv_r\}$.

$(r + 1 - p \le 2(\alpha - p))$

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Chapter 7
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§ 7.3 $L(2, 1)$-labeling on Trees

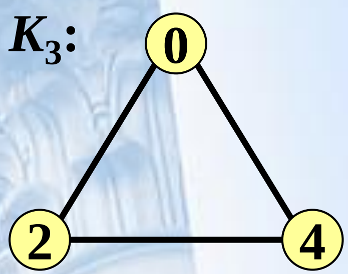
7.3 $L(2, 1)$-labeling on Trees

- **Def:**

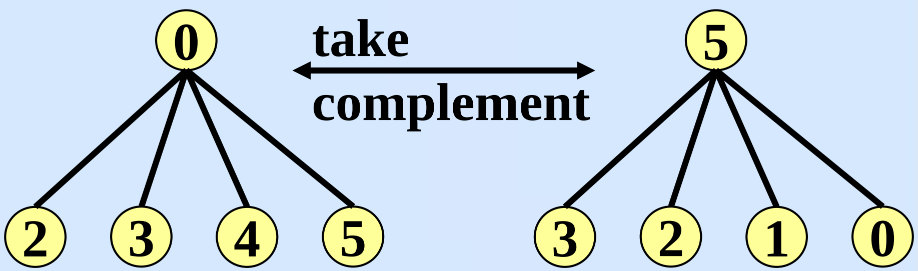
① A ***k*-*L*(2, 1)-labeling** of a graph G is a function $f: V(G) \to \{0, 1, \ldots, k\}$ s.t. (i) $d_G(x, y) = 1 \Rightarrow |f(x) - f(y)| \geq 2$,
(ii) $d_G(x, y) = 2 \Rightarrow |f(x) - f(y)| \geq 1$.

② $\lambda(G) = \min\{k: \exists\ k\text{-}L(2, 1)\text{-labeling of } G\}$.

- **ex:** K_3:



$K_{1,4}$:



$\lambda(K_3) \leq 4$

In fact, $\lambda(K_3) = 4$

$\lambda(K_{1,4}) \leq 5$

7.3 L(2, 1)-labeling on Trees

- **Lemma: G is a graph with max. degree $\Delta(G)$. Then $\lambda(G) \geq \Delta(G) + 1$. Moreover, if $\lambda(G) = \Delta(G) + 1$, then $f(x) = 0$ or $\Delta(G) + 1$ for any optimal $L(2, 1)$-labeling f of G and any vertex x with degree $\Delta(G)$.**

- **Lemma: If T is a tree, then $\lambda(T) \leq \Delta(T) + 2$.**

  Proof. (1/2)

  Prove by induction on $|V|$

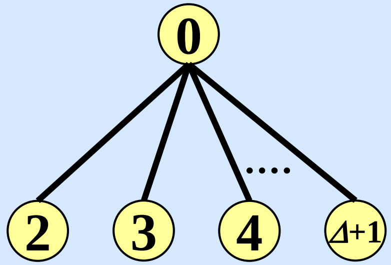
  ① $|V| = 1$, ok

  ② Consider $|V| \geq 2$:

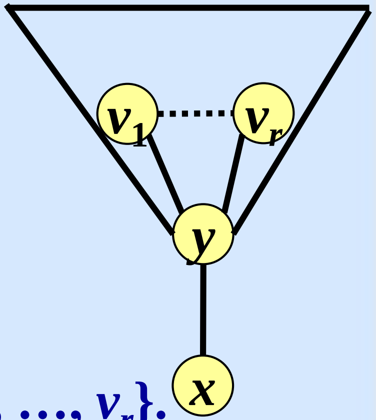
  choose a leaf x of T and $T' = T - x$.

  By I.H.: $\lambda(T') \leq \Delta(T') + 2 \leq \Delta(T) + 2$

  $\therefore \exists f$ is a $(\Delta(T) + 2)$-$L(2, 1)$-labeling of T'.



7.3 $L(2, 1)$-labeling on Trees



- Lemma: If T is a tree, then $\lambda(T) \leq \Delta(T) + 2$.

  Proof. (2/2)

  Prove by induction on $|V|$

  ② Consider $|V| \geq 2$:

  Suppose x is adj. to y in T, and $N(y) = \{x, v_1, v_2, \ldots, v_r\}$.

  $r = \deg_T(y) - 1 \leq \Delta(T) - 1$.

  $\therefore$ $\exists$ at most $r + 3$ coloring can't be used to color x in T.

  $\because$ $r + 3 \leq \Delta(T) + 2$

  Hence,

  $\exists$ at least one color in $\{0, 1, \ldots, \Delta(T) + 2\}$ can be used to color x.

7.3 $L(2, 1)$-labeling on Trees

- **Def:**

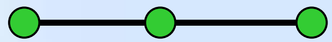
① A ***k*-*L*(*d*, 1)-labeling** of a graph G is a function $f: V(G) \to \{0, 1, \ldots, k\}$ s.t. (i) $d_G(x, y) = 1 \Rightarrow |f(x) - f(y)| \geq d$,
(ii) $d_G(x, y) = 2 \Rightarrow |f(x) - f(y)| \geq 1$.

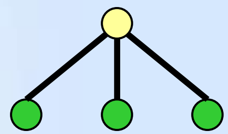
② $\lambda_d(G) = \min\{k: \exists\ k\text{-}L(d, 1)\text{-labeling of } G\}$.

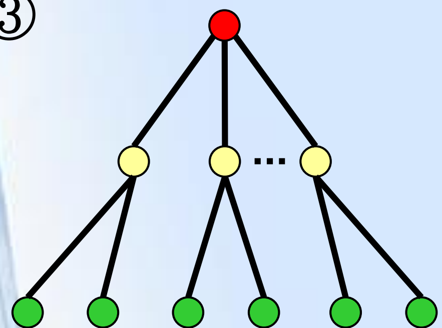
- **Exercise 5 (11/6):**
  Find an upper bound and lower bound of $\lambda_d(T)$ for all tree T.

7.3 *L*(2, 1)-labeling on Trees

- **<u>Note</u>: 若希望 $\lambda(T) = \Delta(T) + 1$**

  ⇒ 不能有①  **max. degree**

  ② 

  ③ 

  $deg = \Delta(T) - 2$

  $\leftarrow \in \{2, 3, \ldots, \Delta(T) - 1\}$

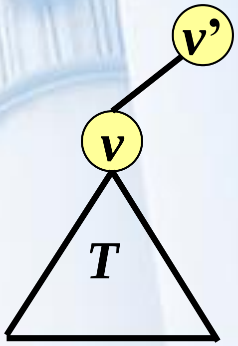
  $\leftarrow = 0, \Delta(T) + 1$

7.3 $L(2, 1)$-labeling on Trees

- **Conjecture: [YG](1992)**
  It is NP-hard to determine if $\lambda(T) = \Delta(T) + 1$.
  Sol.
  No! (By G. J. Chang and David Kuo, 1996)

- **Question: Given T and k, test if $\lambda(T) \leq k$.**

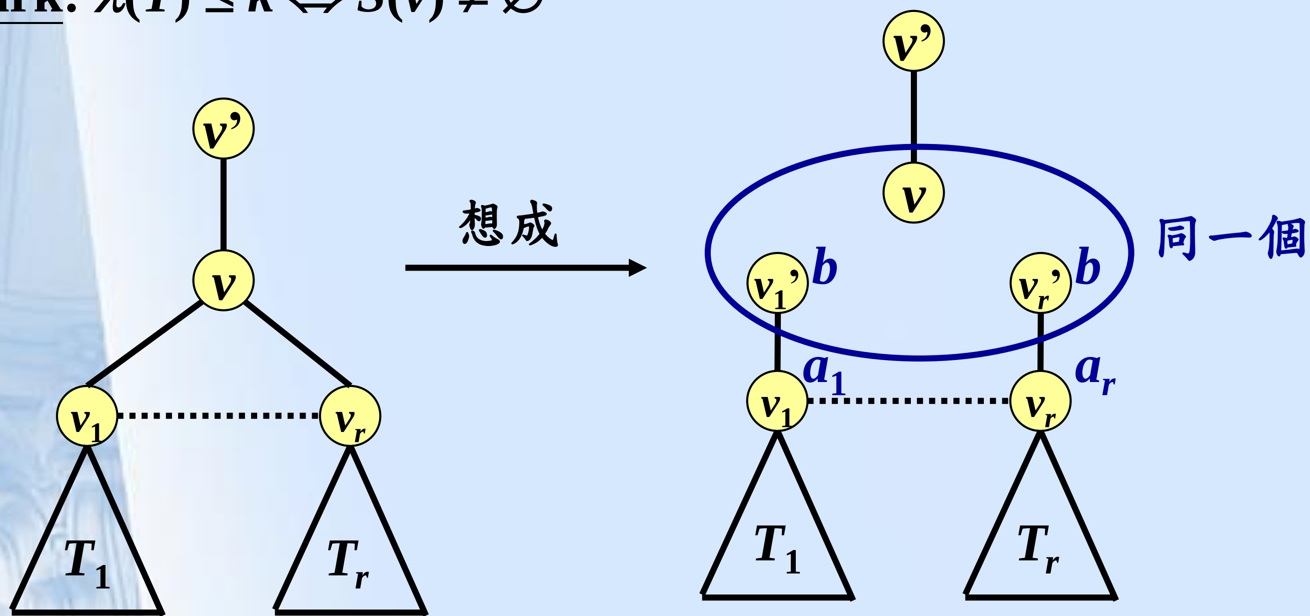
- **Def: $S(v)$ = $\{(a, b)$: $\exists$ k-$L(2, 1)$-labeling f s.t. $f(v') = a$, $f(v) = b\}$.**



7.3 $L(2, 1)$-labeling on Trees

- **Remark:** $\lambda(T) \leq k \Leftrightarrow S(v) \neq \varnothing$

  想法:



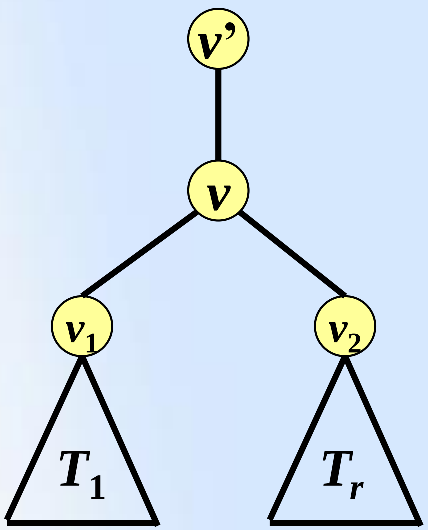
若已知 $S(v_1), \ldots, S(v_r)$, 用之產生 $S(v)$

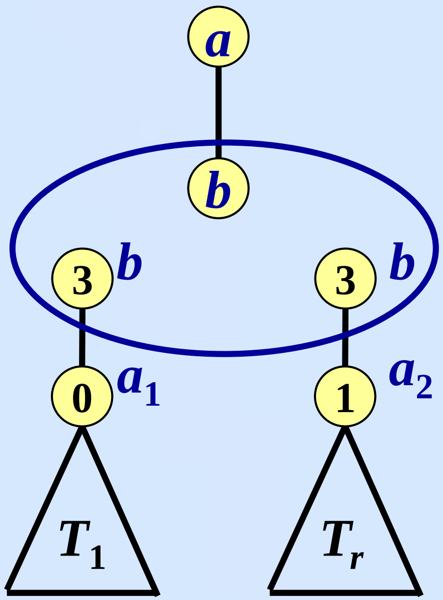
where $S(v_i) = \{(b, a_i): \exists\ k\text{-}L(2, 1)\text{-labeling } f_i$

s.t. $f_i(v_i') = b, f_i(v_i) = a_i\}$

7.3 $L(2, 1)$-labeling on Trees

- **Theorem**: $S(v) = \{(a, b): 0 \le a \le k, 0 \le b \le k, |a - b| \ge 2$ and $\exists$ SDR for $(A_1, \ldots, A_r)$ where $A_i = \{a_i: a_i \ne a, (b, a_i) \in S(v_i)\}\}$
- **Ex**:
  - (0, 3)?
  - (1, 3)?
  - (5, 3)?





7.3 $L(2, 1)$-labeling on Trees

- **Algorithm**:

Initial condition:

$\forall$ leaf v of T: $S(v) = \{(a, b): 0 \leq a \leq k, 0 \leq b \leq k, |a - b| \geq 2\}$.

From leaves and works toward root:

For any vertex v with children $v_1, v_2, \ldots, v_r$

By Theorem, calculate $S(v)$ use $S(v_1), \ldots, S(v_r)$ $O(k^2)$

Construct bipartite graph $G = (X, Y, E)$ for any (a, b), $0 \leq a \leq k, 0 \leq b \leq k, |a - b| \geq 2$ with $X = \{x_1, x_2, \ldots, x_r\}$, $Y = \{0, 1, \ldots, k\}$, $E = \{(x_i, c): c \neq a$ and $(b, c) \in S(v_i)\}$.

Then use algorithm for Max. Matching of Bipartite graph

Then $(a, b) \in S(v) \Leftrightarrow G$ has a matching of size $|X| = S$. $O((2k)^3)$ $\downarrow$ $O((2k)^{2.5})$

$\Rightarrow O(|V|k^2 \cdot (2k)^{2.5}) \rightarrow O(|V|^{5.5})$

$\because k = \Delta(T) + 1 \leq |V|$

7.3 $L(2, 1)$-labeling on Trees

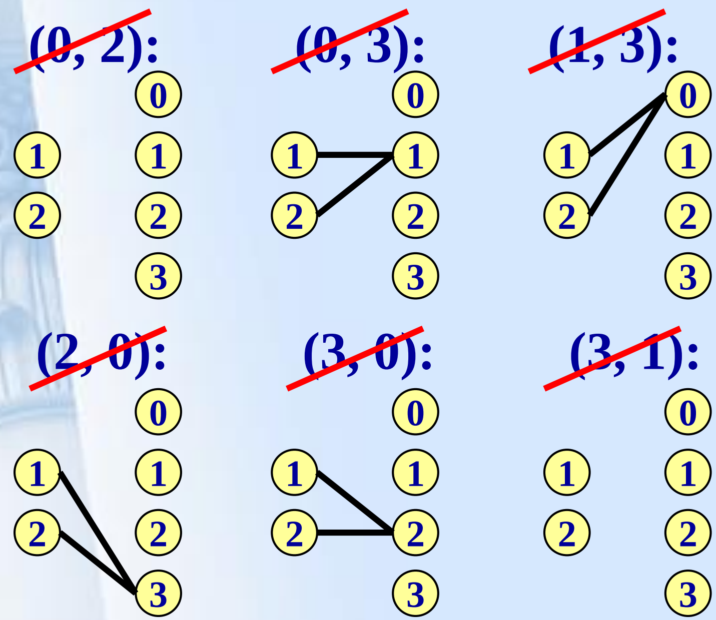
- <u>Ex</u>: $k = \Delta = 3$

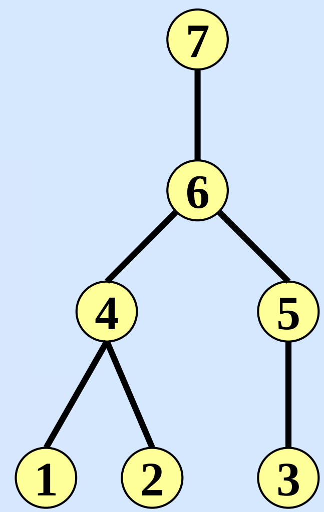
  Sol.

  $S(1) = S(2) = S(3)$

  $= \{(0, 2), (0, 3), (1, 3), (2, 0), (3, 0), (3, 1)\}$

  For $S(4)$:





$\Rightarrow S(4) = \varnothing$

$\Rightarrow \lambda(T) > k$

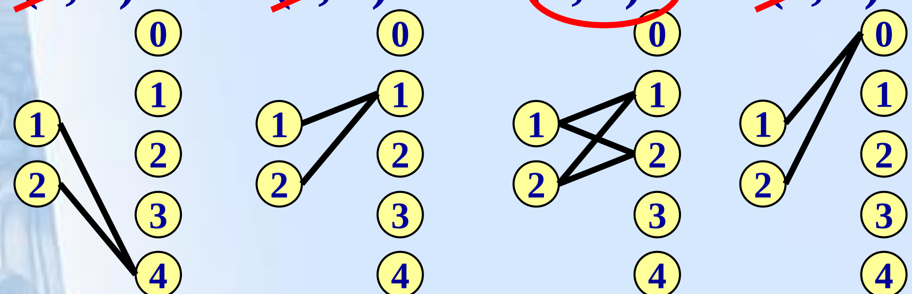
7.3 *L*(2, 1)-labeling on Trees

- **Ex**: $k = \Delta + 1 = 4$

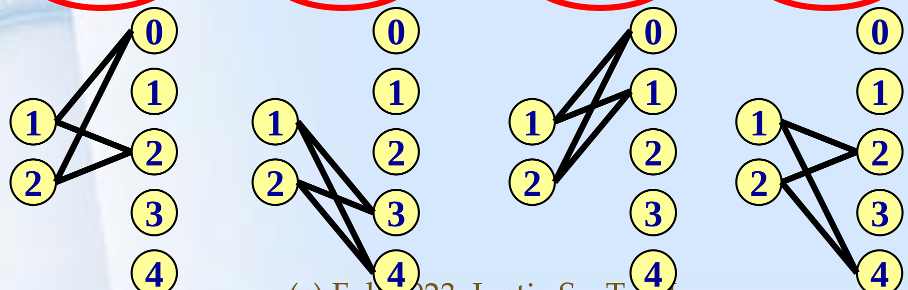
  Sol.

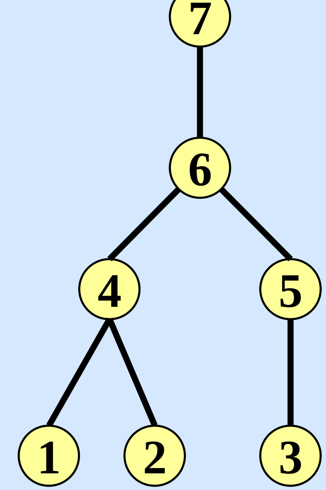
  $S(1) = S(2) = S(3) = \{(0, 2), (0, 3), (0, 4), (1, 3), (1, 4), (2, 0), (2, 4), (3, 0), (3, 1), (4, 0), (4, 1), (4, 2)\}$

  For $S(4)$: (0, 2): (0, 3): (0, 4): (1, 3):



(1, 4): (2, 0): (2, 4): (3, 0):





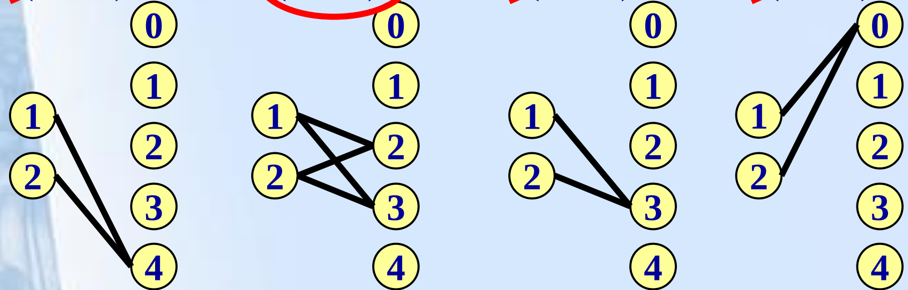
7.3 $L(2, 1)$-labeling on Trees

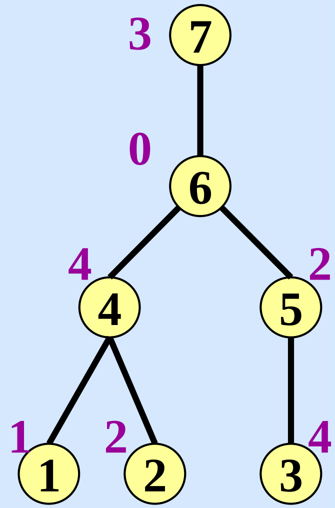
- **Ex:** $k = \Delta + 1 = 4$

  Sol.

  $S(1) = S(2) = S(3) = \{(0, 2), (0, 3), (0, 4), (1, 3),$
  $(1, 4), (2, 0), (2, 4), (3, 0), (3, 1), (4, 0), (4, 1), (4, 2)\}$

  For $S(4)$: (3, 1): (4, 0): (4, 1): (4, 2):





  $\Rightarrow S(4) = \{(0, 4), (1, 4), (2, 0), (2, 4), (3, 0), (4, 0)\}$

  $\Rightarrow \ldots$

  $\Rightarrow \lambda(T) \leq k$

7.3 *L*(2, 1)-labeling on Trees

- **Exercise 2 (11/28): 將L(2, 1)-labeling on Trees *T* = (*V*, *E*)實作出來。**