Computer Science and Information Engineering
National Chi Nan University

Chapter 4
Shortest Paths

§ 4.1 Shortest Path

4.1 Shortest Path

- **Def:**

① **A digraph (directed graph) is an ordered pair $G = (V, E)$, where V is a finite set of elements, called vertices, and E is a set of ordered pairs of distinct vertices, called edges (arcs).**

② **A weighted digraph (or network) is a digraph $G = (V, E)$ with real valued weights (or lengths) assigned to each edge, i.e. $\exists\ w: E \rightarrow R$.**

③ **The length of a path Q in a weighed digraph is the sum of the weights (lengths) of the edges on the path, denoted by $l_Q(x, y)$, i.e. $l_Q(x, y) = \Sigma_{e \in Q} w(e)$.**

④ **A shortest path between a pair of vertices x and y in a weighted digraph is a path Q from x to y of least length, i.e. $l_Q(x, y) = \min\{l_{Q'}(x, y) : Q'$ is a x-y path$\}$.**

4.1 Shortest Path

- **Def:**

⑤ **The distance from x to y in a weighted digraph is denoted by $d(x, y) = \min\{l_Q(x, y): Q \text{ is a } x\text{-}y \text{ path}\}$. (the length of a shortest path from x to y)**

- **Question:**

Let $G = (V, E)$ be a weighted digraph that $\forall e \in E$, $w(e) > 0$, and let $x, y \in V$, find the shortest path from x to y in G and its length, or show there is none.

Note: If $e = (x, y)$, write $w((x, y))$ as $w(x, y)$.

4.1 Shortest Path

- **Algorithm**: **Dijkstra Algorithm**

$f(x) = 0$; father$(x) = 0$;

$\forall z \neq x : \begin{cases} f(z) = w(x, z); \\ \text{father}(z) = x; \end{cases}$

$S = \{x\}$;

do while $(S \neq V)$

choose $z^* \in V \setminus S$ with min $f(z^*)$;

$S \leftarrow S \cup \{z^*\}$;

$\forall z \in V \setminus S$:

if $f(z) > f(z^*) + w(z^*, z)$ **then**

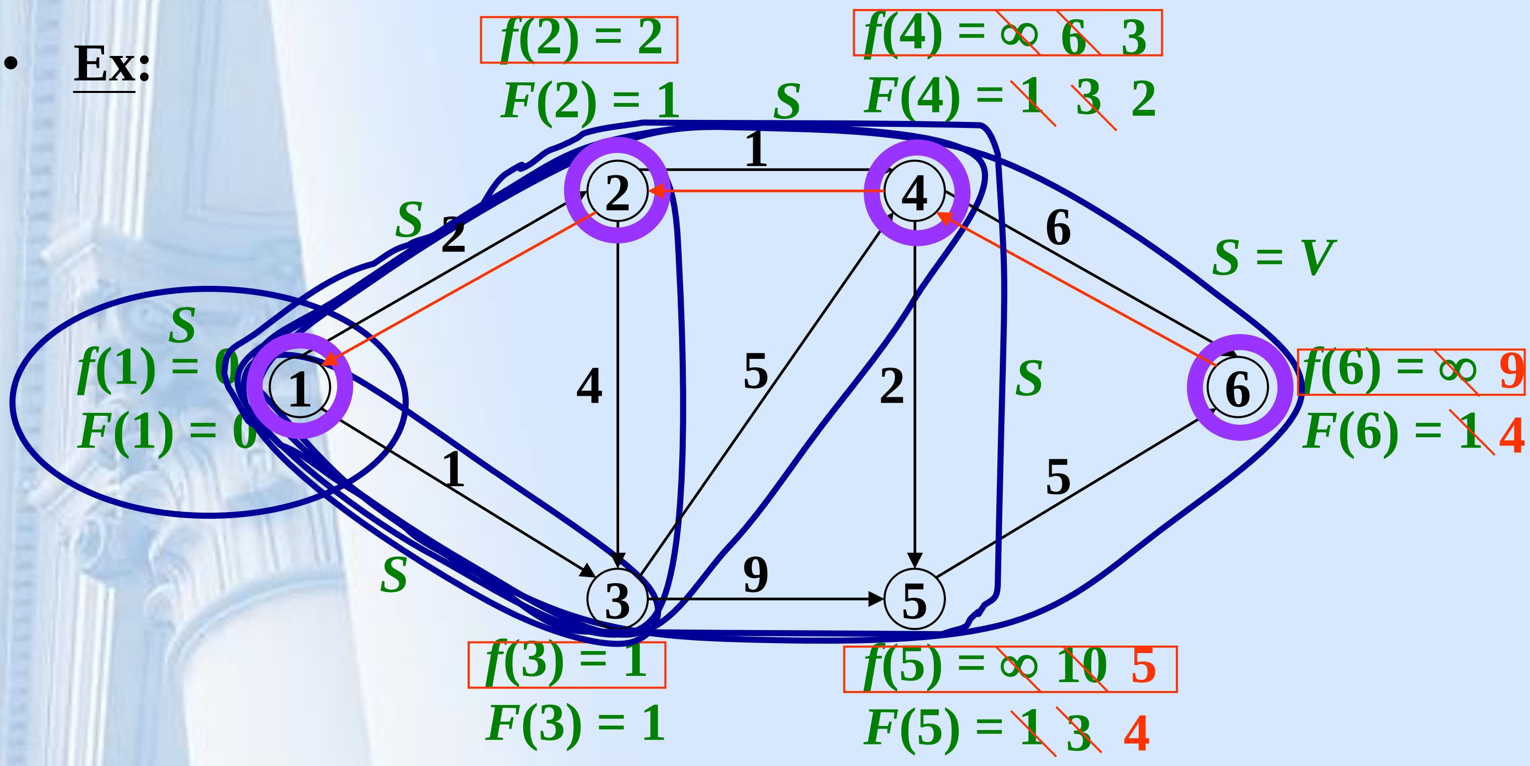
$\begin{cases} f(z) = f(z^*) + w(z^*, z); \\ \text{father}(z) \leftarrow z^*; \end{cases}$

end

4.1 Shortest Path

- **<u>Note</u>: If $(x, y) \notin E(G)$, let $w(x, y) = \infty$, $\forall\ x, y \in V(G)$.**

- **<u>Ex</u>:**



4.1 Shortest Path

- **Note:** ① P: $x = x_0 \to x_1 \to \ldots x_r = y$,

  $\Sigma_{e \in P} w(e) = \Sigma_{i=0}^{r-1} w(x_i, x_{i+1})$.

  ② **Define** $g: V \to R^+$ s.t. $\begin{cases} g(x) = 0, \\ \overrightarrow{zw} \in E \Rightarrow g(w) - g(z) \leq w(z, w). \end{cases}$ ♠

4.1 Shortest Path

- **Thm**(w.d.i): $d(x, y) = \min_{x\text{-}y \text{ path } P} (\Sigma_{e \in P} w(e)) \geq \max_{g \text{ sat. } ♠} g(y)$.

  Proof.

  $\forall$ x, y path P, $\forall$ g satisfy ♠.

  Let $P : x_0 = x \to x_1 \to \ldots \to x_r = y$.

  $$\Sigma_{e \in P} w(e) = \Sigma_{i=0}^{r-1} w(x_i, x_{i+1})$$
  $$\geq \Sigma_{i=0}^{r-1} (g(x_{i+1}) - g(x_i))$$
  $$= g(x_1) - g(x_0) + g(x_2) - g(x_1) + \ldots + g(x_r) - g(x_{r-1})$$
  $$= g(x_r) - g(x_0)$$
  $$= g(y) - g(x)$$
  $$= g(y).$$

  $\therefore \min_{x\text{-}y \text{ path } P} (\Sigma_{e \in P} w(e)) \geq \max_{g \text{ sat } ♠} g(y)$.

4.1 Shortest Path

- <justify Dijkstra Algorithm>

  Assume f^* is the final output f, F^* is the final output father, then from F^*, we can find an x - y path P: $x = x_0, x_1, \ldots, x_r = y$, such that $\Sigma_{e\in P}\, w(e) = \Sigma_{i=1}^{r-1} w(x_i, x_{i+1}) = f^*(y)$.

  claim: f^* satisfies ♠

  Then, $\Sigma_{e\in P}\, w(e) \geq d(x, y) \geq \max_{g \text{ sat } ♠} g(y) \geq f^*(y) = \Sigma_{e\in P}\, w(e)$.

  ∴ all " ≥ " are " = ".

  ⇒ ① $f^*(y) = d(x, y)$.

  ② P is the shortest path from x to y → solve the Question.

- Note: If $f^*(y) = \infty$ means there is none path from x to y.

4.1 Shortest Path

- Proof of claim (f^* satisfies ♠)

  We prove: at any iteration:

  ① $f(z) \le f(z'), \forall z \in S, z' \notin S$.

  ② $\forall \overrightarrow{zw} \in E, \{z, w\} \cap S \ne \phi \Rightarrow f(w) - f(z) \le w(z, w)$

  Proof. (1/6)

  prove by induction on $|S|$:

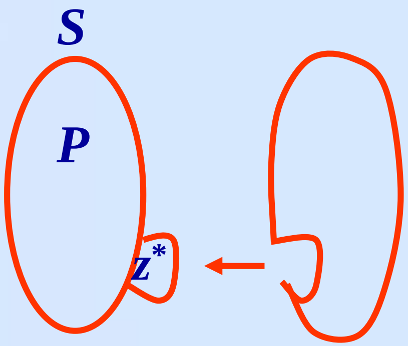
  ①② $|S| = 1$, O.K. Since $w(e) > 0, \forall e \in E$.

  Suppose it's hold for $|S| \le |P|$.

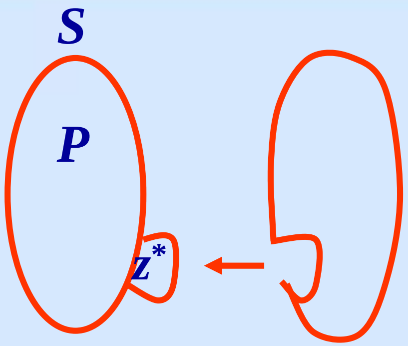
  Now, when $S = P \cup \{z^*\}$.

  Let f means the f-function before $S = P \cup \{z^*\}$ be executed.

  Let f^* means the f-function after $S = P \cup \{z^*\}$ be executed.



4.1 Shortest Path



- **Proof of <u>claim</u> (f^* satisfies ♠)**

  We prove: at any iteration:

  ① $f(z) \le f(z')$, $\forall z \in S, z' \notin S$.

  ② $\forall \overrightarrow{zw} \in E$, $\{z, w\} \cap S \ne \phi \Rightarrow f(w) - f(z) \le w(z, w)$

  Proof. (2/6)

  ① $\forall z \in S, z' \notin S \Rightarrow z' \notin P$; $z \in P$ or $z = z^*$.

  <u>case1</u>: $z \in P$. ($z' \notin P$) $\Rightarrow f^*(z) = f(z)$.

  By the induction hypothesis: $f(z) \le f(z')$.

  If [if] holds: $f^*(z') = f(z^*) + w(z^*z')$

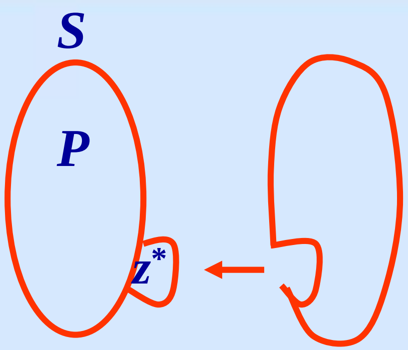
  $\Rightarrow f^*(z') \ge f(z^*) \ge f(z) = f^*(z)$;

  $\because w \ge 0$ $\quad$ $\because$ I.H. for ①, $z^* \notin P$, $z \in P$

  else : $f^*(z') = f(z') \ge f(z) = f^*(z)$.

  $\therefore f^*(z) \le f^*(z')$.

4.1 Shortest Path



- **Proof of <u>claim</u> (f^* satisfies ♠)**

  We prove: at any iteration:

  ① $f(z) \le f(z'), \forall z \in S, z' \notin S.$

  ② $\forall \overrightarrow{zw} \in E, \{z, w\} \cap S \neq \phi \Rightarrow f(w) - f(z) \le w(z, w)$

  Proof. (3/6)

  ① $\forall z \in S, z' \notin S \Rightarrow z' \notin P; z \in P$ or $z = z^*$.

  <u>case2</u>: $z = z^* \ (z' \notin P)$

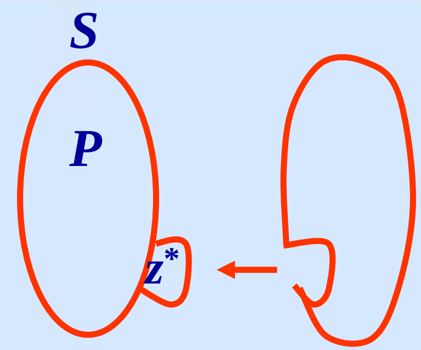
  $\Rightarrow f^*(z) = f^*(z^*) = f(z^*)$.

  If [if] holds: $f^*(z') = f(z^*) + w(z^*, z') \ge f(z^*)$;

  else : $f^*(z') = f(z') \ge f(z^*)$. ($z^*$ with min. f in $V \setminus P$)

  $\therefore f^*(z) \le f^*(z')$.

4.1 Shortest Path



- Proof of <u>claim</u> (f^* satisfies ♠)

  We prove: at any iteration:

  ① $f(z) \le f(z'), \forall z \in S, z' \notin S.$

  ② $\forall \overrightarrow{zw} \in E, \{z, w\} \cap S \ne \phi \Rightarrow f(w) - f(z) \le w(z, w)$

  Proof. (4/6)

  ② $\forall \overrightarrow{zw} \in E, \{z, w\} \cap S \ne \phi.$

  <u>case 1</u>: $\{z, w\} \subseteq S$

  $f^*(w) - f^*(z) = f(w) - f(z) \le w(z, w).$

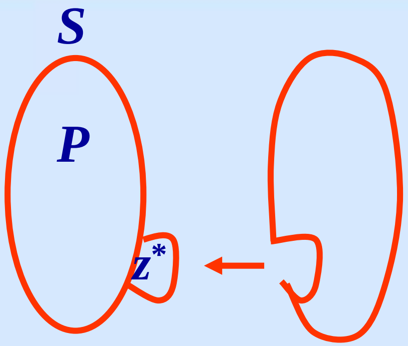
  by I.H.②

  <u>case2</u>: $z \notin S$ and $w \in S$ ($z \notin P$)

  By ①: $f^*(w) \le f^*(z).$

  $\therefore f^*(w) - f^*(z) \le 0 \le w(z, w).$

4.1 Shortest Path



- **Proof of <u>claim</u> (f^* satisfies ♠)**

  We prove: at any iteration:

  ① $f(z) \le f(z'), \forall z \in S, z' \notin S.$

  ② $\forall \overrightarrow{zw} \in E, \{z, w\} \cap S \neq \phi \Rightarrow f(w) - f(z) \le w(z, w)$

  Proof. (5/6)

  ② $\forall \overrightarrow{zw} \in E, \{z, w\} \cap S \neq \phi.$

  <u>case 3</u>: $z \in S$, and $w \notin S$

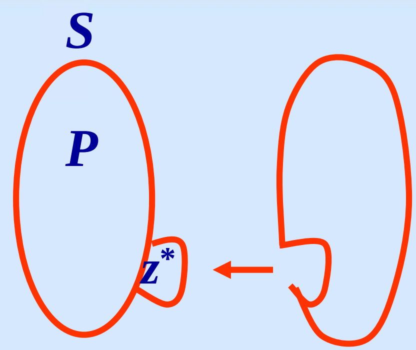
  <u>case 3.1</u>: $z \neq z^*$

  $\because f^*(w) \le f(w); f^*(z) = f(z).$

  $\therefore f^*(w) - f^*(z) \le f(w) - f(z) \le w(z, w).$

  by I.H ② ($\because \{w, z\} \cap (S - z^*) \neq \phi$)

4.1 Shortest Path



- **Proof of <u>claim</u> (f^* satisfies ♠)**

  We prove: at any iteration:

  ① $f(z) \leq f(z'), \forall\, z \in S, z' \notin S.$

  ② $\forall\, \overrightarrow{zw} \in E, \{z, w\} \cap S \neq \phi \Rightarrow f(w) - f(z) \leq w(z, w)$

  Proof. (6/6)

  ② $\forall\, \overrightarrow{zw} \in E, \{z, w\} \cap S \neq \phi.$

  <u>case 3</u>: $z \in S$, and $w \notin S$

  <u>case 3.2</u>: $z = z^*$:

  $$f^*(w) \leq f(z^*) + w(z^*, w)$$
  $$= f(z) + w(z, w);$$

  $\because f^*(z) = f(z).$

  $$\therefore f^*(w) - f^*(z) = f^*(w) - f(z)$$
  $$\leq w(z, w).$$

Computer Science and Information Engineering
National Chi Nan University

Chapter 5
Dynamic Programming

§ 5.1 Knapsack Problem

5.1 Knapsack Problem

- **Def: Knapsack Problem**

  $$f(n, b) = \max v_1x_1 + v_2x_2 + \ldots + v_nx_n$$
  $$\text{s.t. } c_1x_1 + c_2x_2 + \ldots + c_nx_n \le b,$$
  where each x_i = non-negative integer; $v_i \in R^+$;
  $c_i \in R^+$, $b \in Z^+$.

  Sol.

  $$f(k, a) = \max\{f(k-1, a), f(k, a-c_k) + v_k\},$$
  for $0 \le k \le n$, $0 \le a \le b$.

Computer Science and Information Engineering
National Chi Nan University

Chapter 5
Dynamic Programming

§ 5.2 Max. Consecutive Sum Problem

5.2 Max. Consecutive Sum Problem

- **Def: Max. Consecutive Sum Problem**

  Given $x_1, x_2, \ldots, x_n \in R$, sum($i, j$) = $\Sigma_{k=i}^{j} x_k$.

  Find $\max_{1 \le i \le j \le n}$ sum(i, j).

- **Ex:** 1, – 1, 3, 4, – 8, – 2, 3, 7, – 4, 3, 5

Sol 1.

```
for 1 ≤ i ≤ n
    for i ≤ j ≤ n
        sum(i, j) = 0;
        for i ≤ k ≤ j
            sum(i, j) = sum(i, j) + x_k;
取 sum(i, j)的max.   1 ≤ i ≤ j ≤ n
```

$\max_{1 \le i \le j \le n} (\Sigma_{k=i}^{j} x_k)$

→ $O(n^3)$

5.2 Max. Consecutive Sum Problem

Sol 2.

$\text{sum}(i, i) = x_i, \ \forall\, 1 \le i \le n$
<u>for</u> $d = 1$ to $n - 1$
  <u>for</u> $1 \le i \le n - d$
    $\text{sum}(i, i + d) = \text{sum}(i, i + d - 1) + x_{i+d}$
取 sum(i, j)的max. $1 \le i \le j \le n$

⟶ $O(n^2)$

Sol 3.

$O(n)$

- <u>Exercise 4 (10/17)</u>:
  Design an $O(n)$-algorithm for the Max. Consecutive Sum Problem.

Computer Science and Information Engineering
National Chi Nan University

Chapter 5
Dynamic Programming

§ 5.3 Matrix Multiplication Problem

5.3 Matrix Multiplication Problem

- **Def: Matrix Multiplication Problem**

  求$M_1 \cdot M_2 \cdot \ldots \cdot M_n$用最少乘法次數之法,

  where M_i has size $a_{i-1} \times a_i$.

- **Ex 1:**

  $$A_{2\times2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ B_{2\times3} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

  $$AB = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} & a_{11}b_{13}+a_{12}b_{23} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} & a_{21}b_{13}+a_{22}b_{23} \end{pmatrix}$$

  $\Rightarrow$ need $12 = 2 \times 2 \times 3$

  $(\because (AB)_{ij} = \Sigma_{k=1}^{2} a_{ik}b_{kj},\ \forall i = 1, 2;\ j = 1, 2, 3)$

5.3 Matrix Multiplication Problem

- Ex 2:

$A_{1\times 10^5} \cdot B_{10^5\times 10^{10}} \cdot C_{10^{10}\times 1}$

$= (AB)C$ $\quad\quad = A(BC)$

$1\times 10^5\times 10^{10}$ $\quad\quad 10^5\times 10^{10}\times 1$

$1\times 10^{10}\times 1$ $\quad\quad 1\times 10^5\times 1$

$\Rightarrow 10^{15} + 10^{10}$ $\quad\quad \Rightarrow 10^{15} + 10^5$

Sol.

Let $f(i, j) = M_i \ldots M_j$ 最少相乘次數.

$f(i, i) = 0, \forall\ 1 \le i \le n.$

for $d = 1$ to $n - 1$

for $i = 1$ to $n - d$

$f(i, i + d) = \min_{i\le k\le i+d-1} \{f(i, k) + f(k + 1, i + d) + a_{i-1} \cdot a_k \cdot a_{i+d}\};$

end

end $\longrightarrow O(n^3)$