Computer Science and Information Engineering
National Chi Nan University

Chapter 3
Maximum Mathching Algorithm

§ 3.1 Maximum Matching in Bipartite Graph

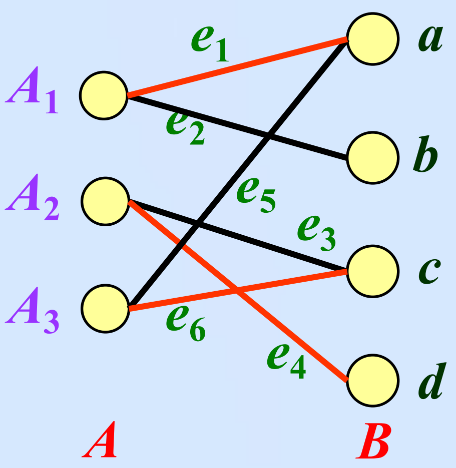
3.1 Maximum Matching in Bipartite Graph

- **Def:**

① **bipartite graph** ≡ V can be partitioned into V_1 and V_2,
$\forall$ edge $\{x, y\} \in E$, $|V_i \cap \{x, y\}| = 1$.

② **matching** ≡ a set of edges M s.t. $e_1, e_2 \in M$, $e_1 \neq e_2$ and $e_1 \cap e_2 = \phi$.

- **Ex:** bipartite graph $G = (A, B, E)$,
matching $M = \{e_1, e_4, e_6\}$,
$\Rightarrow A_1 = \{a, b\} \rightarrow a$
$A_2 = \{c, d\} \rightarrow d$
$A_3 = \{a, c\} \rightarrow c$



3.1 Maximum Matching in Bipartite Graph

- **Def**: $F = (A_1, A_2, \ldots, A_n)$ is a family of sets. An **SDR** of F is a sequence $(a_1, a_2, \ldots, a_n)$ of distinct elements, such that $a_i \in A_i$, $\forall\ 1 \le i \le n$. (SDR = **S**ystem of **D**istinct **R**epresentives)

- **P. Hall's Theorem:**

  $F = (A_1, A_2, \ldots, A_n)$ has an SDR $\Leftrightarrow |\cup_{i \in I} A_i| \ge |I|, \forall I \subset \{1, 2, \ldots, n\}$.(♣)

  Proof. (1/3)

  (⇒) Suppose $(a_1, \ldots, a_n)$ is an SDR of F,

  then $\forall\ I \subset \{1, 2, \ldots, n\}$: $|\cup_{i \in I} A_i| \ge |\cup_{i \in I} \{a_i\}| = |I|$.

3.1 Maximum Matching in Bipartite Graph

- **P. Hall's Theorem:**

$F = (A_1, A_2, \ldots, A_n)$ has an SDR $\Leftrightarrow |\cup_{i \in I} A_i| \geq |I|, \forall I \subset \{1, 2, \ldots, n\}$.(♣)

Proof. (2/3)

(⇐) We may assume that F is a minimal family s.t. Hall's condition(♣) holds.

claim: $|A_i| = 1, \forall\ i = 1, 2, \ldots, n$.

(Then $A_i = \{a_i\}$ and $(a_1, \ldots, a_n)$ is the desired SDR)

Assume $\exists\ |A_j| \geq 2$, say $|A_1| \geq 2$, choose $x \neq y$ in A_1.

Consider $F_x = (A_1 - \{x\}, A_2, \ldots, A_n)$,

$F_y = (A_1 - \{y\}, A_2, \ldots, A_n)$.

3.1 Maximum Matching in Bipartite Graph

- **P. Hall's Theorem:**

  $F = (A_1, A_2, \ldots, A_n)$ **has an SDR** $\Leftrightarrow |\cup_{i \in I} A_i| \geq |I|, \forall I \subset \{1, 2, \ldots, n\}$.(♣)

  Proof. (3/3)

  $\because$ F is minimal, $\therefore$ F_x, F_y does not satisfy (♣).

  i.e. $\exists\, I, J \subseteq \{2, 3, \ldots, n\}$ s.t. $C = (\cup_{i \in I} A_i) \cup (A_1 - \{x\})$,

  $D = (\cup_{i \in J} A_i) \cup (A_1 - \{y\})$

  $\rightarrow |C| < |I| + 1, |D| < |J| + 1 \Rightarrow |C| \leq |I|, |D| \leq |J|$.

  $\because$ $(\cup_{i \in I} A_i) \cap (\cup_{j \in J} A_j) \supseteq \cup_{i \in I \cap J} A_i$,

  $C \cup D = \cup_{i \in I \cup J \cup \{1\}} A_i$,

  $$\begin{aligned} \therefore\ |I| + |J| &\geq |C| + |D| = |C \cap D| + |C \cup D| \\ &\geq |I \cap J| + |I \cup J \cup \{1\}| \\ &= |I| + |J| + 1. \ \rightarrow\leftarrow \end{aligned}$$

3.1 Maximum Matching in Bipartite Graph

- **Def:** $t \geq 0$ is integer. $(A_1, \ldots, A_n)$ is called a **(t, n)-family** if $|\cup_{i\in I} A_i| \geq t + |I|$, $\forall\ \phi \neq I \subseteq \{1, 2, \ldots, n\}$.

- **Ex:** $A_i^* = \{i, n+1, \ldots, n+t\}$, $(1 \leq i \leq n)$

  $F^* = (A_1^*, A_2^*, \ldots, A_n^*)$.

- **Question:** F^* 有多少 SDR?

3.1 Maximum Matching in Bipartite Graph

- **Exercise 1 (10/3): Suppose $F = (A_1, \ldots, A_n)$ is a minimal (t, n)-family,**
  i.e. ① F is a (t, n)-family,
  ② the removal of any element from any set A_i results a family that isn't (t, n)-family.
  Then $|A_i| = t + 1$, $\forall\, i \in \{1, 2, \ldots, n\}$.

- **Conjecture: F^* 是有最少 SDR 的(t, n)-family, 當$t \geq 2$.**
  ($t = 2$ 已證出) (Europ. J. Comb. 10(1989) 231-234)

3.1 Maximum Matching in Bipartite Graph

- **Thm**: $G = (X, Y, E)$: bipartite graph,

  G has a matching of size $|X| \Leftrightarrow |\mathrm{Adj}(S)| \geq |S|, \forall\, S \subseteq X$.

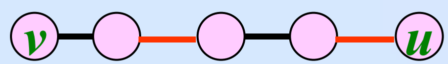
  Proof. (1/4)

  ($\Rightarrow$) o.k.

  ($\Leftarrow$) Suppose M is a maximum matching.

  If $|M| = |X|$, then we are done, otherwise, suppose $|M| < |X|$.

  Choose $v \in X$ s.t. v is not incident to any edge in M.

  Let $U = \{u \in X \cup Y$: $\exists$ 

  (alternating path)$\}$.

  Let $S^* = U \cap X$, $T^* = U \cap Y$.

  claim: $|\mathrm{Adj}(S^*)| < |S^*|$ (then $\rightarrow\leftarrow$)

3.1 Maximum Matching in Bipartite Graph

- <u>Thm</u>: $G = (X, Y, E)$: bipartite graph,

  G has a matching of size $|X| \Leftrightarrow |\mathrm{Adj}(S)| \geq |S|, \forall S \subseteq X$.

  Proof. (2/4)

  <u>claim</u>: $|\mathrm{Adj}(S^*)| < |S^*|$ (then $\rightarrow\leftarrow$)

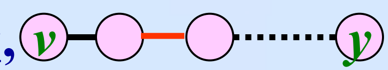
  check: ① $\mathrm{Adj}(S^*) \subseteq T^*$ ② $|S^*| \geq |T^*| + 1$

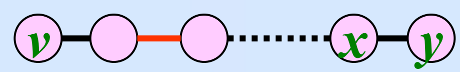
  (then $|S^*| > |T^*| \geq |\mathrm{Adj}(S^*)|$.)

  Proof. ① $\forall y \in \mathrm{Adj}(S^*)$

  $\Rightarrow \exists x \in S^* = U \cap X, xy \in E$

  let $P =$ 

  <u>case 1</u>: $y \in P$: $\exists$ P的一段, 

  <u>case 2</u>: $y \notin P$: $\exists$ $P \cup \{xy\}$: 

  i.e. $y \in U, y \in T^*$.

3.1 Maximum Matching in Bipartite Graph

- Thm: $G = (X, Y, E)$: bipartite graph,

  G has a matching of size $|X| \Leftrightarrow |\text{Adj}(S)| \geq |S|, \forall S \subseteq X$.

  Proof. (3/4)

  Check: ② $|S^*| \geq |T^*| + 1$

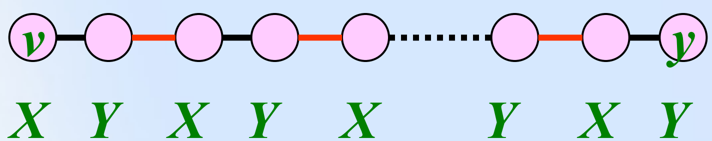
  Proof. $\forall y \in T^*, \exists P =$ 

.

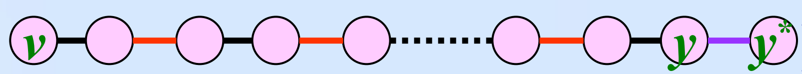
  If y is not incident to any edge in M (exposed),

  then $\exists M' = M \oplus P \equiv (M - P) \cup (P - M)$,

  M' is a matching of size $|M| + 1$. →←

  P: 

  Hence $\forall y \in T^*, \exists yy^* \in M$. Also $y^* \in S^*$.

  (since $\exists P$: 

)

3.1 Maximum Matching in Bipartite Graph

- **Thm**: $G = (X, Y, E)$: bipartite graph,

  G has a matching of size $|X| \Leftrightarrow |\mathrm{Adj}(S)| \geq |S|, \forall\, S \subseteq X$.

  Proof. (4/4)

  Check: ② $|S^*| \geq |T^*| + 1$

  Proof. Consider $f: T^* \rightarrow S^*$ by $f(y) = y^*$.

  f is 1-1: $\because$ M matching.

  f is not onto: $\because$ v exposed.

  $\therefore$ $|T^*| < |S^*|$, i.e. $|S^*| \geq |T^*| + 1$.

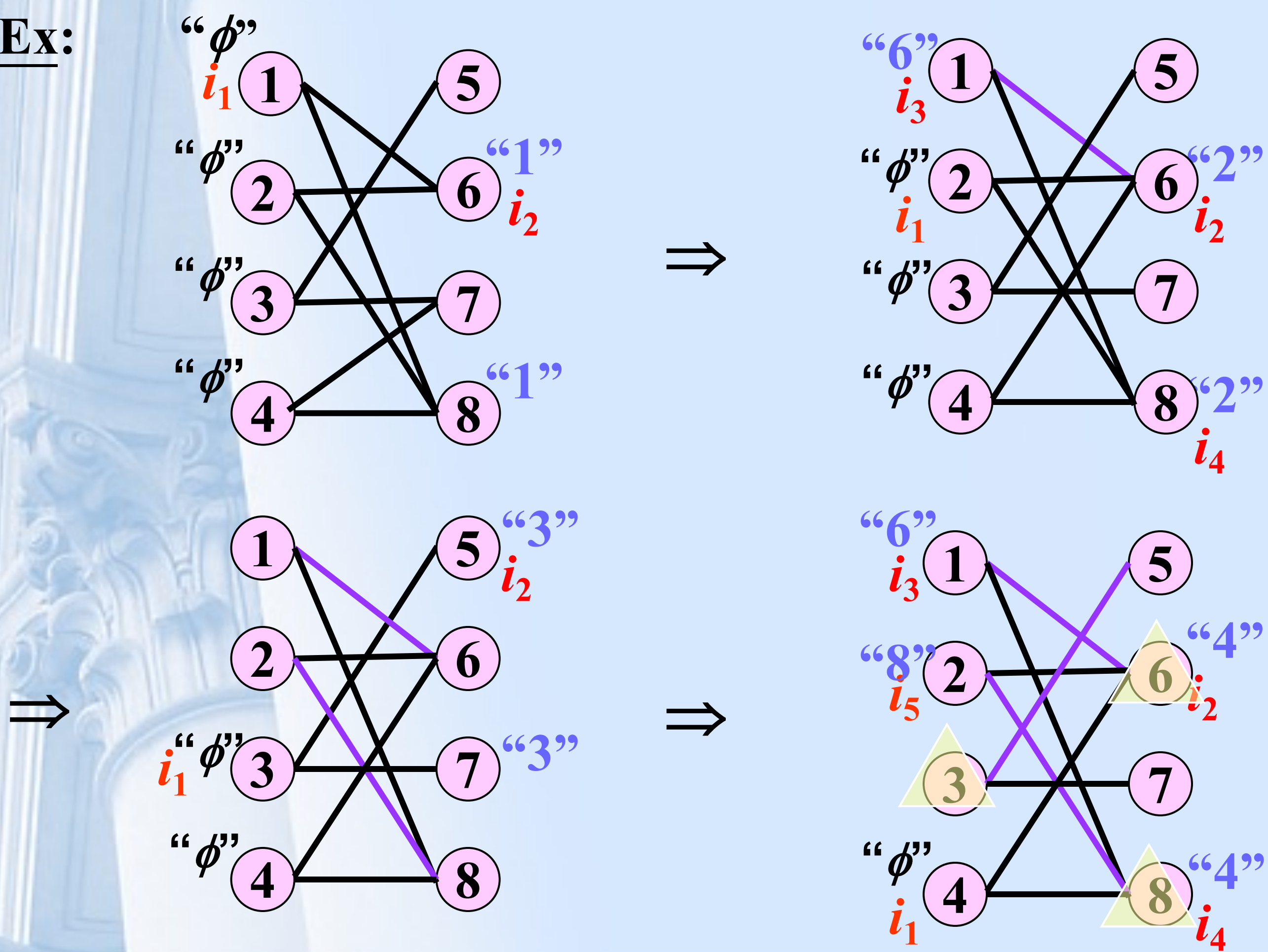
3.1 Maximum Matching in Bipartite Graph

- **Algorithm: Maximum Matching Algorithm for $G = (X, Y, E)$**

> (0) $M \leftarrow \phi$;
>
> **(1.0)** Given label "ϕ" to all M-exposed vertex in X;
>
> **(1.1)** If $\exists$ no unscanned labels then STOP,
>
> otherwise find a vertex i with unscanned label;
>
> If $i \in X$ then goto (1.2), otherwise goto (1.3);
>
> **(1.2)** Scan $i \in X$ by: $\forall$ edge $ij \in E$ with j has no label,
>
> label j by "i"; Goto (1.1);
>
> **(1.3)** Scan $i \in Y$ by: if i is exposed then goto (2),
>
> otherwise identify the unique $ij \in M$, label j by "i"; Goto (1.1);
>
> **(2)** Find P: 
>
> ; $M \leftarrow M \oplus P$;
>
> Remove all labels; Goto (1.0);

3.1 Maximum Matching in Bipartite Graph

- **Ex:**



3.1 Maximum Matching in Bipartite Graph

- **Def**: C is a **vertex-cover** of $G = (V, E)$
  if $C \subseteq V$ and every edge $xy \in E$ either $x \in C$ or $y \in C$.

- **Thm**: (Weak Duality Inequality, **w.d.i.**)
  max $|M| \leq$ min $|C|$

Proof. (1/2)

$\forall$ matching M; $\forall$ vertex cover C;

Define $f: M \to C$ by $f(xy) = \begin{cases} x, \text{ if } x \in C, \\ y, \text{ o.w..} \end{cases}$

① well-define:

If $x \notin C$ then C is vertex cover.

$\therefore$ by definition, $y \in C$. (o.w. $xy \in E$, $x \notin C$ and $y \notin C$)

3.1 Maximum Matching in Bipartite Graph

- **<u>Thm</u>: (Weak Duality Inequality, w.d.i.)**

  $\max |M| \le \min |C|$

Proof. (2/2)

② **<u>1-1</u>:**

If $f(xy) = f(x'y')$, but $xy \neq x'y'$ in M,

then ∃ two different edges in M have a common end vertex.

→← to M is a matching

∴ f is 1-1.

Hence $|M| \le |C|$, ∴ $\max|M| \le \min|C|$.

3.1 Maximum Matching in Bipartite Graph

- <justify Max. Matching Algorithm>

  Assume M^* is the final output M, and L^* is the set of all labeled vertices at final iteration.

  Let $C^* = (X - L^*) \cup (Y \cap L^*)$

  claim ①: M^* is a matching.

  claim ②: C^* is a vertex cover.

  claim ③: $|C^*| \le |M^*|$

  Then $|C^*| \le |M^*| \le \max|M| \le \min|C| \le |C^*|$,

  ∴ all "≤" are "="

  ⇒ ①′ M^* is a max matching.

  ②′ C^* is a min vertex cover.

  ③′ $\max_M |M| = \min_C |C|$.

3.1 Maximum Matching in Bipartite Graph

- **Proof of claim. (1/2)**
  - ① M^* is a matching by (0) and (2)
  - ② $\forall\ xy \in E, x \in X, y \in Y.$

    Suppose $x \notin C^*, y \notin C^*$

    $\Rightarrow x \in L, y \notin L$ when we scan the labeled vertex x,

    we MUST labeled y in (1.2).

    $\Rightarrow C^*$ is a vertex cover.
  - ③ ★ $\begin{cases} \forall\ x \in C^* \cap X = X - L \Rightarrow \exists\ e \in M^* \text{ incident to } x \text{ by (1.0).} \\ \forall\ y \in C^* \cap Y = Y \cap L \Rightarrow \exists\ e \in M^* \text{ incident to } y \text{ by (1.3).} \end{cases}$

    [∵是最後一次iteration, ∴只會在(1.1) ~ (1.3)跑, 不會到(2)]

    Define f: $C^* \rightarrow M^*$ by $f(x)$ = the edge in M^* incident to x.

3.1 Maximum Matching in Bipartite Graph

- **Proof of claim. (1/2)**

  Define f: $C^* \to M^*$ by $f(x)$ = the edge in M^* incident to x.

  (a) well-define:

  M^* is a matching 及(★), ∃! edge incident to x.

  (b) 1-1:

  Suppose $f(x) = f(y) = e$,

  i.e. $e = xy$ with $x \in X$, $y \in Y$

  when we scan y, we MUST label x by "y"

  $\to\leftarrow$ in (1.3) otherwise.

  $\Rightarrow |C^*| \leq |M^*|$

  Then ①′, ②′, ③′ holds.

3.1 Maximum Matching in Bipartite Graph

- **Time-Complexity for Max. Matching Algorithm for bipartite graph: $O(|V| \cdot |E|) = O(|V|^3)$.**

- **Homework 1: (Due day: 10/3)**

  將Maximum Matching Algorithm for $G = (X, Y, E)$實作出來。

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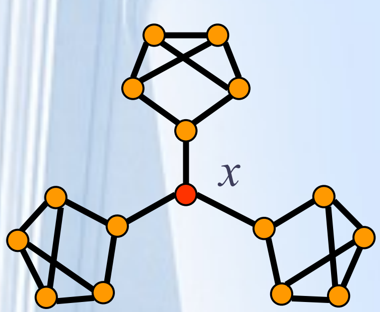
Chapter 3
Maximum Mathching Algorithm

§ 3.2 Maximum Matching in General Graph

3.2 Maximum Matching in General Graph

Ex:



沒有 perfect matching;
(由 cut vertex 觀之)

Def: 1. A matching M is called a **perfect matching** if $2|M| = |V|$.
2. A component of a graph G is **odd** or **even** iff it has an odd or even number of vertices.
3. Denote by $o(G)$ the number of odd components of G.

Theorem 5.2: (**Tutte's theorem**) A graph G has a perfect matching

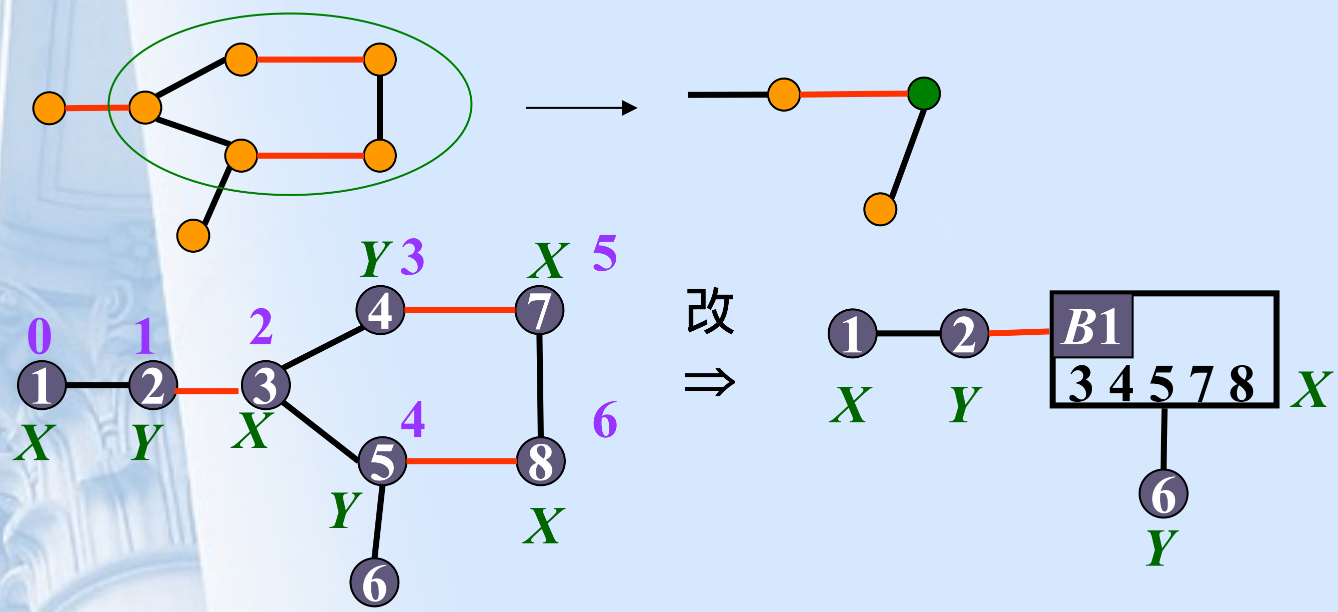
$$\Leftrightarrow o(G - S) \leq |S|, \forall\ S \subseteq V(G)$$

上Ex: Let $S = \{x\}$, $\because o(G - S) = 3 > 1 = |S|$

$\therefore$ G having no perfect matching.

3.2 Maximum Matching in General Graph

- <u>Def</u>: J. Edmonds (Blossom) Algorithm: C_{2k+1}: k edges $\in M$.
  (因為是odd cycle，兩條path中必有一條為even!)

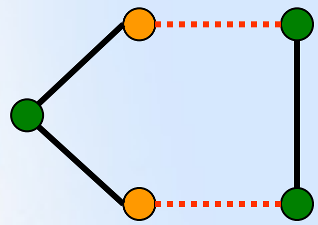


1 – 2 … 3 – 4 … 7 – 8 … 5 – 6

1 – 2 … *B*1 – 6

3.2 Maximum Matching in General Graph

- **<u>Note</u>: $\max_M |M| \neq \min_C |C|$ (非bipartite graph)**
- **<u>Ex</u>:**



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Maximum Mathching Algorithm

§ 3.3 Odd-Set Cover

3.3 Odd-Set Cover

- **Def**: $G = (V, E)$: graph,

  $\ell = \{v_1, v_2, \ldots, v_r, S_1, S_2, \ldots, S_t\}$ is an **odd-set cover** of G iff

  ① $r \geq 0$, $v_i \in V$ $\forall$ $1 \leq i \leq r$; $t \geq 0$, $S_j \subseteq V$, $|S_j|$: odd ≥ 3 $\forall$ $1 \leq i \leq t$.

  ② $\forall$ $xy \in E$, either $x =$ some v_i

  or $y =$ some v_i

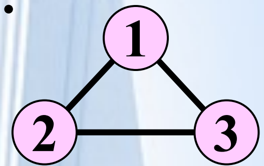
  or $\{xy\} \subseteq$ some S_i.

- **Note**: Vertex cover is an odd-set cover.

- **Def**: **value(ℓ)** $= r + \Sigma_{j=1}^{t} (|S_j| - 1) / 2$.

3.3 Odd-Set Cover

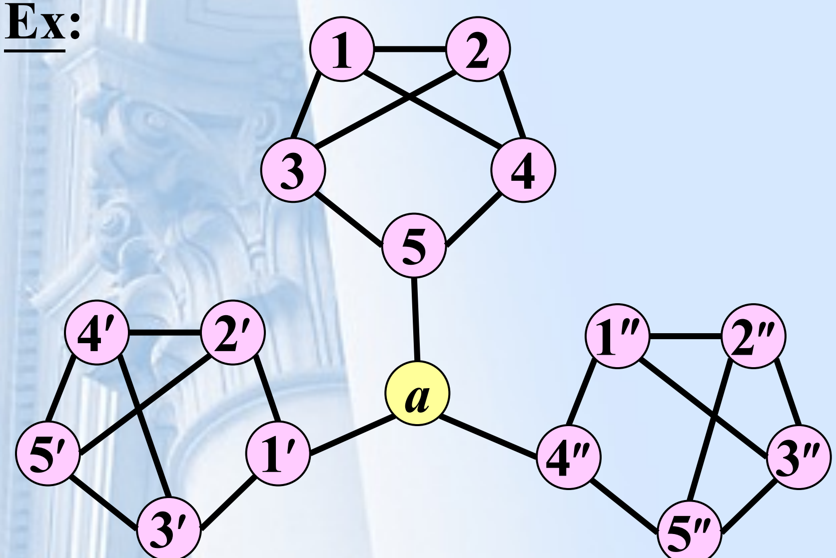
- **Ex:**



$\ell_1 = \{1, 2\}$ √ $\ell_2 = \{\{1, 2, 3\}\}$ √

$\ell_3 = \{1, \{2, 3\}\}$ ×

Ex:



① $\ell = \{a, \{1, 2, 3, 4, 5\}, \{1', 2', 3', 4', 5'\}, \{1'', 2'', 3'', 4'', 5''\}\}$

② $\text{value}(\ell) = 1 + 3 \times (5 - 1) / 2 = 7$

3.3 Odd-Set Cover

- **Weak Duality Inequality: $\max_M |M| \le \min_\ell$ value (ℓ).**

- **Exercise 2 (10/17):**

  Prove Weak Duality Inequality: $\max_M |M| \le \min_\ell$ value $|\ell|$.

- **Exercise 3 (10/17):**

  Use strong duality equality (for matching in general graph) $\max_M |M| = \min_\ell$ value $|\ell|$ to prove Tutte theorem.

分組

- **A(奇立), B(國城), C(聿辰), D(俊傑), E(翊豪), F(昀卲), G(文廷), H(烜嘉), I(安惠)**

- **第一次: AB, CE, DH, GI, F**
- **第二次: AC, BE, DF, GH, I**
- **第三次: AI, BC, DE, FG, H**