Chapter 5
Matchings and Independent Sets

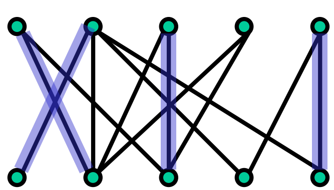
§ 5.1 Matchings

5.1 Matchings

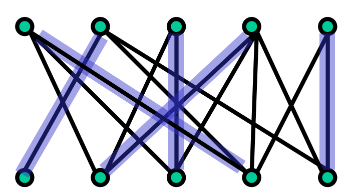
- **<u>Def</u>: G: nonempty and loopless graph**
  - **$M \neq \phi \subseteq E(G)$ is called a matching in G iff $\forall\ e_1, e_2, \in M$, e_1, e_2 are not adjacent in G.**
  - **$\forall\ e \in M$, if $e = (x, y)$, x, y are said to be matched under M.**
  - **$\forall\ x \in V(G)$, if $\exists\ e \in M$ s.t. $e = (x, y)$, then x is M-saturated, or say M saturates x; otherwise, x is M-unsaturated.**
  - **A matching M is perfect if it saturates every vertices of G.**
  - **M is maximum if $\forall$ matching M' in G, $|M| \geq |M'|$**

- **<u>Note</u>: Only discuss undirected graph.**

- **ex: (a)**



(b)



5.1 Matchings

- **Theorem 5.1: (Hall's theorem) Let $G = (X, Y, E)$ be a bipartite graph. Then**

  G contains a matching M that $|M| = |X| \Leftrightarrow |S| \leq |N_G(S)|\ \forall\ S \subseteq X$.

  Proof. (1/2)

  ($\Rightarrow$) Let $M = \{(x_i, y_i)|\ x_i \in X, y_i \in Y\}$ be a matching of G which saturates every $x_i \in X \Rightarrow$ all y_i are distinct

  $\therefore\ \forall\ S \subseteq X, N_G(S) \supseteq \{y_i|\ \forall\ x_i \in S\}$

  $\Rightarrow \forall\ S \subseteq X, |N_G(S)| \geq |S|$

5.1 Matchings

- **<u>Theorem 5.1</u>: (Hall's theorem) Let $G = (X, Y, E)$ be a bipartite graph. Then**

  G contains a matching M that $|M| = |X| \Leftrightarrow |S| \leq |N_G(S)|\ \forall\ S \subseteq X$.

  Proof. (2/2)

  ($\Leftarrow$) Suppose M is a maximum matching in G.

  Construct a digraph D: $V(D) = V(G) \cup \{x, y\}$

  $E(D) = \{(x, x_i)|\ x_i \in X\} \cup \{(x_i, y_j): x_i \in X, y_j \in Y, x_iy_j \in E(G)\} \cup \{(y_j, y)|\ y_j \in Y\}$

  $\zeta_D(x, y) = |M|$. Let T be a minimum (x, y)-separating set of D.

  By Menger's theorem, $|M| = \zeta_D(x, y) = \kappa_D(x\ y) = |T|$

  Let $T_1 = T \cap X$, $T_2 = T \cap Y$.

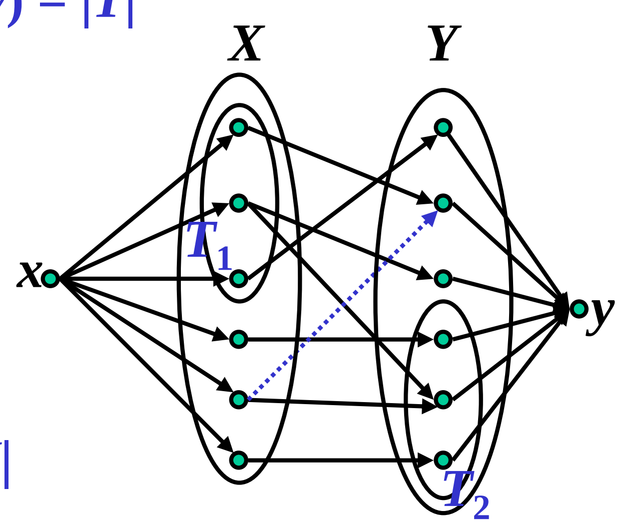
  $\Rightarrow E_D(X \backslash T_1, Y \backslash T_2) = \phi$

  $\Rightarrow N_D^+(X \backslash T_1) \subseteq T_2$

  $\Rightarrow |M| = |T| = |T_1| + |T_2|$

  $\geq |T_1| + |N_D^+(X \backslash T_1)|$

  $= |T_1| + |N_G(X \backslash T_1)| \geq |T_1| + |X \backslash T_1| = |X|$



5.1 Matchings

- **Def: $F = (A_1, A_2, \ldots, A_n)$ is a family of sets. An SDR of F is a sequence $(a_1, a_2, \ldots, a_n)$ of distinct elements, such that $a_i \in A_i$, $\forall\ 1 \le i \le n$. (SDR 是 System of Distinct Representires)**

- **P. Hall's Theorem:**

  $F = (A_1, A_2, \ldots, A_n)$ has an SDR $\Leftrightarrow |\cup_{i\in I} A_i| \ge |I|, \forall I \subset \{1, 2, \ldots, n\}$.(♣)

  Proof. (1/3) (略)

  (⇒) Suppose $(a_1, \ldots, a_n)$ is an SDR of F,

  then $\forall\ I \subset \{1, 2, \ldots, n\}$: $|\cup_{i\in I} A_i| \ge |\cup_{i\in I} \{a_i\}| = |I|$.

5.1 Matchings

- **P. Hall's Theorem:**

  $F = (A_1, A_2, \ldots, A_n)$ has an SDR $\Leftrightarrow |\cup_{i \in I} A_i| \geq |I|, \forall I \subset \{1, 2, \ldots, n\}$.(♣)

  Proof. (2/3)

  (⇐) We may assume that F is a minimal family s.t. Hall's condition(♣) holds.

  claim: $|A_i| = 1, \forall\, i = 1, 2, \ldots, n$.

  (Then $A_i = \{a_i\}$ and $(a_1, \ldots, a_n)$ is the desired SDR)

  Assume $\exists\, |A_j| \geq 2$, say $|A_1| \geq 2$, choose $x \neq y$ in A_1.

  Consider $F_x = (A_1 - \{x\}, A_2, \ldots, A_n)$,

  $F_y = (A_1 - \{y\}, A_2, \ldots, A_n)$.

5.1 Matchings

- **P. Hall's Theorem:**

  $F = (A_1, A_2, \ldots, A_n)$ **has an SDR** $\Leftrightarrow |\cup_{i\in I} A_i| \geq |I|, \forall I \subset \{1, 2, \ldots, n\}$.(♣)

  Proof. (3/3)

  $\because$ F **is minimal**, $\therefore$ F_x, F_y **does not satisfy (♣).**

  i.e. $\exists\ I, J \subseteq \{2, 3, \ldots, n\}$ **s.t.** $C = (\cup_{i\in I} A_i) \cup (A_1 - \{x\})$,

  $D = (\cup_{i\in J} A_i) \cup (A_1 - \{y\})$

  $\rightarrow |C| < |I| + 1, |D| < |J| + 1 \Rightarrow |C| \leq |I|, |D| \leq |J|$.

  $\therefore (\cup_{i\in I} A_i) \cap (\cup_{j\in J} A_j) \supseteq \cup_{i\in I\cap J} A_i$,

  $C \cup D = \cup_{i\in I\cup J\cup\{1\}} A_i$,

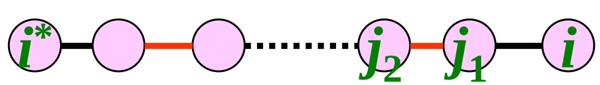
  $|I| + |J| \geq |C| + |D| = |C \cap D| + |C \cup D|$

  $\geq |I \cap J| + |I \cup J \cup \{1\}|$

  $= |I| + |J| + 1.$ $\rightarrow\leftarrow$

5.1 Matchings

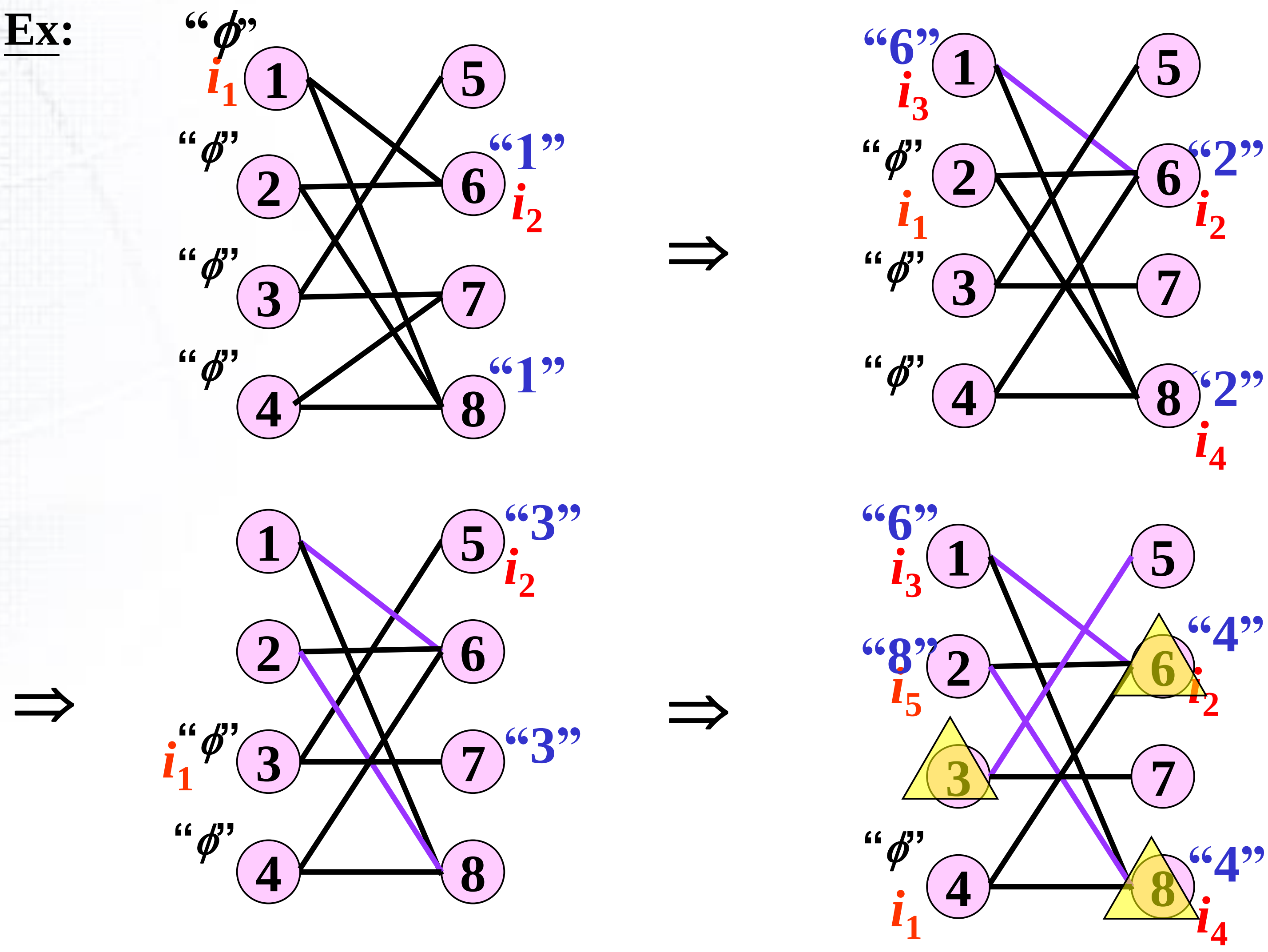
- **Algorithm: Maximum Matching Algorithm for $G = (X, Y, E)$**

> (0) $M \leftarrow \phi$;
>
> **(1.0)** Given label "ϕ" to all M-unsaturated vertex in X;
>
> **(1.1)** If $\exists$ no unscanned labels then STOP,
> otherwise find a vertex i with unscanned label;
> If $i \in X$ then goto (1.2), otherwise goto (1.3);
>
> **(1.2)** Scan $i \in X$ by: $\forall$ edge $ij \in E$ with j has no label,
> label j by "i"; Goto (1.1);
>
> **(1.3)** Scan $i \in Y$ by: if i is exposed then goto (2),
> otherwise identify the unique $ij \in M$, label j by "i"; Goto (1.1);
>
> **(2)** Find P: 
>
> ; $M \leftarrow M \oplus P$;
> Remove all labels; Goto (1.0);

- **Time-Complexity for Max. Matching Algorithm for bipartite graph: $O(|V| \cdot |E|) = O(|V|^3)$.**

5.1 Matchings

- **Ex:**



5.1 Matchings

- **Def: C is a vertex-covering of $G = (V, E)$**
  if $C \subseteq V$ and every edge $xy \in E$ either $x \in C$ or $y \in C$.

- **Thm: (Weak Duality Inequality, w.d.i.)**
  max $|M| \leq$ min $|C|$

Proof. (1/2)

$\forall$ matching M; $\forall$ vertex cover C;

Define f: $M \to C$ by $f(xy) = \begin{cases} x, \text{ if } x \in C, \\ y, \text{ o.w.} \end{cases}$

① well-define:

If $x \notin C$ then C is vertex cover.

$\therefore$ by definition, $y \in C$. (o.w. $xy \in E$, $x \notin C$ and $y \notin C$)

5.1 Matchings

- **Thm: (Weak Duality Inequality, w.d.i.)**

  max $|M| \leq$ min $|C|$

Proof. (2/2)

② 1-1:

If $f(xy) = f(x'y')$, but $xy \neq x'y'$ in M,

then ∃ two different edges in M have a common end vertex.

→← to M is a matching

∴ f is 1-1.

Hence $|M| \leq |C|$, ∴ max$|M| \leq$ min$|C|$.

5.1 Matchings

- **<justify Max. Matching Algorithm>**

Assume M^* is the final output M, and L^* is the set of all labeled vertices at final iteration.

Let $C^* = (X - L^*) \cup (Y \cap L^*)$

claim ①: M^* is a matching.

claim ②: C^* is a vertex cover.

claim ③: $|C^*| \le |M^*|$

Then $|C^*| \le |M^*| \le \max|M| \le \min|C| \le |C^*|$,

∴ all "≤" are "="

⇒ ①′ M^* is a max matching.

②′ C^* is a min vertex cover.

③′ $\max_M |M| = \min_C |C|$.

5.1 Matchings

- **Proof of claim. (1/2) (略)**

  ① M^* is a matching by (0) and (2)

  ② $\forall\ xy \in E, x \in X, y \in Y.$

  Suppose $x \notin C^*, y \notin C^*$

  $\Rightarrow x \in L, y \notin L$ when we scan the labeled vertex x,

  we MUST labeled y in (1.2).

  $\Rightarrow C^*$ is a vertex cover.

  ③ ★ $\begin{cases} \forall\ x \in C^* \cap X = X - L \Rightarrow \exists\ e \in M^* \text{ incident to } x \text{ by (1.0).} \\ \forall\ y \in C^* \cap Y = Y \cap L \Rightarrow \exists\ e \in M^* \text{ incident to } y \text{ by (1.3).} \end{cases}$

  [∵是最後一次iteration, ∴只會在(1.1) ~ (1.3)跑, 不會到(2)]

  Define $f: C^* \to M^*$ by $f(x)$ = the edge in M^* incident to x.

5.1 Matchings

- **Proof of claim. (2/2)**

  Define f: $C^* \to M^*$ by $f(x)$ = the edge in M^* incident to x.

  (a) well-define:

  M^* is a matching 及(★), ∃! edge incident to x.

  (b) 1-1:

  Suppose $f(x) = f(y) = e$,

  i.e. $e = xy$ with $x \in X$, $y \in Y$

  when we scan y, we MUST label x by "y"

  $\rightarrow\leftarrow$ in (1.3) otherwise.

  $\Rightarrow |C^*| \leq |M^*|$

  Then ①′, ②′, ③′ holds.

5.1 Matchings

- **Corollary 5.1.1**: $G = (X, Y, E)$ is a bipartite graph.
  G has a perfect matching $\Leftrightarrow |X| = |Y|$ and $|S| \le |N_G(S)|\ \forall\ S \subseteq X$ or Y.

- **Corollary 5.1.2**: If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.

Proof.

Let $G = (X, Y, E)$ be a k-regular bipartite graph.

① $\because\ k|X| = k|Y| = |E|.\ \therefore\ |X| = |Y|$

② Let $S \subseteq X$, let $E_1 = \{xy \mid x \in S \text{ and } xy \in E\}$
let $E_2 = \{xy \mid y \in N_G(S) \text{ and } xy \in E\}$ $\Rightarrow E_1 \subseteq E_2$

$\Rightarrow k|S| = |E_1| \le |E_2| = k|N_G(S)|$

$\Rightarrow |S| \le |N_G(S)|$

$\therefore$ By ① ② and Hall's theorem, M is a perfect matching in G.
(Corollary 5.1.1)

5.1 Matchings

- **Corollary 5.1.3: Let G be an equally bipartite simple graph of order $2n$.**

  $\delta(G) \geq n/2 \Rightarrow G$ has a perfect matching.

  Proof.

  Let $G = (X, Y, E)$ be an equally bipartite simple graph.

  Suppose $\exists\ S \subseteq X$ s.t. $|S| > |N_G(S)|$

  $\because |X| = |Y| \quad \therefore Y \backslash N_G(S) \neq \phi$

  $\because G$ **is simple** $\quad \therefore |S| > |N_G(S)| \geq \delta(G) \geq n/2$

  Let $u \in Y \backslash N_G(S)$, then $N_G(u) \subseteq X \backslash S$

  $\Rightarrow \delta(G) \leq d_G(u) = |N_G(u)| \leq |X| - |S| < n/2 \quad \rightarrow\leftarrow$

  $\therefore \forall\ S \subseteq X, |S| \leq |N_G(S)|$

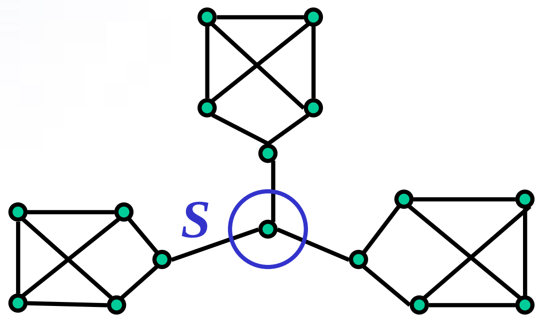
  By Hall's theorem, and $|X| = |Y| \quad \therefore G$ has a perfect matching.

5.1 Matchings

- **Def:**
  - A component of a graph G is odd or even iff it has an odd or even number of vertices.
  - Denote by $o(G)$ the number of odd components of G.

- **Theorem 5.2: (Tutte's theorem) A graph G has a perfect matching**
  $$\Leftrightarrow o(G - S) \leq |S|, \forall S \subseteq V(G)$$

- **ex:**



$\because o(G - S) = 3 > 1 = |S|$

$\therefore$ **G having no perfect matching.**

5.1 Matchings

- **<u>Corollary 5.2.1</u>: Every k-regular $(k - 1)$-edge connected graph of even order has a perfect matching $\forall\, k \geq 1$.**

Proof. (1/2) (略)

Let G be a k-regular $(k - 1)$-dege connected graph of even order.

If $k = 1$, the result holds clearly.

Suppose $k \geq 2$, let $S \subseteq V(G)$ and $S \neq V(G)$

1. If $S = \phi$, $\because$ G is even order, $\therefore$ $o(G - S) = 0 \leq 0 = |S|$.

2. If $S \neq \phi$, let $G_1, G_2, \ldots, G_n$ be all odd components of $G - S$, and let $m_i = |(V(G_i), S)|$, $v_i = |V(G_i)|$.

$\because$ $\lambda(G) \geq k - 1$, $\therefore$ $m_i \geq k - 1$ $\forall\, i = 1, 2, \ldots, n$

If $\exists\, i$ $(1 \leq i \leq n)$ s.t. $m_i = k - 1$, then

$\varepsilon(G_i) = (kv_i - k + 1)/2 = k(v_i - 1)/2 + 1/2 \notin Z \rightarrow\leftarrow$

$\Rightarrow m_i \geq k$ $\forall\, i = 1, 2, \ldots, n$

5.1 Matchings

- **Corollary 5.2.1: Every k-regular $(k-1)$-edge connected graph of even order has a perfect matching $\forall\, k \geq 1$.**

Proof. (2/2)

Suppose $k \geq 2$, let $S \subseteq V(G)$ and $S \neq V(G)$

2. If $S \neq \phi$, let $G_1, G_2, \ldots, G_n$ be all odd components of $G - S$, and let $m_i = |(V(G_i), S)|$, $v_i = |V(G_i)|$.

$$\Rightarrow m_i \geq k \;\forall\, i = 1, 2, \ldots, n$$

$$\Rightarrow o(G-S) = n \leq \frac{1}{k}\sum_{i=1}^{n} m_i \leq \frac{1}{k}\sum_{u\in S} d_G(u) = |S|.$$

By Tutte's theorem $\therefore$ G has a perfect matching.

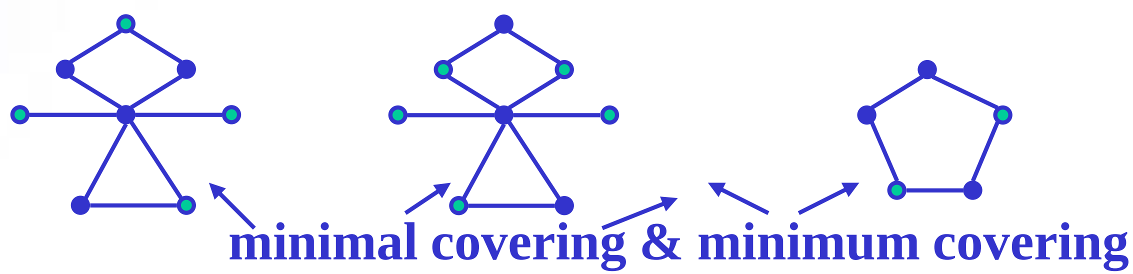
- **Corollary 5.2.2: Every 2-edge connected and 3-regular graph has a perfect matching.**

- **Note: Hall's theorem $\Leftrightarrow$ Menger's theorem $\Leftrightarrow$ Tutte's theorem $\Leftrightarrow$ König's theorem**

5.1 Matchings

- **Def**: G: a loopless graph.
  - ① $K \neq \phi \subseteq V(G)$ is a (**vertex-**) **covering** of G if $\forall e \in E(G)$, $\exists\, x \in K$ s.t. x is an end-vertex of e.
  - ② A covering K is **minimum** if $|K| \leq |K'|$ $\forall$ any covering K' of G.
  - ③ A covering K is **minimal** if $K \backslash \{x\}$ is not a covering of G, $\forall\, x \in K$.
  - ④ The **covering number** of G, $\beta(G) = |\{x|\ x \in K, K \text{ is a minimum covering}\}|$
  - ⑤ The **matching number** of G, $\alpha'(G) = |\{e|\ e \in M, M \text{ is a maximum matching}\}|$

- ex: ①

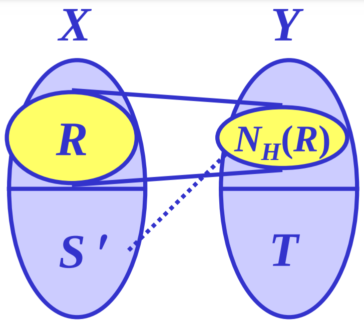


② $\beta(K_n) = n - 1$; $\alpha'(K_n) = \lfloor n/2 \rfloor$

$\beta(C_n) = \lceil n/2 \rceil$; $\alpha'(C_n) = \lfloor n/2 \rfloor$

$\beta(K_{m,n}) = \min\{m, n\}$; $\alpha'(K_{m,n}) = \min\{m, n\}$

5.1 Matchings



- **Remark**: ∀ loopless graph G, $\alpha'(G) \le \beta(G)$.
- **Theorem 5.3: (König's theorem) ∀ bipartite graph G, $\alpha'(G) = \beta(G)$.**

Proof. (略)

By remark, we need to only prove $\alpha'(G) \ge \beta(G)$.

Let $\{X, Y\}$ be a bipartition of G, K is a minimum covering of G.

Let $S = K \cap X$, $T = K \cap Y$, $S' = X \backslash S$, $T' = Y \backslash T$

By definition of S, $[S', T'] = \phi$.

Consider $H = G[S \cup T']$, ∵ K is minimum ∴ $\forall R \subseteq S$, $|R| \le |N_H(R)|$

$\Rightarrow$ By Hall's theorem, H has a matching M_1 saturating S.

Similarly, $G[S' \cup T]$ has a matching M_2 saturating T.

$\Rightarrow M_1 \cup M_2$ is a matching of G and $M_1 \cap M_2 = \phi$.

$\therefore \beta(G) = |K| = |S| + |T| = |M_1| + |M_2| = |M_1 \cup M_2| \le \alpha'(G)$

5.1 Matchings

Theorem 5.3: (König's theorem) $\forall$ bipartite graph G, $\alpha'(G) = \beta(G)$.

- **Corollary 5.3: Let G be an equally bipartite simple graph of order $2n$,**

  $$\varepsilon > (k-1)n \text{ for } k \geq 1 \Rightarrow \alpha'(G) = \beta(G) \geq k.$$

  Proof.

  By König's theorem, need to only prove $\beta(G) \geq k$.

  $\because$ G is simple and equally bipartite, $\therefore$ $\forall\, x \in V(G)$, $d_G(x) \leq n$.

  Suppose $\beta(G) \leq k - 1$, then $$(k-1) \cdot n < \varepsilon(G) \leq \beta(G) \cdot \Delta \leq \beta(G) \cdot n \leq (k-1) \cdot n \quad \rightarrow\leftarrow$$

  $\therefore$ $\beta(G) \geq k$.

5.1 Matchings

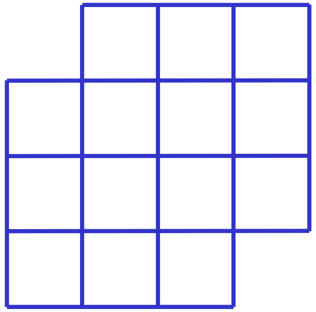
•**Corollary 5.1.1**: $G = (X, Y, E)$ is a bipartite graph.

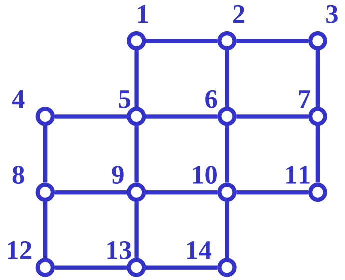
G has a perfect matching $\Leftrightarrow |X| = |Y|$ and $|S| \leq |N_G(S)|\ \forall\ S \subseteq X$ or Y.

- **Example 5.1.1**: It is impossible, using 1×2 rectangles, to exactly cover an 4×4 square from which two opposite 1×1 corner squares have been removed.

Sol.

Construct a simple graph G as:





The problem can be reduced to proving that G has no perfect matching.

$\Rightarrow G$ is a bipartite graph with $X = \{1, 3, 4, 6, 9, 11, 12, 14\}$, $Y = \{2, 5, 7, 8, 10, 13\}$.

$\because$ and $|X| = 8 > 6 = |Y|$,

$\therefore G$ has no perfect matching by Corollary 5.1.1.

5.1 Matchings

- **Example 5.1.2: skip**
- **課本附Hall's Thm, Tutte's Thm 及 König's Thm的直接證明，請參考。**
- **Note: Exercise 5.1.4, 5.1.5 are Hall’s Thm 的變形**
- **Exercise: 5.1.2(a)**

加: 5.1.2(e); 5.1.3; 5.1.6; 5.1.11(c)

Chapter 5
Matchings and Independent Sets

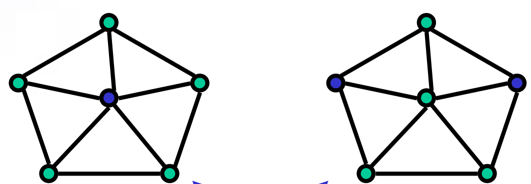
§ 5.2 Independent Sets

5.2 Independent Sets

- **Def**: G: a loopless graph.
  - $I \neq \phi \subseteq V(G)$ is called an **independent set** of G iff $\forall\, x, y \in I, xy \notin E(G)$
  - I is called **maximum** if $\forall$ independent set I' of G, $|I| \geq |I'|$
  - I is called **maximal** if $\forall\, x \in V(G)\backslash I$, $I \cup \{x\}$ is not an independent set.
  - The **independent number** of G, $\alpha(G) \equiv |\{x|\ x \in I, I$ is a maximum indep. set$\}|$

- **Note**: Only discuss simple undirected graph.

- ex: (a)



maximal independent set & maximum independent set

(b) $\alpha(K_n) = 1$, $\alpha(C_{2n}) = n = \alpha(C_{2n+1})$ (or $\alpha(C_n) = \lfloor n/2 \rfloor$)

$\alpha(K_{m,n}) = \max\{m, n\}$

5.2 Independent Sets

- **<u>Theorem 5.4</u>: $I \subseteq V(G)$ is an independent set of a loopless graph $G \Leftrightarrow$ $V(G) \backslash I$ is a covering of G.**

  Proof.

  I is an independent set of G

  $\Leftrightarrow \forall\ x, y \in I,\ xy \notin E(G)$

  $\Leftrightarrow \forall\ xy \in E(G),\ x \notin I$ or $y \notin I$

  $\Leftrightarrow \forall\ xy \in E(G),\ x \in V(G) \backslash I$ or $y \in V(G) \backslash I$

  $\Leftrightarrow V(G) \backslash I$ is a covering of G.

- **<u>Corollary 5.4.1</u>: $I \subseteq V(G)$ is a maximal (maximum) independent set of $G \Leftrightarrow$ $V(G) \backslash I$ is a minimal (minimum) covering of G.**

- **<u>Corollary 5.4.2</u>: $\forall$ loopless graph G, $\alpha(G) + \beta(G) = \nu(G)$**

5.2 Independent Sets

- **Def:**
  - $L \subseteq E(G)$ is called an **edge-covering** of G iff
    $\forall\, x \in V(G), \exists\, e \in L$ s.t. x is an end-vertex of e.
  - The **edge-covering number** of G, $\beta'(G) = |\{e|\ e \in L, L$ is a minimum edge-covering of $G\}|$

- **Note:** G has an edge-covering $\Leftrightarrow$ G contains no isolated vertex

- **ex:** $\beta'(K_n) = \lceil n/2 \rceil$,
  $\beta'(C_n) = \lceil n/2 \rceil$,
  $\beta'(K_{m,n}) = \max\{m, n\}$.

5.2 Independent Sets

- **<u>Theorem 5.5</u>: ∀ graph G with $\delta(G) > 0$, $\alpha'(G) + \beta'(G) = \nu(G)$.**

Proof. (1/2) (略)

① Let M be a maximum matching of G and

U be the set of M-unsaturated vertices of G.

$\because \delta(G) > 0$,

$\therefore \forall x \in U, \exists e_x$ s.t. x incident to e_x

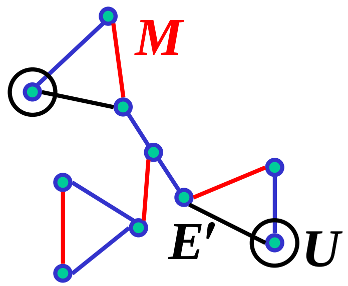
Let $E' = \{e_x \mid \forall x \in U\}$

$\Rightarrow M \cup E'$ is an edge-covering of G.

$\therefore \alpha' + \beta' \leq |M| + |M \cup E'|$

$= \alpha' + [\alpha' + (\nu - 2\alpha')]$

$= \nu$



5.2 Independent Sets

- **Theorem 5.5**: $\forall$ graph G with $\delta(G) > 0$, $\alpha'(G) + \beta'(G) = \nu(G)$.

Proof. (2/2)

② Let L be a minimum edge-covering of G,

Let $H = G[L]$,

Let M be a maximum matching of H and

Let U be the set of M-unsaturated vertices in H.

$\because$ $H[U]$ has no edge,

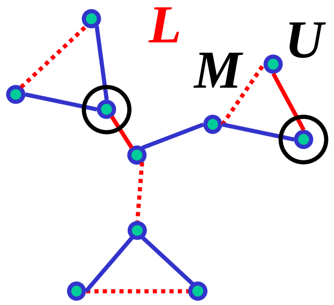
$\therefore$ $|L| - |M| = |L \backslash M|$

$\geq |U|$

$= \nu - 2|M|$

$\Rightarrow \alpha' + \beta' \geq |L| + |M| \geq \nu$

By ① ②, $\alpha'(G) + \beta'(G) = \nu(G)$.



5.2 Independent Sets

Theorem 1.9: G: a simple undirected graph of $\nu \geq 3$.

$$d_G(x) + d_G(y) \geq \nu, \forall x, y \in V(G), xy \notin E(G) \Rightarrow G \text{ is hamiltonian}$$

- **Corollary 5.5**: $\forall$ bipartite graph with $\delta(G) > 0$, $\alpha(G) = \beta'(G)$
- **Note**: If G is nonempty, $\alpha'(G) = \alpha(L(G))$
- **Theorem 5.6**: G: simple undirected graph
  $$\forall x, y \in V(G), xy \notin E(G), d_G(x) + d_G(y) \geq \nu \Rightarrow \alpha(G) \leq \kappa(G)$$
- **Corollary 5.6**: G: simple graph with $\delta(G) \geq \nu/2 \Rightarrow \alpha(G) \leq \kappa(G)$
- **Theorem 5.7**: G: simple undirected graph of order $\nu \geq 3$.
  $$\alpha(G) \leq \kappa(G) \Rightarrow G \text{ is hamiltonian.}$$
- **Note: Theorem 5.6 + 5.7 $\Rightarrow$ Theorem 1.9**

5.2 Independent Sets

- **<u>Def</u>:**
  - $S \neq \phi \subseteq V(G)$ **is called a** **dominating set** **of** $G \Leftrightarrow$
    $\forall\, x \in V(G) \backslash S, \exists\, y \in S$ **s.t.** $xy \in E(G)$.
  - S **is called to be** **minimal** **if** $\forall\, S' \subseteq S$, S' **is not a dominating set.**
  - **The** **domination number** **of** G, $\gamma(G) \equiv \min \{|S|: S$ **is a dominating set of** $G\}$

- **Note:** ① S **is a minimal dominating set of** $G \Rightarrow V(G) \backslash S$ **is a dominating set** $(\delta(G) > 0)$

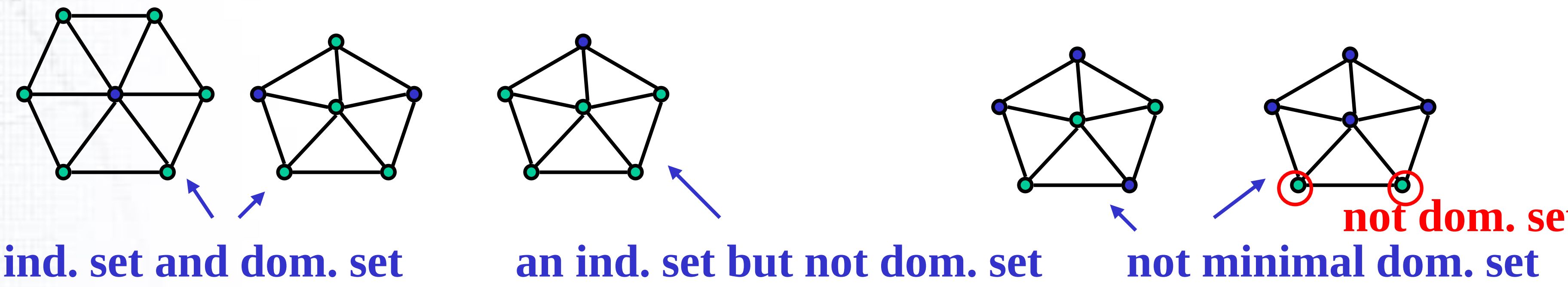
  ② $\gamma(G) \leq (1/2)\nu(G)$

  ③ **An independent set** S **of** G **is a dominating set of** $G \Leftrightarrow$
  S **is a maximal independent set.**

  ④ $\alpha(G) \geq \gamma(G)$

5.2 Independent Sets

- ex:



- exercise: 5.2.1

- 加: 5.2.5, 5.2.6, 5.2.7