Chapter 2
Trees and Graphic

§ 2.2 Vector Spaces of Graphs

2.2 Vector Spaces of Graphs

- **Def:** ① $\begin{cases} \text{vertex-space } \mathcal{V}(G) \equiv \text{the vector space of all functions from } V(G) \text{ into } R. \\ \text{edge-space } \mathcal{E}(G) \equiv \text{the vector space of all functions from } E(G) \text{ into } R. \end{cases}$

  ② (G, w) is called a **weighted graph**: G: a loopless graph, $\mathrm{w} \in \mathcal{E}(G)$.
  - w is called a **weighted function** and
  - $\mathrm{w}(a)$ is called a **weight** of the edge a of G
  - write $\mathrm{w}(x, y)$ for $\mathrm{w}((x, y))$ if $(x, y) \in E(G)$

  ③ $\forall B \subseteq E(G)$ in (G, w) write $\mathrm{w}(B) = \sum_{a \in B} \mathrm{w}(a)$

  $\forall S \subseteq V(G)$, $S \neq \phi$ in (G, w), write $\begin{cases} \mathrm{w}^+(S) = \mathrm{w}(E_G^+(S)); \\ \mathrm{w}^-(S) = \mathrm{w}(E_G^-(S)). \end{cases}$

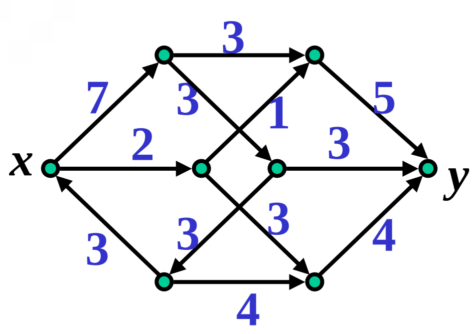
Chapter 4
Flows and Connectivity

§ 4.1 Network Flows

4.1 Network Flows

- **Def:**
  - A connected weighted loopless graph (G, w) with two specified vertices x, y, called the **source** and the **sink**, respectively, is called a **network (N = (G_{xy}, w))**.
  - W.L.O.G., a network is a simple digraph.
  - If w is a nonnegative **capacity function** c, then the network N = (G_{xy}, c) is called a **capacity network**, and the value c(a) is the **capacity** of a.
  - If c(a) is an integer for any $a \in E(G)$, then N is called an **integral capacity network**.

- **ex:**



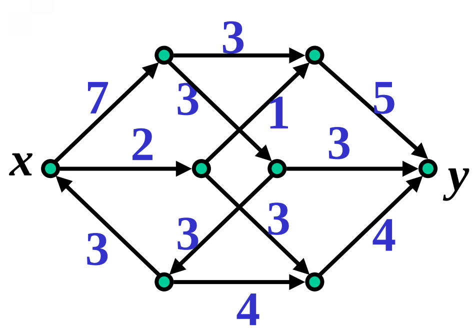
4.1 Network Flows

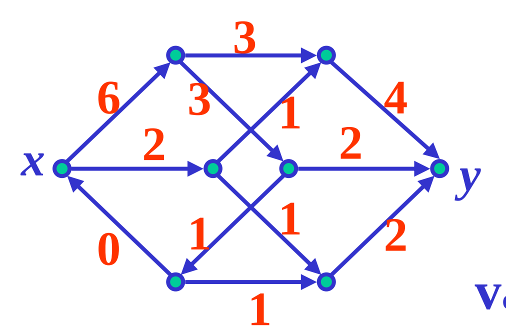
- **Def:**
  - $N = (G_{xy}, c)$ is a capacity network. A function $f: E(G) \to \mathbb{R}$ is called a **flow** in N from x to y, in short **(x, y)-flow**, if it satisfies:

    (4.1) $0 \le f(a) \le c(a), \forall a \in E(G)$, **the capacity constraint condition**

    (4.2) $f^+(u) = f^-(u), \forall u \in V(G) \setminus \{x, y\}$, **the conservation condition**
  - **zero flow**: $f(a) = 0, \forall a \in E(G)$
  - The **value** of f, **val f** $\equiv f^+(x) - f^-(x) = f^-(y) - f^+(y)$
  - An (x, y)-flow f in N is **maximum** if $\nexists$ (x, y)-flow f′ in N s.t. val f′ > val f.

- **ex:**



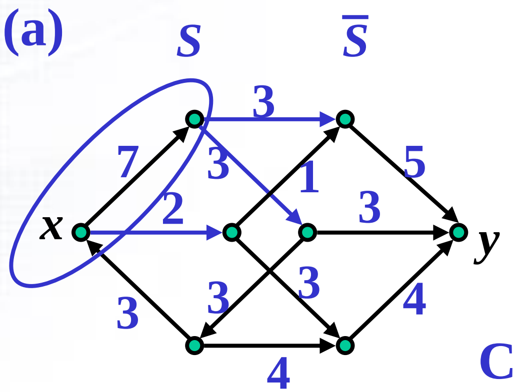


val f = 8

4.1 Network Flows

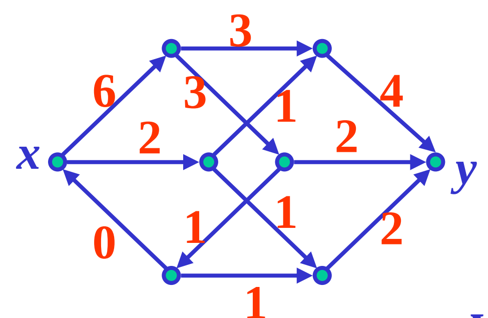
- **<u>Def</u>:**
  - An *(x, y)*-cut in *N* is a set of edges of the form $(S, \bar{S})$, where $x \in S$ and $y \in \bar{S}$.
  - The capacity of an (x, y)-cut B, cap $B \equiv c(B) = \sum_{a \in B} c(a)$
  - An (x, y)-cut B in N is minimum if $\nexists$ (x, y)-cut B' in N s.t. cap B' < cap B.

- **ex:** (a)



Cap B = 8

(b)



val f = 8

4.1 Network Flows

- **Theorem 4.1: (max-flow min-cut theorem) In any capacity network, the value of a maximum flow is equal to the capacity of a minimum cut.**

- **Corollary 4.1: In any integral capacity network, there must be an integral maximum flow, and its value is equal to the capacity of a minimum cut.**

Chapter 4
Flows and Connectivity

§ 4.2 Menger's Theorem

4.2 Menger's Theorem

- **Def**: ① $x, y \in V(G)$, (x, y)-paths $P_1, P_2, \ldots, P_n$ in G is called
  - **internally disjoint** if $V(P_i) \cap V(P_j) = \{x, y\}$, $\forall\ 1 \le i \ne j \le n$.
  - **edge-disjoint** if $E(P_i) \cap E(P_j) = \phi$, $\forall\ 1 \le i \ne j \le n$.

  ② The maximum numbers of internally disjoint and edge-disjoint (x, y)-paths in G is denoted by $\zeta_G(x, y)$ and $\eta_G(x, y)$, respectively. /zeta/, /eta/

- **Def**: ① $\lambda_G(x, y) \equiv$ minimum number of edges in an (x, y)-cut in G, which is called the **local edge-connectivity** of G.

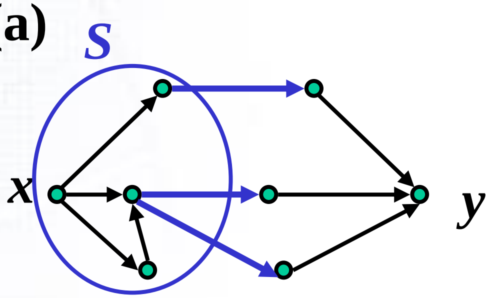
  ② $\phi \ne S \subseteq V(G) \setminus \{x, y\}$ is said to be an **(x, y)-separating set** in G if $\nexists$ (x, y)-path in $G - S$.

  ③ $\kappa_G(x, y) \equiv$ minimum cardinality of an (x, y)-separating set in G, which is called the **local (vertex-)connectivity** of G.

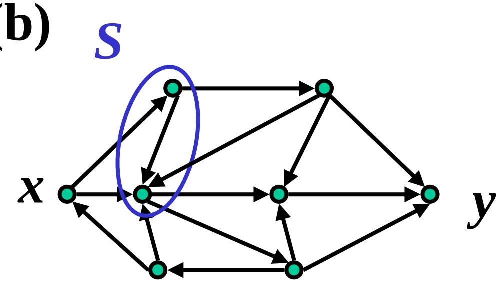
4.2 Menger's Theorem

- **Remark**: ① $\eta_G(x, y) \leq \lambda_G(x, y), \forall\, x, y \in V(G)$.
  ② $\zeta_G(x, y) \leq \kappa_G(x, y), \forall\, x, y \in V(G)$.

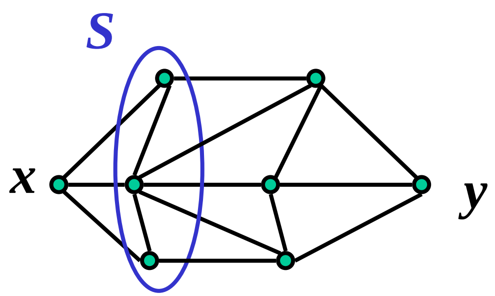
- ex: (a)



(b)



(c)



4.2 Menger's Theorem

- **Corollary 4.1: In any integral capacity network, there must be an integral maximum flow, and its value is equal to the capacity of a minimum cut.**

- **Theorem 4.2: Let x, y be two distinct vertices in a graph G. Then $\eta_G(x, y) = \lambda_G(x, y)$.**

Proof.

Consider a capacity network $N = (G_{xy}, c)$ with $c(e) = 1, \forall e \in E(G)$.

By Corollary 4.1, $\exists$ a max. (x, y)-integral flow f and a min. (x, y)- cut B s.t.

$$\text{val } f = \text{cap } B.$$

$\Rightarrow \lambda_G(x, y) \le |B| = \text{cap } B = \text{val } f$

Let $H = G_f$, the support of f (= the subgraph of G induced by the set of edges at which the value of f is nonzero.)

$\because$ $c(e) = 1, \forall e \in E(G)$, $\therefore$ $f(a) = 1 \ \forall e \in E(H)$.

$$\Rightarrow \begin{cases} d_H^+(x) - d_H^-(x) = \text{val } f = d_H^-(y) - d_H^+(y), \\ d_H^+(u) = d_H^-(u), \forall u \in V(G) \setminus \{x, y\}. \end{cases}$$

$\Rightarrow$ $\exists$ at least val f edge-disjoint (x, y)-paths in H

$\Rightarrow (\lambda_G(x, y) \le)$ val f $\le \eta_G(x, y)$

By Remark: ①, $\eta_G(x, y) = \lambda_G(x, y)$

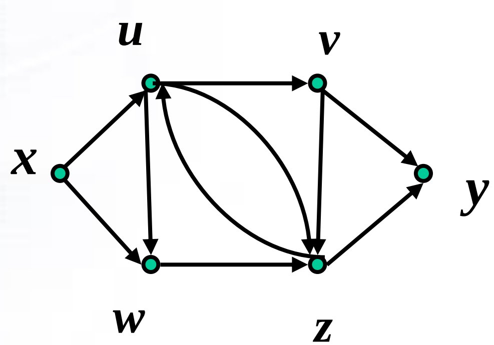
Remark: ① $\eta_G(x, y) \le \lambda_G(x, y), \forall x, y \in V(G)$.

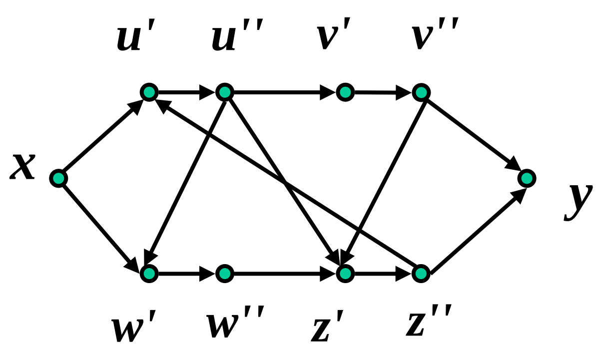
4.2 Menger's Theorem

- **Def:** Let $u \in V(G)$, the **split** of u is a graph $G' = (V', E')$ s.t.

  1. $V' = V \setminus \{u\} \cup \{u', u''\}$
  2. $E' = \{(x, y) \mid (x, y) \in E(G) \text{ and } x \neq u \text{ and } y \neq u\} \cup \{(u', u'')\}$
     $\cup \{(u'', y) \mid (u, y) \in E(G)\} \cup \{(x, u') \mid (x, u) \in E(G)\}$

- **ex:**





- **Theorem 4.3:** (**Menger's theorem**) $x \neq y \in V(G)$ and $(x, y) \notin E(G)$.

  Then $\zeta_G(x, y) = \kappa_G(x, y)$

Chapter 4
Flows and Connectivity

§ 4.3 Connectivity

4.3 Connectivity

- **Def:** ① G is a strongly connected digraph, $S \neq \phi$, $S \subseteq V(G)$ is said to be a **separating set** if $G - S$ is not strongly connected.

  ② The (**vertex-**) **connectivity** of G,

$$\kappa(G) = \begin{cases} 0 & \text{, if } G \text{ is not strongly connected;} \\ \nu - 1 & \text{, if } G \text{ contains a complete spanning subgraph;} \\ \min \{|S|: S \text{ is a separating set of } G\}, \text{ o.w.} \end{cases}$$

- **Note:** ① Every strongly connected digraph contains a separating set provided it contains no complete graph as a spanning subgraph.

  ② $\kappa(G) = \begin{cases} \nu - 1, \text{ if } \forall\, x, y \in V(G), E_G(x, y) \neq \phi; \\ \min \{\kappa_G(x, y): \forall\, x, y \in V(G), E_G(x, y) = \phi\}, \text{ o.w.} \end{cases}$

4.3 Connectivity

- **<u>Def</u>: ③ If a separating set S of G, $|S| = \kappa(G)$, then S is called a κ-separating set.**
  ④ A graph G is said to be k-connected if $\kappa(G) \geq k$.

- **ex: ① $\kappa(K_n) = n - 1$**
  ② $\kappa(C_n) = 1$, if C_n is directed for $n \geq 3$.
  ③ $\kappa(C_n) = 2$, if C_n is undirected for $n \geq 3$.
  ④ All nontrivial connected undirected graphs and strongly connected digraph are 1-connected.

4.3 Connectivity

- **Def: ① $B \neq \phi$, $B \subset E(G)$ is said to be a directed cut if $G - B$ is not strongly connected.**

  ② the edge-connectivity of G,

  $$\lambda(G) = \begin{cases} 0, & \text{if } G \text{ is trivial or not strongly connected;} \\ \min\{|B| : B \text{ is a directed cut of } G\}, & \text{o.w.} \end{cases}$$

- **Note: ① every nontrivial strongly connected digraph must contain a directed cut.**

  ② $\lambda(G) = \min\{\lambda_G(x, y) : \forall\, x, y \in V(G)\}$.

- **Def: ③ A directed cut B of G is a λ-cut if $|B| = \lambda(G)$.**

  ④ G is said to be k-edge-connected if $\lambda(G) \geq k$.

4.3 Connectivity

- **ex: ① $\lambda(K_n) = n - 1$**
  ② $\lambda(C_n) = 1$, if C_n is directed for $n \geq 3$.
  ③ $\lambda(C_n) = 2$, if C_n is undirected for $n \geq 3$.
  ④ All nontrivial connected undirected graphs and strongly connected digraph are 1-edge-connected.

- **Remark: ① If B is a directed cut of G, then $\exists\, S \neq \phi, S \subset V(G)$, s.t. $(S, \bar{S}) \subseteq B$.**
  ② recall: cut is the form $[S, \bar{S}]$

4.3 Connectivity

- **Thm 4.4**: (**Whitney's inequality**) **For any graph** G, $\kappa(G) \le \lambda(G) \le \delta(G)$.

  Proof. (1/3)

  ① **If** G **is trivial or empty; then** $\kappa(G) = \lambda(G) = \delta(G) = 0$.

  ② **We need to only prove this theorem for a loopless digraph** G.

  W.L.O.G. say $\delta(G) = \delta^+(G)$.

  ⟨a⟩ Let $x \in V(G)$ **s.t.** $d_G^+(x) = \delta(G)$.

  $\because$ $E_G^+(x)$ **is a directed cut of** G.

  $\therefore$ $\lambda(G) \le |E_G^+(x)| = \delta(G)$.

4.3 Connectivity

- **Thm 4.4: (Whitney's inequality) For any graph G, $\kappa(G) \le \lambda(G) \le \delta(G)$.**

  Proof. (2/3)

  ⟨b⟩ Prove $\kappa(G) \le \lambda(G)$ by induction on $\lambda(G) \ge 0$.

  When $\lambda(G) = 0$, G is no strongly connected. ∴ $\kappa(G) = 0 = \lambda(G)$.

  Suppose $\kappa(H) \le \lambda(H)$ $\forall$ digraph H with $\lambda(H) < \lambda$ and $\lambda \ge 0$.

  Now, consider a digraph G with a directed cut B s.t.

  $|B| = \lambda(G) = \lambda$.

  Let $a = (x, y) \in B$, and $H = G - a$. Then $\lambda(H) \le \lambda - 1$.

  By I.H., $\kappa(H) \le \lambda(H) \le \lambda - 1 < \lambda(G)$

  case 1: If $\exists$ $K_{\nu(H)} \subseteq H$, then

  so does G. i.e. $\exists$ $K_{\nu(G)} \subseteq G$

  $\Rightarrow \kappa(G) = \nu - 1 = \kappa(H) \le \lambda(H) \le \lambda - 1 < \lambda(G)$.

  case 2: If $\nexists$ $K_{\nu(H)} \subseteq H$ then $\exists$ κ-separating set S in H.

4.3 Connectivity

- **Thm 4.4: (Whitney's inequality) For any graph G, $\kappa(G) \le \lambda(G) \le \delta(G)$.**

  Proof. (3/3)

  case 2: If $\nexists K_{\nu(H)} \subseteq H$ then $\exists$ κ-separating set S in H.

  case 2.1: If $G - S$ is not strongly connected, then

  $$\kappa(G) \le |S| = \kappa(H) \le \lambda(H) < \lambda(G).$$

  case 2.2: If $G - S$ is strongly connected:

  case 2.2.1: If $\nu(G - S) = 2$, then

  $$\kappa(G) \le \nu - 1 = \nu(G - S) + |S| - 1 = |S| + 1$$
  $$= \kappa(H) + 1 \le \lambda(H) + 1 \le \lambda(G).$$

  case 2.2.2: If $\nu(G - S) > 2$, then

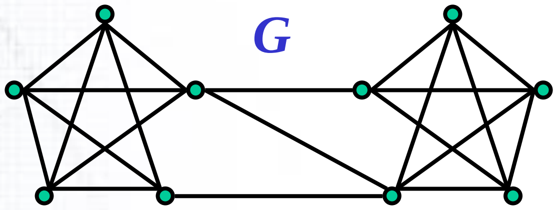
  either $S \cup \{x\}$ or $S \cup \{y\}$ is a separating set of G.

  $$\therefore\ \kappa(G) \le |S| + 1 = \kappa(H) + 1 \le \lambda(H) + 1 \le \lambda(G).$$

  $\therefore\ \kappa(G) \le \lambda(G)$ and the theorem following by the principle of induction.

4.3 Connectivity

- **ex:**



$\kappa(G) = 2$,
$\lambda(G) = 3$,
$\delta(G) = 4$.

- **Remark**: $\forall\ a, b, c \in \mathbb{N}$ with $0 < a \leq b \leq c$, $\exists$ 2 graphs G_1, G_2 (undirected and directed), s.t. $\kappa(G_1) = \kappa(G_2) = a$, $\lambda(G_1) = \lambda(G_2) = b$, $\delta(G_1) = \delta(G_2) = c$.

- **Theorem 4.5**: Let G be a graph of order at least $k + 1$. Then
  (a) $\kappa(G) \geq k \Leftrightarrow \zeta_G(x, y) \geq k, \forall\ x, y \in V(G)$,
  (b) $\lambda(G) \geq k \Leftrightarrow \eta_G(x, y) \geq k, \forall\ x, y \in V(G)$.

4.3 Connectivity

- **<u>Theorem 4.6</u>: If $\kappa(G_i) > 0$, $\forall$ $i = 1, 2, \ldots, n$, then**

  $$\kappa(G_1 \times G_2 \times \ldots \times G_n) \geq \kappa(G_1) + \kappa(G_2) + \ldots + \kappa(G_n).$$

  Furthermore, if $\kappa(G_i) = \delta(G_i) > 0$ $\forall$ $i = 1, 2, \ldots, n$, then

  $$\kappa(G_1 \times G_2 \times \ldots \times G_n) = \kappa(G_1) + \kappa(G_2) + \ldots + \kappa(G_n).$$

  Particularly, $\kappa(Q_n) = n$.

- **Exercise: 4.1.1, 4.2.2(b)**

- **加: 4.1.6, 4.2.4, 4.2.5, 4.3.10, 4.3.11**

Chapter 5
Matchings and Independent Sets

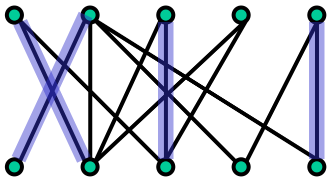
§ 5.1 Matchings (1)

5.1 Matchings

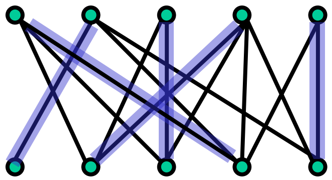
- **<u>Def</u>: G: nonempty and loopless graph**
  - **$M \neq \phi \subseteq E(G)$ is called a matching in G iff $\forall\, e_1, e_2, \in M$, e_1, e_2 are not adjacent in G.**
  - **$\forall\, e \in M$, if $e = (x, y)$, x, y are said to be matched under M.**
  - **$\forall\, x \in V(G)$, if $\exists\, e \in M$ s.t. $e = (x, y)$, then x is M-saturated, or say M saturates x; otherwise, x is M-unsaturated.**
  - **A matching M is perfect if it saturates every vertices of G.**
  - **M is maximum if $\forall$ matching M' in G, $|M| \geq |M'|$**

- **<u>Note</u>: Only discuss undirected graph.**

- **ex: (a)**



(b)



5.1 Matchings

- **<u>Theorem 5.1</u>: (Hall's theorem) Let $G = (X, Y, E)$ be a bipartite graph. Then**

  G contains a matching M that $|M| = |X| \Leftrightarrow |S| \leq |N_G(S)|\ \forall\ S \subseteq X$.

  Proof. (1/2)

  ($\Rightarrow$) Let $M = \{(x_i, y_i)|\ x_i \in X, y_i \in Y\}$ be a matching of G which saturates every $x_i \in X \Rightarrow$ all y_i are distinct

  $\therefore\ \forall\ S \subseteq X, N_G(S) \supseteq \{y_i|\ \forall\ x_i \in S\}$

  $\Rightarrow \forall\ S \subseteq X, |N_G(S)| \geq |S|$

5.1 Matchings

- **Theorem 5.1**: (Hall's theorem) Let $G = (X, Y, E)$ be a bipartite graph. Then G contains a matching M that $|M| = |X| \Leftrightarrow |S| \le |N_G(S)|\ \forall\ S \subseteq X$.

Proof. (2/2)

($\Leftarrow$) Suppose M is a maximum matching in G.

Construct a digraph D: $V(D) = V(G) \cup \{x, y\}$

$E(D) = \{(x, x_i) | x_i \in X\} \cup \{(x_i, y_j): x_i \in X, y_j \in Y, x_iy_j \in E(G)\} \cup \{(y_j, y) | y_j \in Y\}$

$\zeta_D(x, y) = |M|$. Let T be a minimum (x, y)-separating set of D.

By Menger's theorem, $|M| = \zeta_D(x, y) = \kappa_D(x\ y) = |T|$

Let $T_1 = T \cap X$, $T_2 = T \cap Y$.

$\Rightarrow E_D(X \backslash T_1, Y \backslash T_2) = \phi$

$\Rightarrow N_D^+(X \backslash T_1) \subseteq T_2$

$\Rightarrow |M| = |T| = |T_1| + |T_2|$

$\ge |T_1| + |N_D^+(X \backslash T_1)|$

$= |T_1| + |N_G(X \backslash T_1)| \ge |T_1| + |X \backslash T_1| = |X|$