Chapter 3
Plane Graphs and Planar Graphs

§ 3.1 Plane Graphs and Euler Formula

3.1 Plane Graphs and Euler Formula

- **Def:**
  - G **is said to be embeddable on the surface** $S \equiv G$ **can be drawn in** S **s.t. its edges intersection only at their end-vertices.**
  - **Such a drawing of** G **is called an embedding of** G**, denoted by** $\tilde{G}$**. [tilde]**

- **Note: ① In this chapter, only discuss undirected graph.**

  ② The surface we consider, only plane or sphere.

  ③ $G \cong \tilde{G}$**.**

3.1 Plane Graphs and Euler Formula

- **Thm 3.1: A graph *G* is embeddable on the sphere *S***
  $\Leftrightarrow$ *G* is embeddable on the plane *P*.

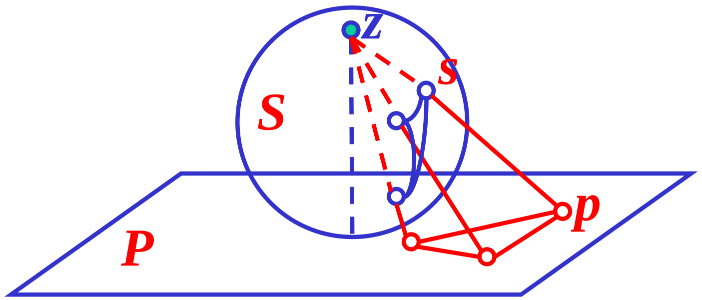
Proof.

see textbooks.

① Define ϕ: $S \rightarrow P$

② Prove "$\Rightarrow$"

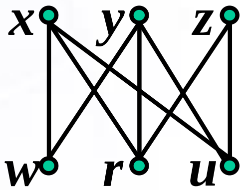
"$\Leftarrow$"



3.1 Plane Graphs and Euler Formula

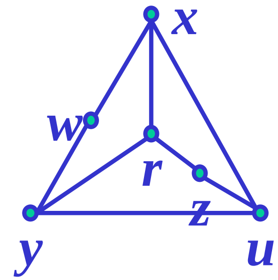
- **Def: ① If G is embeddable on the plane (or the sphere), G is called a planar graph, otherwise G is called a non-planar graph.**
  ② $\tilde{G}$ is called a plane graph.

- **ex: ①**

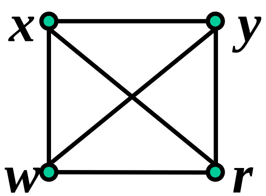


$G_1 = K_{3,3}^{-}$

$\tilde{G}_1$:

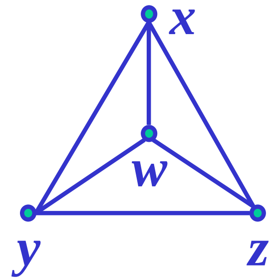


②



$G_2 = K_4$

$\tilde{G}_2$:



3.1 Plane Graphs and Euler Formula

- **Def: *G* is a nonempty plane graph.**

  ① **Faces is the connected regions in plane. (be partitioned by *G*)**

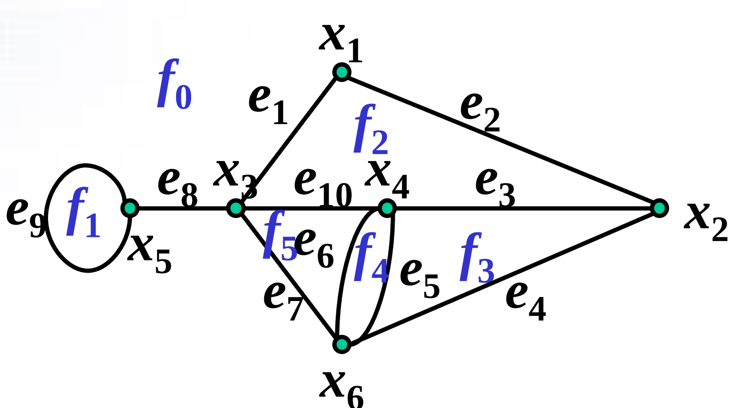
  ② **The set and the number of faces of *G* ≡ $F(G)$ and $\phi(G)$**

  ③ **∀ $f \in F(G)$, $B_G(f)$, the boundary of f ≡ the closed walk around f.**

  ④ **The number of edges in $B_G(f)$ is called the degree of f ≡ $d_G(f)$**

  ⑤ **exterior face ≡ unbounded face**

- **ex:**



G

① $F(G) = \{f_0, f_1, f_2, f_3, f_4, f_5\}$, $|F(G)| = \phi(G) = 6$

② $B_G(f_0) = x_1e_2x_2e_4x_6e_7x_3e_8x_5e_9x_5e_8x_3e_1x_1$

③ $d_G(f_0) = 7$; $d_G(f_1) = 1$

④ f_0 **is the exterior face**

3.1 Plane Graphs and Euler Formula

- **Note: ① ∀ plane graph G, $\phi(G) \geq 1$, and $\phi(G) = 1 \Leftrightarrow G$ is a forest.**

  ② Any planar embedding of a planar graph has exactly one exterior face.

  ③ G: a planer graph, $\forall x \in V(G)$, $\forall e \in E(G)$,

  G can be embedded in the plane s.t.

  x or e is on the boundary of the exterior face of the embedding

  (= x or $e \in B_G(f_0)$, where f_0 is the exterior face)

3.1 Plane Graphs and Euler Formula

- **Thm 3.2: For any plane graph G, $\sum_{f \in F(G)} d_G(f) = 2\varepsilon(G)$**

Proof.

If G is empty, it's true.

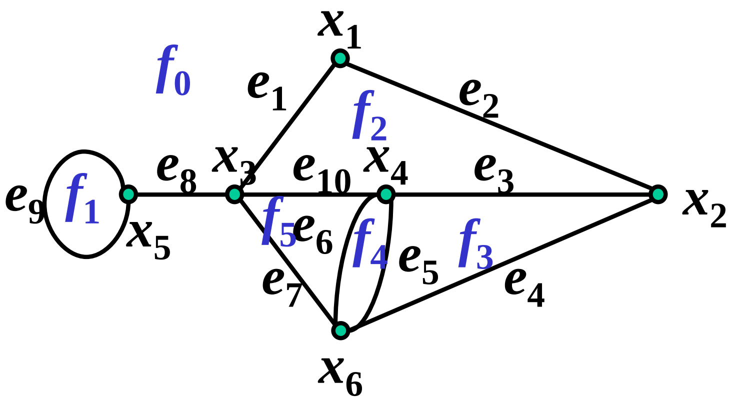
If G is nonempty, $\forall\, e \in E(G)$:

either $\exists$ 2 faces f_i, f_j s.t. $e \in B_G(f_i) \cap B_G(f_j)$

or $\exists$ face f_i, e appear in $B_G(f_i)$ twice.

$\therefore\ 2\varepsilon(G) = \sum_{f \in F(G)} d_G(f)$

ex:



3.1 Plane Graphs and Euler Formula

Theorem 2.3: F is a spanning forest of G and $E(G)\backslash E(F) \neq \phi$
$\Rightarrow \forall e \in E(G)\backslash E(F)$, $F + e$ contains a unique cycle.

- **Thm 3.3: (Euler's formula) If G is a connected plane graph, then $\nu - \varepsilon + \phi = 2$**

Proof.

Let T be a spanning tree of G.

$\Rightarrow \phi(T) = 1$ and $\varepsilon(\overline{T}) = \varepsilon - \varepsilon(T) = \varepsilon - (\nu - 1) = \varepsilon - \nu + 1$

$\forall e \in E(\overline{T})$, $T + e$ contains unique cycle $\Rightarrow \phi(T + e) = 2$ by Thm 2.3

$\therefore \phi(G) \geq \phi(T) + \varepsilon - \nu + 1$

Obtain a new face, must added $e \in E(\overline{T})$ to T

$\therefore \phi(G) \leq \phi(T) + \varepsilon - \nu + 1$

$\Rightarrow \phi(G) = \phi(T) + \varepsilon - \nu + 1 = 1 + \varepsilon - \nu + 1$

i.e. $\nu - \varepsilon + \phi = 2$

3.1 Plane Graphs and Euler Formula

- **Corollary 3.3.1: *G* is a plane graph ⇒ $\nu - \varepsilon + \phi = 1 + \omega$**

- **Corollary 3.3.2: *G*: a planar graph, all embedding of *G* has the same number of faces.**

3.1 Plane Graphs and Euler Formula

Corollary 1.6.2: G: an undirected graph, G is bipartite $\Leftrightarrow$ G contains no odd cycle.

- **Corollary 3.3.3: G: a simple connected planar bipartite graph of order $\nu \geq 3$,**

  $\Rightarrow \varepsilon \leq 2\nu - 4$

  Proof.

  Let $\tilde{G}$ is a planar embedding of G.

  ① If $\tilde{G}$ is a tree, then by Thm 2.3, $\varepsilon = \nu - 1 = 2\nu - 1 - \nu \leq 2\nu - 4$ ($\because \nu \geq 3$)

  ② If $\tilde{G}$ contains a cycle:

  $\because \tilde{G} \cong F$ is a simple bipartite graph.

  $\therefore$ By Corollary 1.6.2, G contains no odd cycle

  $\Rightarrow \forall f \in F(\tilde{G}), d_{\tilde{G}}(f) \geq 4$

  By Thm 3.2, $4\phi \leq \sum_{f \in F(\tilde{G})} d_{\tilde{G}}(f) = 2\varepsilon \Rightarrow \varepsilon \geq 2\phi$

  By Euler's formula, $\nu - \varepsilon + \phi = 2$

  $\therefore \varepsilon \geq 2\phi = 2(2 - \nu + \varepsilon) = 4 - 2\nu + 2\varepsilon$

  $\Rightarrow \varepsilon \leq 2\nu - 4$

3.1 Plane Graphs and Euler Formula

Corollary 3.3.3: *G*: a simple connected planar bipartite graph of order $\nu \geq 3$,

$$\Rightarrow \varepsilon \leq 2\nu - 4$$

- **Corollary 3.3.4**: $K_{3,3}$ is non-planar.

Proof.

$K_{3,3}$ is a simple connected bipartite graph.

If $K_{3,3}$ is planar, then By Coro. 3.3.3,

$$\varepsilon(K_{3,3}) \leq 2\nu(K_{3,3}) - 4$$

but $\varepsilon(K_{3,3}) = 9$, $\nu(K_{3,3}) = 6$

$$9 > 2 \cdot 6 - 4 = 8 \quad \rightarrow\leftarrow$$

$\therefore$ $K_{3,3}$ is non-planar.

3.1 Plane Graphs and Euler Formula

- **<u>Def</u>: ① A simple planar graph G is called to be maximal if**
  $\forall\, x, y \in V(G)$, $xy \notin E(G)$, $G + xy$ is non-planar.

- **<u>Note</u>: G: a maximal planar graph, $\forall\, \tilde{G}$: the planar embedding of G,**
  $\forall\, f \in F(\tilde{G})$, $d_{\tilde{G}}(f) = 3$

- **<u>Def</u>: ② A planar embedding of a maximal planar graph is called a**
  plane triangulation.

3.1 Plane Graphs and Euler Formula

Thm 3.2: For any plane graph G, $\sum_{f \in F(G)} d_G(f) = 2\varepsilon(G)$

- **Thm 3.4: G: a simple planar graph of order $\nu \geq 3$**

  G is maximal $\Leftrightarrow \varepsilon(G) = 3\nu(G) - 6$

  Proof.

  Let $\tilde{G}$ be a planar embedding of G.

  G is maximal $\Leftrightarrow \forall f \in F(\tilde{G}), d_{\tilde{G}}(f) = 3$

  $\Leftrightarrow 2\varepsilon = \sum_{f \in F(\tilde{G})} d_{\tilde{G}}(f) = 3\phi$ (by Thm 3.2)

  $\Leftrightarrow 2\varepsilon = 3(2 - \nu + \varepsilon) = 6 - 3\nu + 3\varepsilon$ (by Euler's Formula)

  $\Leftrightarrow \varepsilon = 3\nu - 6$

- **Corollary 3.4.1: G: a simple planar graph of order $\nu \geq 3 \Rightarrow \varepsilon \leq 3\nu - 6$**

3.1 Plane Graphs and Euler Formula

Corollary 3.4.1: *G*: a simple planar graph of order $\nu \geq 3 \Rightarrow \varepsilon \leq 3\nu - 6$

- **Corollary 3.4.2: K_5 is non-planar.**

  Proof.

  If K_5 is planar, by Coro 3.4.1,

  $$10 = \varepsilon(K_5) \leq 3\nu(K_5) - 6 = 3 \cdot 5 - 6 = 9 \quad \rightarrow\leftarrow$$

  $\therefore K_5$ is non-planar.

- **Corollary 3.4.3: *G*: a simple planar graph $\Rightarrow \delta(G) \leq 5$**

  Proof.

  ① $\nu = 1 \sim 6$ is true.

  ② $\nu \geq 7$: by Corollary 1.1 and Corollary 3.4.1:

  $$\delta\nu \leq \sum_{x \in V(G)} d_G(x) = 2\varepsilon \leq 6\nu - 12$$

  $$\Rightarrow \delta \leq 6 - (12/\nu)$$

  $$\therefore \delta \leq 5$$

3.1 Plane Graphs and Euler Formula

- **<u>Thm 3.5</u>: Any simple planar graph can be embedded in the plane s.t. each edge is a straight line segment.**

- **ex:**

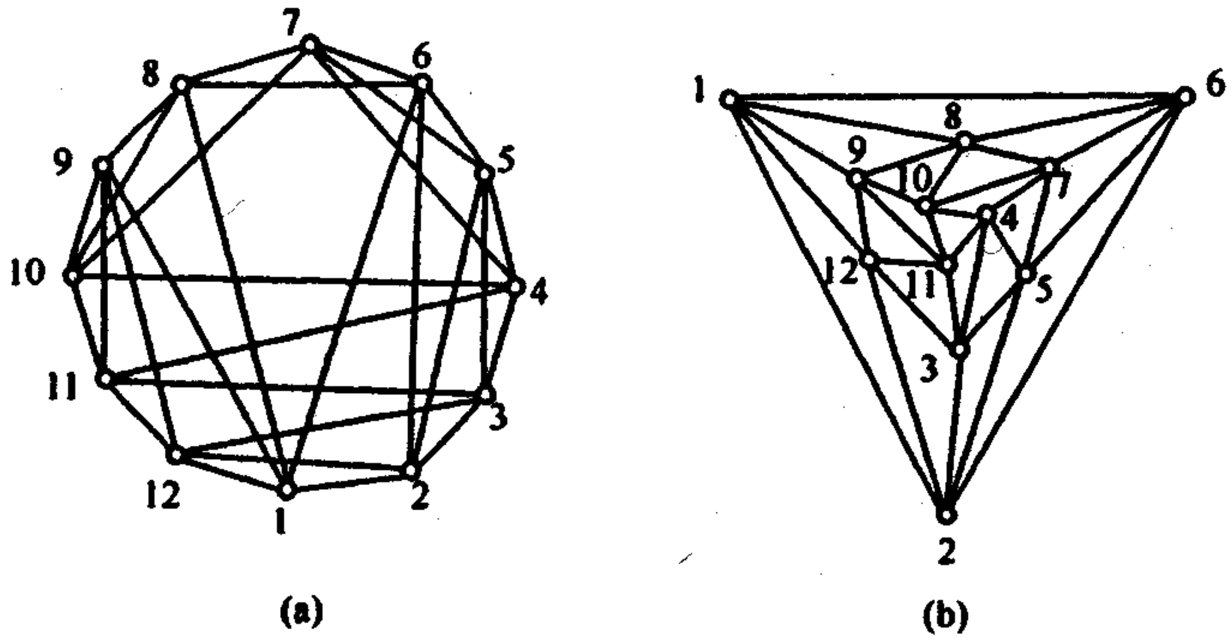


Figure 3.5: a planar graph and its planar embedding with straight line segments

3.1 Plane Graphs and Euler Formula

- **exercise: 3.1.1(b); 3.1.5**

- **加: 3.1.7; 3.1.9; 3.1.14; 3.1.17**

Chapter 3
Plane Graphs and Planar Graphs

§ 3.2 Kuratowski's Theorem

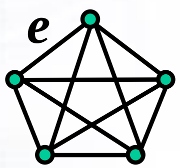
3.2 Kuratowski's Theorem

- **Def: ① $e = xy \in E(G)$ is said to be subdivided $\equiv ((G \backslash e) \cup z) \cup \{xz, zy\}$**

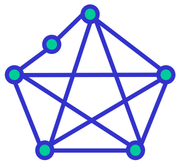
  ② A subdivision of a graph G is a graph obtained from G by a sequence of edge subdivisions.

- **ex:**

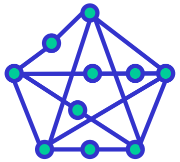
G



***e* is subdivided**



subdivision of G



- **Thm 3.6: (Kuratowski's theorem) A graph is planar**

  ⇔ it contains no subdivision of K_5 or $K_{3,3}$ as its subgraph.

Chapter 3
Plane Graphs and Planar Graphs

§ 3.3 Dual Graphs

3.3 Dual Graphs

- **Def**: G: a plane graph with $E(G) = \{e_1, e_2, \ldots, e_\varepsilon\}$. $F(G) = \{f_1, f_2, \ldots, f_\phi\}$

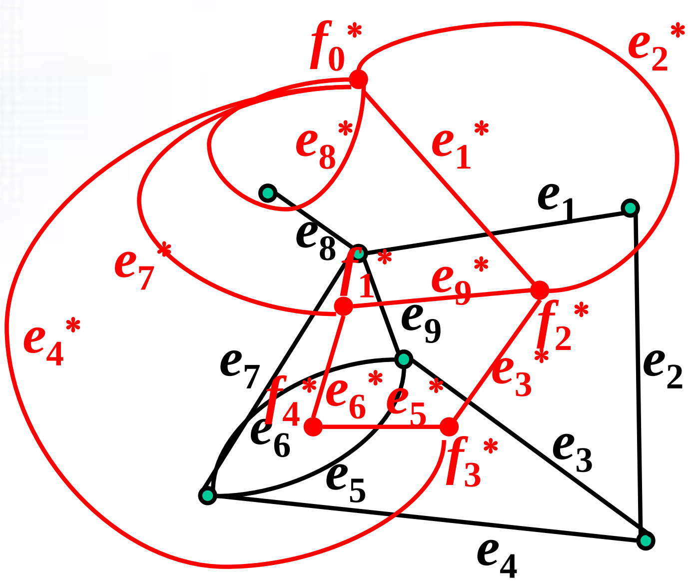
  G^* is called the **geometric dual** of G if

  $V(G^*) = \{f_1^*, f_2^*, \ldots, f_\phi^*\}$

  $E(G^*) = \{e_1^*, e_2^*, \ldots, e_\varepsilon^*\}$, s.t.

  $f_i^* f_j^* \in E(G^*) \Leftrightarrow \exists\, e_i \in E(G)$ s.t. e_i is a common boundary of f_i and f_j of G.

- **ex:**

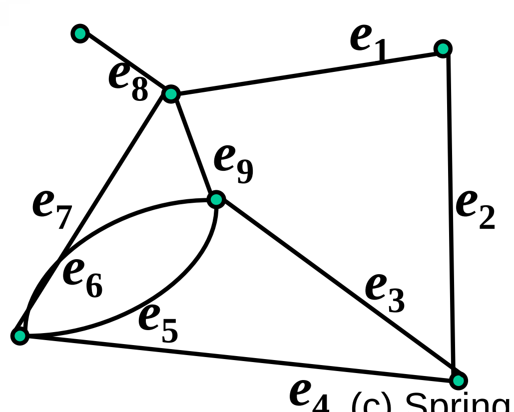


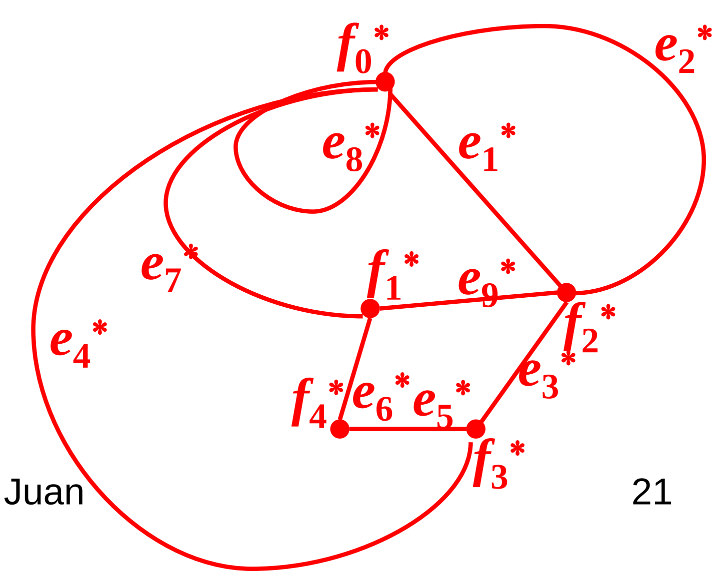
3.3 Dual Graphs

- **Note:** $\begin{cases} \nu(G^*) = \phi(G) \\ \varepsilon(G^*) = \varepsilon(G) \\ d_{G^*}(f^*) = d_G(f),\ \forall f \in F(G) \end{cases}$

- **Thm 3.7:** G : a plane graph, G^* : the geometric dual of G, $B \subseteq E(G)$ and $B^* = \{e^* \in E(G^*) : e \in B\}$. Then
  (a) $G[B]$ is a cycle of $G \Leftrightarrow B^*$ is a bond of G^*.
  (b) B is a bond of $G \Leftrightarrow G^*[B^*]$ is a cycle of G^*.

- **ex:**

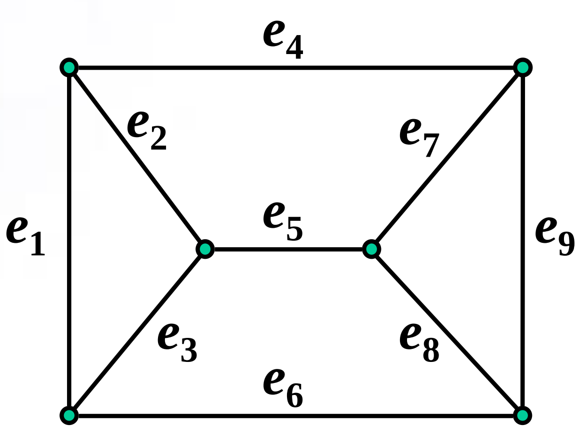


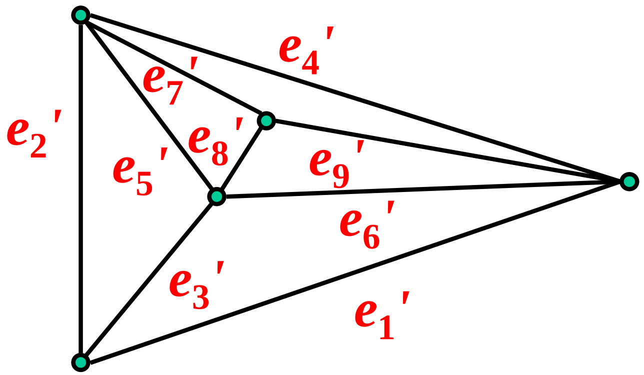


3.3 Dual Graphs

- **Def**: G, G': two graphs. G' is called the **combinatorial dual** of G if
  $\exists$ a bijective mapping $\varphi : E(G) \rightarrow E(G')$ s.t.
  $\forall\, B \subseteq E(G)$, $G[B]$ is a cycle of G
  $\Leftrightarrow \varphi(B) = \{e' \in E(G') : \varphi(e) = e',\ e \in B\}$ is a bond of G'.

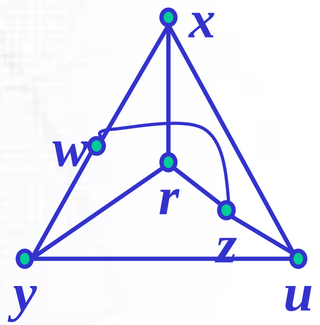
- ex:

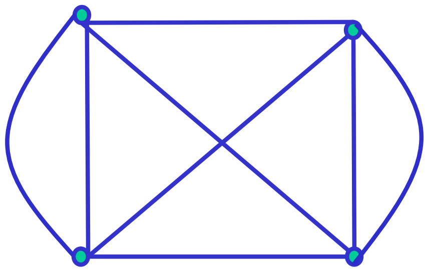




3.3 Dual Graphs

- **ex:**





3.3 Dual Graphs

- **<u>Thm 3.8</u>: *G* : a plane graph, G^* : the geometric dual of *G*. Then**
  (a) G^* is the combinatorial dual of *G*.
  (b) *G* is the combinatorial dual of G^*.

- **<u>Thm 3.9</u>: (Whitney's theorem)**
  A graph is planar ⇔ it has combinatorial dual.