Chapter 2
Trees and Graphic Spaces

§ 2.1 Trees and Spanning Trees (2)

2.1 Trees and Spanning Trees

- **Theorem 2.1: The following statements are equivalent**

  (a) G is a tree.

  (b) G has no loop and $\forall\ x, y \in V(G)$, $\exists!$ xy-path.

  (c) G is connected and $\forall e \in E(G)$, $\omega(G - e) = 2$

  (d) G is connected and $\varepsilon = \nu - 1$.

- **Corollary 2.1: A graph G is a forest $\Leftrightarrow \varepsilon = \nu - \omega$**

- **Example 2.1.1: G: a forest and $\delta(G) \geq 1$, G contains $\geq 2\omega$ 1-degree vertices.**

2.1 Trees and Spanning Trees

Corollary 2.1: A graph G is a forest $\Leftrightarrow \varepsilon = \nu - \omega$

- **Example 2.1.2**: Let $A = \{A_1, A_2, \ldots, A_n\}$ be a family of n distinct subsets of $X = \{1, 2, \ldots, n\}$. Then $\exists\, x \in X$ s.t. $A_1\backslash\{x\}, A_2\backslash\{x\}, \ldots, A_n\backslash\{x\}$ are all distinct.

Proof. (1/2) (略)

Note: $B, C \in A$, $B \neq C$ and $B\backslash\{i\} = C\backslash\{i\} \Rightarrow B \,\Delta\, C=\{i\}$.

If not, i.e. $\forall\, i \in X, \exists\, 1 \leq k(i) < l(i) \leq n$ s.t. $A_{k(i)}\backslash\{i\} = A_{l(i)}\backslash\{i\}$

i.e. $A_{k(i)} \,\Delta\, A_{l(i)} = \{i\}$

$\therefore$ Let $k, l: X \to X$ s.t. $k(i) = k < l(i) = l$ where $A_{k(i)} \,\Delta\, A_{l(i)} = \{i\}$.

Construct a simple undirected graph G where $V(G) = X$,

$E(G) = \{k(i)l(i) = x_i \mid A_{k(i)} \,\Delta\, A_{l(i)} = \{i\}\ \forall\, i \in X\}$

$\because \forall\, i \in X, \exists!\, 1 \leq k(i) < l(i) \leq n$ s.t. $A_{k(i)} \,\Delta\, A_{l(i)} = \{i\}$.

$\therefore\ \varepsilon(G) = n \Rightarrow$ By Coro. 2.1, G contains a cycle.

Suppose $(i_1, i_2, \ldots, i_s, i_1)$ is a cycle in G.

$\Rightarrow i_j i_{j+1} = x_{p_j}, 1 \leq j \leq s-1;\ i_s i_1 = x_{p_s}$

$\therefore\ k(p_j) = i_j, 1 \leq j \leq s;\ l(p_j) = i_{j+1}, 1 \leq j \leq s-1,\ l(p_s) = i_1$.

2.1 Trees and Spanning Trees

- **Example 2.1.2**: Let $A = \{A_1, A_2, \ldots, A_n\}$ be a family of n distinct subsets of $X = \{1, 2, \ldots, n\}$. Then $\exists\, x \in X$ s.t. $A_1 \backslash \{x\}, A_2 \backslash \{x\}, \ldots, A_n \backslash \{x\}$ are all distinct.

  Proof. (2/2)

  Suppose $(i_1, i_2, \ldots, i_s, i_1)$ is a cycle in G.

  $\Rightarrow i_j i_{j+1} = x_{p_j}, 1 \leq j \leq s-1;\ i_s i_1 = x_{p_s}$

  $\therefore\ k(p_j) = i_j, 1 \leq j \leq s;\ l(p_j) = i_{j+1}, 1 \leq j \leq s-1, l(p_s) = i_1$.

  But $\{p_s\} = A_{k(p_s)} \Delta A_{l(p_s)} = A_{i_s} \Delta A_{i_1} = A_{i_1} \Delta A_{i_s}$

  $= A_{i_1} \Delta A_{i_2} \Delta A_{i_2} \Delta A_{i_3} \Delta A_{i_3} \Delta \ldots \Delta A_{i_{s-1}} \Delta A_{i_{s-1}} \Delta A_{i_s}$

  $= (A_{i_1} \Delta A_{i_2}) \Delta (A_{i_2} \Delta A_{i_3}) \Delta \ldots \Delta (A_{i_{s-1}} \Delta A_{i_s})$

  $= \{p_1\} \Delta \{p_2\} \Delta \{p_3\} \Delta \ldots \Delta \{p_{s-1}\}$

  $= \bigcup_{j=1}^{s-1} p_j = \{p_1, p_2, \ldots, p_{s-1}\}\quad \rightarrow\leftarrow$

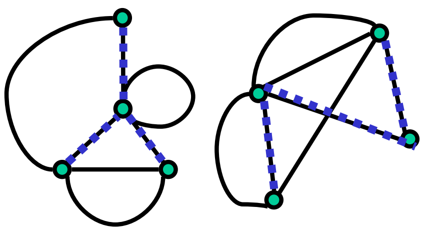
  $\therefore\ \exists\, x \in X$ s.t. $A_1 \backslash \{x\}, A_2 \backslash \{x\}, \ldots, A_n \backslash \{x\}$ are all distinct.

2.1 Trees and Spanning Trees

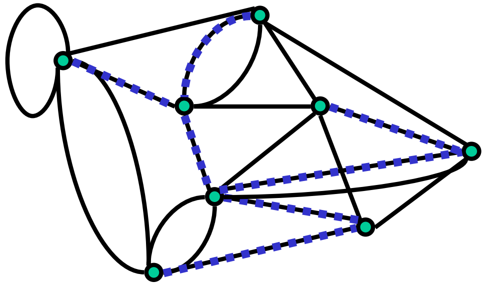
- <u>Def</u>: G: a digraph is a tree.
  - ① G is called an **out-tree rooted at x**, if $\forall\, y \in V(G)$, $y \neq x$, $\exists!$ (x, y)-dipath in G.
  - ② x is called a **root** of G.
  - ③ **in-tree**
  - ④ **rooted tree** $\equiv$ in-tree or out-tree.

- <u>Def</u>: F is a spanning subgraph of G with $\omega(F) = \omega(G)$.
  - ① F is called a **spanning forest** if F is a forest.
  - ② F is called a **spanning tree** if F is a tree.

- ex:



A spanning forest



A spanning tree

2.1 Trees and Spanning Trees

Theorem 2.1: The following statements are equivalent:

(a) G is a tree.

(c) G is connected and $\forall e \in E(G)$, $\omega(G - e) = 2$

- **Theorem 2.2: G: a graph. G contains a spanning tree $\Leftrightarrow$ G is connected**

  Proof.

  ($\Rightarrow$) clear.

  ($\Leftarrow$) Let T be a connected spanning subgraph with edges as few as possible.

  $\therefore$ $\omega(T) = 1$, $\omega(T - e) = 2$ for any $e \in E(T)$.

  By Thm 2.1(c) $\Rightarrow$ (a), T is a tree.

- **Corollary 2.2.1: ① Every graph contains a spanning forest and**

  ② Every connected graph contains a spanning tree.

2.1 Trees and Spanning Trees

Corollary 2.2.1: ① Every graph contains a spanning forest and ② Every connected graph contains a spanning tree.

- **Corollary 2.2.2**: $\varepsilon \geq \nu - \omega$ for every graph.

  Proof.

  Let G be a graph.

  By Corollary 2.2.1, every connected component of G contains a spanning tree and G contains a spanning forest F.

  By Corollary 2.1, $\varepsilon(F) = \nu(F) - \omega(F) = \nu(G) - \omega(G)$.

  $\therefore\ \varepsilon = \varepsilon(G) \geq \varepsilon(F) = \nu - \omega$.

2.1 Trees and Spanning Trees

- **Theorem 2.3**: F is a spanning forest of G and $E(G)\backslash E(F) \neq \phi$
  $\Rightarrow \forall\, e \in E(G)\backslash E(F)$, $F + e$ contains a unique cycle.

  Proof.

  By definition of spanning forest, F contains no cycle, and
  $\forall\, e \in E(G)\backslash E(F)$, let $e = xy$.
  Since x, y in the same component T of G, and T is a tree.
  Let P be the unique xy-path in F by Thm 2.1(b).
  Then $P + e$ is a unique cycle in $F + e$.

> **Theorem 2.1**: The following statements are equivalent:
> (a) G is a tree.
> (b) G has no loop and $\forall\, x, y \in V(G)$, $\exists!$ xy-path.
> (c) G is connected and $\forall e \in E(G)$, $\omega(G - e) = 2$
> (d) G is connected and $\varepsilon = \nu - 1$.

2.1 Trees and Spanning Trees

Corollary 2.1: A graph G is a forest $\Leftrightarrow \varepsilon = \nu - \omega$

- **Corollary 2.3: Any loopless graph contains at least $\varepsilon - \nu + \omega$ distinct cycles.**

- **Def: G: a graph,**

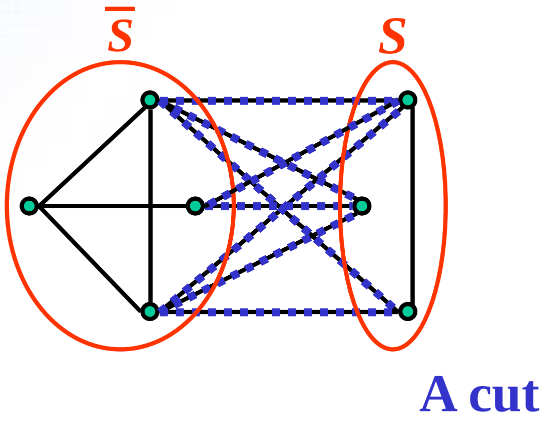
  ① $B\ (\neq \phi) \subseteq E(G)$ is called a (edge) cut of $G \equiv$

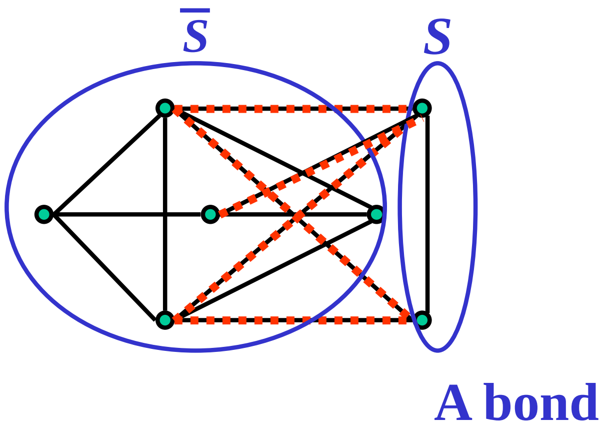
  $\exists\ S \subseteq V(G),\ 1 \leq |S| \leq \nu - 1$ s.t. $B = [S, \bar{S}]$

  ② A cut B is called a minimal cut (or bond) $\equiv$

  $\forall\ B' \subset B$, B' is not a cut of G.

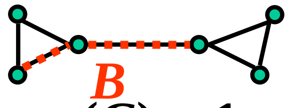
- **ex:**





2.1 Trees and Spanning Trees

- <u>Note</u>: ① $B \subseteq E(G)$ is a cut of G, $\Rightarrow \omega(G - B) > \omega(G)$.

  ② "$\Leftarrow$" is not always true. ex: 

  ③ B is a bond of $G \Rightarrow \omega(G - B) = \omega(G) + 1$

  ④ If B is a cut, then B is a bond $\Leftrightarrow \omega(G - B) = \omega(G) + 1$

- <u>Def</u>: If F is a spanning subgraph of G.

  ① The **cograph** of F in G, $\overline{F}(G)$ ($\overline{F}$) $\equiv G - E(F)$.

  ② If F is a spanning forest (tree), then $\overline{F}(G)$ is called the **coforest** (**cotree**) of G.

  ③ Write $\overline{F}$ instead of $E(\overline{F})$.

- <u>Note</u>: $\overline{F}(K_v) = F^C$

2.1 Trees and Spanning Trees

- Theorem 2.4: G: a nonempty (undirected) graph G, F: a spanning forest of G.
  Let $e \in E(F)$. Then,
  ① $\overline{F}$ contains no bond of G; ② $\overline{F} + e$ contains a unique bond of G.

Proof. (1/2)

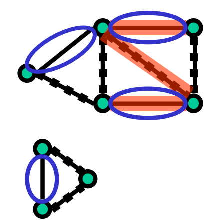
① Let B be any bond of G.

$\therefore\ \omega(G - B) - 1 = \omega(G) = \omega(F)$

$\Rightarrow E(F) \cap B \neq \phi$

$\Rightarrow B \not\subseteq E(\overline{F}(G))$

ex:



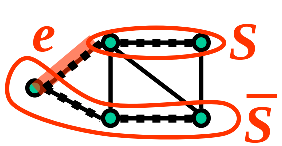
·····: spanning forest
▬: bond
⬭: $\overline{F}(G)$

② Let $S \subseteq V(F) = V(G)$ s.t. S is the vertex-set of some component of $F - e$.

$\exists$: Let $[S, \overline{S}] = B$, then B is a cut of G, and $B \subseteq E(\overline{F}(G)) + e$

$\therefore\ E(\overline{F}(G)) + e$ contains a cut of G.

ex:



2.1 Trees and Spanning Trees

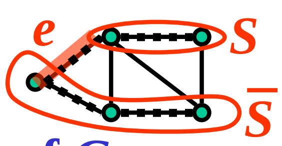
- **Theorem 2.4**: G: a nonempty (undirected) graph G, F: a spanning forest of G.
  Let $e \in E(F)$. Then,
  ① $\overline{F}$ contains no bond of G; ② $\overline{F} + e$ contains a unique bond of G.

Proof. (2/2)

② Let $S \subseteq V(F) = V(G)$ s.t. S is the vertex-set of some component of $F - e$.

(∃): Let $[S, \overline{S}] = B$, then B is a cut of G, and $B \subseteq E(\overline{F}(G)) + e$

$\therefore E(\overline{F}(G)) + e$ contains a cut of G. ex:



(!): Suppose $E(\overline{F}(G)) + e$ contains two distinct bonds B and B' of G.

By ①, $e \in B \cap B'$, but $\because B \neq B'$, $\therefore (B \cup B') - e$ contains a bond (Ex 2.1.13)

$\because (B \cup B') - e \subseteq E(\overline{F}(G))$

$\therefore E(\overline{F}(G))$ contains a bond →←

$\therefore E(\overline{F}(G)) + e$ contains a unique bond of G.

2.1 Trees and Spanning Trees

- **Corollary 2.4: Any loopless (undirected) graph G contains at least $\nu - \omega$ distinct bonds.**

  Proof.

  $\because$ G has ω component, $\therefore$ the spanning forest F has $\nu - \omega$ edges.

  By Thm 2.4, $\forall\, e \in E(F)$, $\overline{F} + e$ contains a unique bond of G.

  $\therefore$ G contains $\geq \nu - \omega$ distinct bonds.

2.1 Trees and Spanning Trees

- **<u>Note</u>: ① The relationship between spanning forest (trees) and coforests (cotrees) is complementary in a graph. ∴ <u>Thm 2.3</u> and <u>Thm 2.4</u> explores the relationship between cycles and bonds is analogous to that between forests and coforests. (§ 2.2).**

P100: **Def: $\varsigma(G)$ ≡ the number of spanning trees in G. (ς: sigma)**

② § 2.3. see P103: **<u>Corollary 2.7.3</u>: $S(T_n) = n^{n-2}$, T_n: tournament**

P105: **<u>Corollary 2.7.5</u>: $S(K_n) = n^{n-2}$, for $n \geq 2$.**

③ § 2.4. see The minimum connector problem. (Prim's Algorithm)

④ § 2.5. see The shortest path problem. (Dijsktra's Algorithm)

2.1 Trees and Spanning Trees

- **exercise: 2.1.2 (a); 2.1.11**
- **加: 2.1.4; 2.1.13; 2.1.14**