Chapter 1
Basic Concepts of Graphs

§ 1.9 Hamiltonian Graphs (2)

1.9 Hamiltonian Graphs

- **__Theorem 1.9__: G: a simple undirected graph of $\nu \geq 3$**

  $d_G(x) + d_G(y) \geq \nu$, $\forall\, x, y \in V(G)$, $xy \notin E(G)$ (★) $\Rightarrow$ G is hamiltonian

  <Proof 2> (1/2)

  By __exercise 1.5.6(c)__, $\because$ G satisfies (★). $\therefore$ G is connected and contains a cycles.

  Let $C = (x_1, x_2, \ldots, x_k, x_1)$ be a longest cycle in G.

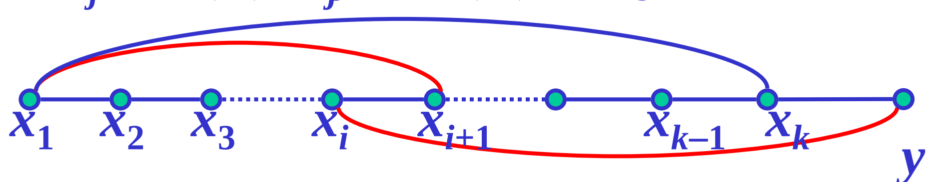
  Suppose $k < \nu$, let $R = V(G) \backslash V(C)$.

  $\because$ G is connected, $\therefore$ W.L.O.G., $\exists\, y \in R$, s.t. $yx_k \in E(G)$.

  $\because$ C is largest, $\therefore$ $x_1y \notin E(G)$.

  Let $S = \{x_i \in V(C)$: $x_1x_{i+1} \in E(G)$, $1 \leq i \leq k-1\}$,

  $T = \{x_j \in V(C)$: $x_jy \in E(G)$, $2 \leq j \leq k\}$.



$\Rightarrow$ $|S \cup T| \leq k$ and $|S \cap T| = 0$

$\Rightarrow$ $d_{G[V(C)\cup\{y\}]}(x_1) + d_{G[V(C)\cup\{y\}]}(y) = |S| + |T| = |S \cup T| \leq k$ ——— ①

1.9 Hamiltonian Graphs

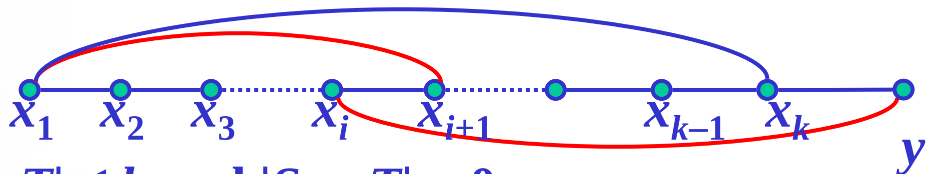
- **<u>Theorem 1.9</u>: G: a simple undirected graph of $\nu \geq 3$**

  $d_G(x) + d_G(y) \geq \nu,\ \forall\ x, y \in V(G),\ xy \notin E(G)$ (★) $\Rightarrow$ G is hamiltonian

  <Proof 2> (2/2)

  Let $S = \{x_i \in V(C): x_1 x_{i+1} \in E(G),\ 1 \leq i \leq k-1\}$,

  $T = \{\ x_j \in V(C): x_j y \in E(G),\ 2 \leq j \leq k\}$.



  $\Rightarrow |S \cup T| \leq k$ and $|S \cap T| = 0$

  **$\Rightarrow d_{G[V(C)\cup\{y\}]}(x_1) + d_{G[V(C)\cup\{y\}]}(y) = |S| + |T| = |S \cup T| \leq k$ ——① **

  $\because$ C is largest. $\therefore$ $\forall\ z \in R\backslash\{y\}$, either $x_1 z \notin E(G)$ or $yz \notin E(G)$

  $\Rightarrow |(\{x_1\}, R - \{y\})| + |(\{y\}, R - \{y\})| \leq \nu - k - 1$ ———— ②

  ① + ② $\Rightarrow d_G(x_1) + d_G(y) \leq k + \nu - k - 1 = \nu - 1$ $\rightarrow\leftarrow$

  $\therefore$ $k = \nu$, i.e. G is hamiltonian.

1.9 Hamiltonian Graphs

- **Corollary 1.9: Every simple graph with $\nu \geq 3$ and $\delta \geq (1/2)\nu$ is hamiltonian.**

- **Def: For a digraph G, $x \in V(G)$, $d_G(x) = d_G^+(x) + d_G^-(x)$.**

- **Theorem 1.10: Let $C = (x_1, x_2, \ldots, x_k, x_1)$ be a longest directed cycle in a strongly connected simple digraph G. (the index calculate in mod k, and $0 = k$)**
  If $k < \nu$, then $\exists\, x \in V(G) \setminus V(C)$ and two integral $a \in [1, k]$, $b \in [1, k - 1]$ s.t.
  (i) $(x_a, x) \in E(G)$
  (ii) $(x_{a+i}, x), (x, x_{a+i}) \notin E(G)$, $\forall\, i \in [1, b]$
  (iii) $d_G(x) + d_G(x_{a+b}) \leq 2\nu - 1 - b$

1.9 Hamiltonian Graphs

- **Corollary 1.10.1**: G: a strongly connected simple digraph.
  $\forall$ nonadjacent vertices $x, y \in V(G)$, $d_G(x) + d_G(y) \geq 2\nu - 1$
  $\Rightarrow G$ is hamiltonian.

  Proof.

  If not, then by Thm 1.10, $\exists\ x$ and $x_{a+b} \in V(G)$, s.t.
  (ii) $(x, x_{a+b}), (x_{a+b}, x) \notin E(G)$, and
  (iii) $d_G(x) + d_G(x_{a+b}) \leq 2\nu - 1 - b \leq 2\nu - 2$ $\rightarrow\leftarrow$

- **Corollary 1.10.2**: G: a strongly connected simple digraph,
  $\forall\ x \in V(G)$, $d_G(x) \geq \nu \Rightarrow G$ is hamiltonian.

- **Corollary 1.10.3**: G: a simple digraph, $\delta \geq (1/2)\nu > 1 \Rightarrow G$ is hamiltonian.
  By exercise 1.5.8(a) and Corollary 1.10.1.

1.9 Hamiltonian Graphs

- **Theorem 1.10: Let $C = (x_1, x_2, \ldots, x_k, x_1)$ be a longest directed cycle in a strongly connected simple digraph G. (the index calculate in mod k, and $0 = k$)**

  If $k < \nu$, then $\exists\, x \in V(G) \setminus V(C)$ and two integral $a \in [1, k]$, $b \in [1, k-1]$ s.t.

  (i) $(x_a, x) \in E(G)$

  (ii) $(x_{a+i}, x), (x, x_{a+i}) \notin E(G), \forall\, i \in [1, b]$

  (iii) $d_G(x) + d_G(x_{(a+b) \bmod k}) \le 2\nu - 1 - b$

  Proof. (1/6) (略)

  Let $S = V(C)$,

  $\because$ G is strongly connected and $|S| < \nu$.

  $\therefore$ $\exists\, x_i, x_j \in S$ and (x_i, x_j)-path P in G s.t. $V(P) - S \neq \phi$ and $V(P) \cap S = \{x_i, x_j\}$

  If $x_i \neq x_j$, we say P is an S-path, else say an S-cycle.

  Case 1: G contains no S-path.

  Case 2: G contains an S-path.

1.9 Hamiltonian Graphs

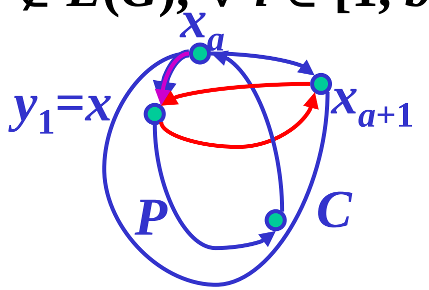
- **Theorem 1.10**: Let $C = (x_1, x_2, \ldots, x_k, x_1)$ be a longest directed cycle in a strongly connected simple digraph G. (the index calculate in mod k, and $0 = k$)

  If $k < \nu$, then $\exists\, x \in V(G) \setminus V(C)$ and two integral $a \in [1, k]$, $b \in [1, k-1]$ s.t.

  (i) $(x_a, x) \in E(G)$ (ii) $(x_{a+i}, x), (x, x_{a+i}) \notin E(G), \forall\, i \in [1, b]$

  (iii) $d_G(x) + d_G(x_{(a+b) \bmod k}) \leq 2\nu - 1 - b$

  Proof. (2/6)

  Case 1: G contains no S-path.

  Let $P = (x_a, y_1, y_2, \ldots, y_t, x_a)$ be an S-cycle with $x_a \in S$.

  Let $x = y_1$ and $b = 1$

  Note that $(x_a, x) \in E(G)$ (i), and $\because$ no S-path. $\therefore$ $(x_{a+1}, x), (x, x_{a+1}) \notin E(G)$ (ii)

  Obviously, $|[\{x_{a+1}\}, S]| \leq 2(k-1)$ —— ①

  $\because$ G contains no S-path. $\therefore$ $\forall\, i \in [1, k]$ and $i \neq a$, $(x, x_i), (x_i, x) \notin E(G)$

  $\Rightarrow |[\{x\}, S]| \leq 2$ —— ②

1.9 Hamiltonian Graphs

$|[\{x_{a+1}\}, S]| \le 2(k-1)$ —— ①

$|[\{x\}, S]| \le 2$ —— ②

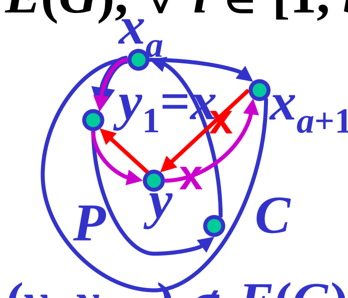
- **Theorem 1.10**: Let $C = (x_1, x_2, \ldots, x_k, x_1)$ be a longest directed cycle in a strongly connected simple digraph G. (the index calculate in mod k, and $0 = k$)

  If $k < \nu$, then $\exists\, x \in V(G) \setminus V(C)$ and two integral $a \in [1, k]$, $b \in [1, k-1]$ s.t.

  (i) $(x_a, x) \in E(G)$ (ii) $(x_{a+i}, x), (x, x_{a+i}) \notin E(G)$, $\forall\, i \in [1, b]$

  (iii) $d_G(x) + d_G(x_{(a+b) \bmod k}) \le 2\nu - 1 - b$

  Proof. (3/6)



$\because$ G contains no S-path

$\therefore$ $\forall\, y \in V \setminus (S \cup \{x\})$: (x_{a+1}, y) or $(y, x) \notin E(G)$ and (x, y) or $(y, x_{a+1}) \notin E(G)$.

$\Rightarrow |[\{y\}, \{x_{a+1}, x\}]| \le 2$, $\forall\, y \in V \setminus (S \cup \{x\})$ —— ③

by ①②③: $d_G(x) + d_G(x_{a+1}) = |[\{x, x_{a+1}\}, V]| = |[\{x, x_{a+1}\}, V \setminus \{x, x_{a+1}\}]|$

$= |[\{x, x_{a+1}\}, V \setminus (S \cup \{x\})]| + |[\{x, x_{a+1}\}, S \setminus \{x_{a+1}\}]|$

$= |[\{x, x_{a+1}\}, V \setminus (S \cup \{x\})]| + |[\{x\}, S]| + |[\{x_{a+1}\}, S]|$

$\le 2(\nu - k - 1) + 2 + 2(k-1) = 2\nu - 2$

i.e. $d_G(x) + d_G(x_{a+b}) \le 2\nu - 1 - b$ —— (iii)

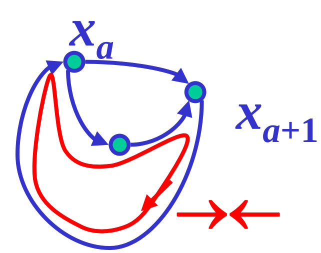
1.9 Hamiltonian Graphs

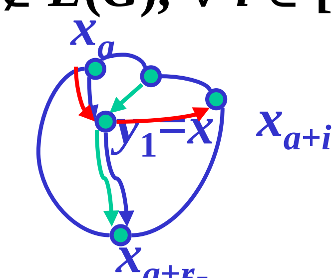
- **<u>Theorem 1.10</u>: Let $C = (x_1, x_2, \ldots, x_k, x_1)$ be a longest directed cycle in a strongly connected simple digraph G. (the index calculate in mod k, and $0 = k$)**

If $k < \nu$, then $\exists\, x \in V(G) \setminus V(C)$ and two integral $a \in [1, k]$, $b \in [1, k-1]$ s.t.

(i) $(x_a, x) \in E(G)$ **(ii) $(x_{a+i}, x), (x, x_{a+i}) \notin E(G)$, $\forall\, i \in [1, b]$**

(iii) $d_G(x) + d_G(x_{(a+b) \bmod k}) \le 2\nu - 1 - b$





Proof. (4/6)

<u>Case 2</u>: G contains an S-path.

Let $P = (x_a, y_1, \ldots, y_t, x_{a+r})$ be an S-path with $x_a \in S$, s.t. r is <u>as small as possible.</u>

Let $x = y_1$ (Note that, <u>$(x_a, x) \in E(G)$</u>.) **(i)**

$\because$ C is longest. $\therefore$ $r \ge 2$ and <u>$(x, x_{a+1}) \notin E(G)$</u>. **(ii)**

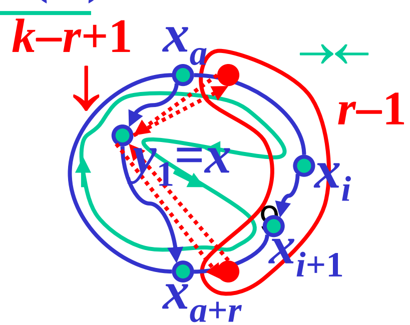
$\because$ r is as small as possible, $\therefore$ <u>$\forall\, i \in [2, r-1]$, $(x, x_{a+i}), (x_{a+i}, x) \notin E(G)$</u>

<u>and $(x_{a+1}, x) \notin E(G)$</u>

$\because$ C is longest, $\therefore$ $\forall\, i \in [1, k]$, (x_i, x) or $(x, x_{i+1}) \notin E(G)$.

$\Rightarrow |[\{x\}, S]| \le 2k - [(k - r) + 2(r - 1)]$

$= k - r + 2$ ——①



1.9 Hamiltonian Graphs

- **Theorem 1.10: Let $C = (x_1, x_2, \ldots, x_k, x_1)$ be a longest directed cycle in a strongly connected simple digraph G. (the index calculate in mod k, and $0 = k$)**

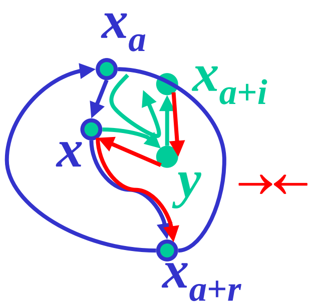
  If $k < \nu$, then $\exists\, x \in V(G) \setminus V(C)$ and two integral $a \in [1, k]$, $b \in [1, k-1]$ s.t.

  (i) $(x_a, x) \in E(G)$

  (ii) $(x_{a+i}, x), (x, x_{a+i}) \notin E(G)$, $\forall\, i \in [1, b]$

  (iii) $d_G(x) + d_G(x_{(a+b) \bmod k}) \le 2\nu - 1 - b$

  Proof. (5/6)



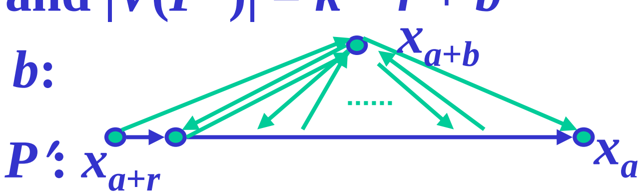
  $\forall\, y \in V \setminus (S \cup \{x\})$ and $\forall\, i \in [1, r-1]$.

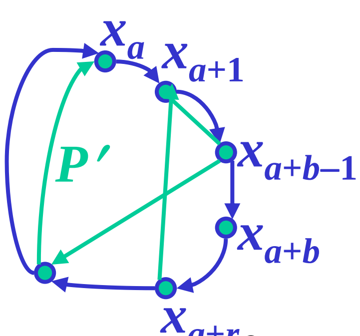
  $\because$ The choice of r, $|[\{y\}, \{x, x_{a+i}\}]| \le 2$ ——— ②

  Let b be the maximum $i \in [1, r-1]$ s.t. $G[S]$ contains an (x_{a+r}, x_a)-path P' and $V(P') = \{x_{a+r}, x_{a+r+1}, \ldots, x_{a-1}, x_a, x_{a+1}, \ldots, x_{a+i-1}\}$

  $\Rightarrow x_{a+b} \notin V(P')$ and $|V(P')| = k - r + b$

  $\because$ the choice of b:





  $|[\{x_{a+b}\}, V(P')]| \le (k - r + b - 1) + 2 = k - r + b + 1$ ———③

1.9 Hamiltonian

$\|[\{x\}, S]\| \le k - r + 2$ ——— ①

$\|[\{y\}, \{x, x_{a+i}\}]\| \le 2$ ——— ②

$\|[\{x_{a+b}\}, V(P')]\| \le k - r + b + 1$ ——— ③

- **Theorem 1.10**: Let $C = (x_1, x_2, \ldots, x_k, x_1)$ be a longest directed cycle in a strongly connected simple digraph G. (the index calculate in mod k, and $0 = k$)

  If $k < \nu$, then $\exists\, x \in V(G) \setminus V(C)$ and two integral $a \in [1, k]$, $b \in [1, k-1]$ s.t.

  (i) $(x_a, x) \in E(G)$ (ii) $(x_{a+i}, x), (x, x_{a+i}) \notin E(G), \forall\, i \in [1, b]$

  (iii) $d_G(x) + d_G(x_{(a+b)\bmod k}) \le 2\nu - 1 - b$

  Proof. (6/6)

  $\because |S \setminus (V(P') \cup \{x_{a+b}\})| = (r-1) - (b-1) - 1 = r - b - 1$

  $\therefore \|[\{x_{a+b}\}, S \setminus (V(P') \cup \{x_{a+b}\})]\| \le 2(r - b - 1)$ ——— ④

  By ①②③④:

  $d_G(x) + d_G(x_{a+b}) = \|[\{x, x_{a+b}\}, V]\|$

  $= \|[\{x, x_{a+b}\}, S]\| + \|[\{x, x_{a+b}\}, V \setminus (S \cup \{x\})]\|$

  $= \|[\{x\}, S]\| + \|[\{x_{a+b}\}, S \setminus \{x_{a+b}\}]\| + \|[\{x, x_{a+b}\}, V \setminus (S \cup \{x\})]\|$

  $\le (k - r + 2) + [(k - r + b + 1) + 2(r - b - 1)] + 2(\nu - (k+1))$

  $= 2\nu - b - 1$ $\therefore$ (iii) hold

1.9 Hamiltonian Graphs

- **Corollary 1.10.4: Every strongly connected tournament is hamiltonian.**
  By Thm 1.5 or Thm 1.10

- **Corollary 1.10.5:** G**: simple digraph.**
  $\forall\, x, y \in V(G)$ **either** $(x, y) \in E(G)$ **or** $d_G^{+}(x) + d_G^{-}(y) \geq \nu$
  $\Rightarrow G$ **is hamiltonian.**

- **Corollary 1.10.6: Every tournament contains a Hamilton path.**

- **Exercise: 1.9.1**

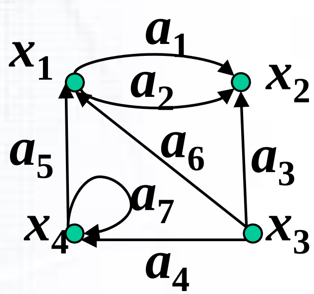
- **加: 1.9.3, 1.9.4, 1.9.5, 1.9.9**

Chapter 1
Basic Concepts of Graphs

§ 1.10 Matrix Presentation of Graphs

1.10 Matrix Presentation of Graphs

- **<u>Def</u>: The adjacency matrix of a graph $G = (V, E)$, where $V = \{x_1, x_2, \ldots, x_\nu\}$ is a $\nu \times \nu$ matrix $A(G) = (a_{ij})$, where $a_{ij} = \mu(\{x_i\}, \{x_j\}) = |E_G(\{x_i\}, \{x_j\})|$**
- **ex:**



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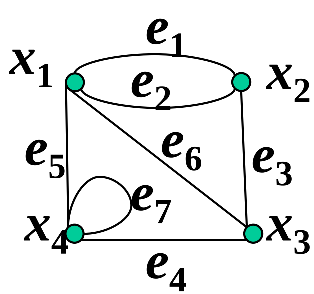


G

$$A(D) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad A(G) = \begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

- **<u>Def</u>: The incidence matrix of a loopless graph G is a $\nu \times \varepsilon$ matrix, $M(G) = (m_x(e))$, $x \in V(G)$ and $e \in E(G)$,**

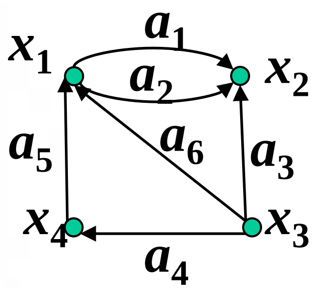
  where, if G is directed, then

$$m_x(e) = \begin{cases} 1, & \text{if } x \text{ is the tail of } e; \\ -1, & \text{if } x \text{ is the head of } e; \\ 0, & \text{o.w.} \end{cases}$$

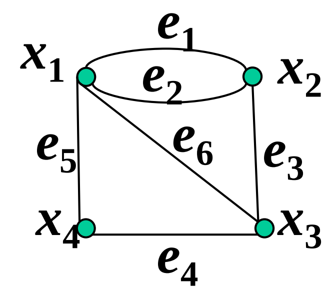
  and if G is undirected, then $m_x(e) = \begin{cases} 1, & \text{if } e \text{ is incident with } x; \\ 0, & \text{o.w.} \end{cases}$

1.10 Matrix Presentation of Graphs

- **ex:**



D_1



G_1

$$M(D_1) = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

$$M(G_1) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

- **<u>Note</u>: For computer**

1.10 Matrix Presentation of Graphs

- Def: ① $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$ be a permutation of $\{1, 2, \ldots, n\}$.
  ② an $n \times n$ permutation matrix $P = (p_{ij})$ defined by $p_{ij} = \begin{cases} 1, & \text{if } j = \sigma(i); \\ 0, & \text{o.w.} \end{cases}$
- ex: $\sigma = (2\ 3\ 1\ 4)$, $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- Remark: ① If A, B are the adjacency matrices of G, H,
  $G \cong H \Leftrightarrow \exists\ \nu \times \nu$ permutation matrix P s.t. $A = P^{-1}BP$
  ② If M, N are incidence matrices of G, H,
  $G \cong H \Leftrightarrow \exists\ \nu \times \nu$ permutation matrix P and $\varepsilon \times \varepsilon$ permutation matrix Q s.t. $M = PNQ$

1.10 Matrix Presentation of Graphs

- **<u>Theorem 1.11</u>: Let A be the adjacency matrix of a digraph G with $V(G) = \{x_1, x_2, \ldots, x_\nu\}$ and $A^k = (a_{ij}^{(k)})$ for $k \geq 1$. Then $a_{ij}^{(k)}$ = the number of different (x_i, x_j)-walks of length k in G.**

Proof.

Prove by induction on k.

① When $k = 1$, it's trivial.

② Assume $a_{ij}^{(k-1)}$ is the number of different (x_i, x_j)-walks of length $k - 1$.

$\because A^k = A^{k-1} \cdot A$, $\therefore a_{ij}^{(k)} = \sum_{l=1}^{\nu} a_{il}^{(k-1)} \cdot a_{lj}$ —— (★)

$\because$ Every (x_i, x_j)-walk of length k in G

$= (x_i, x_l)$-walk of length $k - 1$ in G + an edge (x_l, x_j)

$\therefore$ By I.H and (★), $a_{ij}^{(k)}$ = the number of different (x_i, x_j)-walks of length k.

1.10 Matrix Presentation of Graphs

- **<u>Note</u>: ① ∵ $\exists!$ (x, y)-walk of length n for any $x, y \in V(B(d, n))$.**
  ∴ $A(B(d, n))^n = J$ (= an square matrix all of whose entries are 1)
  ② $A(K(d, n))^n + A(K(d, n))^{n-1} = J$

- **<u>Example 1.10.1</u>: $\nexists$ (Δ, k)-Moore digraph for $\Delta \geq 2$ and $k \geq 2$.**
  Proof. 略

max. degree ↘ **diameter** ↙

- **<u>Note</u>: $K(\Delta, 2)$ is a maximum $(\Delta, 2)$-digraph. (目前唯一知道的)**
  Proof. By <u>Example 1.10.1</u>, $\nu(\Delta, 2)$-diagaph $\leq \Delta^2 + \Delta$, and
  $K(\Delta, 2)$ had order $\Delta^2 + \Delta$, and maximum degree = Δ, diameter = 2.

1.10 Matrix Presentation of Graphs

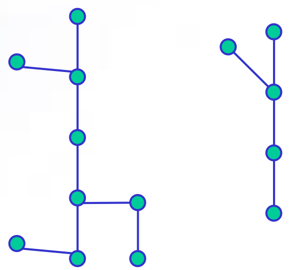
- 後略
- **exercise: 1.10.1; 1.10.2**
- **加: 1.10.8**

Chapter 2
Trees and Graphic Spaces

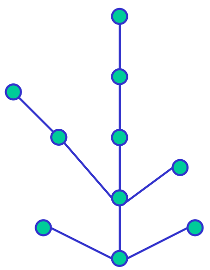
§ 2.1 Trees and Spanning Trees (1)

2.1 Trees and Spanning Trees

- **<u>Def</u>: ① A graph is called a forest (or acyclic graph) if it contains no cycle.**
  ② A connected forest is called a tree.
- **<u>Note</u>: ① Forests and trees both are bipartite simple graphs.**
  ② Restrict to undirected graphs.
- **ex:**



(a)



(b)

2.1 Trees and Spanning Trees

- **Theorem 2.1: The following statements are equivalent**

  (a) G is a tree.

  (b) G has no loop and $\forall\ x, y \in V(G)$, $\exists!$ xy-path.

  (c) G is connected and $\forall e \in E(G)$, $\omega(G - e) = 2$

  (d) G is connected and $\varepsilon = \nu - 1$.

Proof. (1/7)

(a) $\Rightarrow$ (b):

$\because$ G is a tree, $\therefore$ G is a connected simple graph contains no cycle.

Suppose $\exists$ $u, v \in V(G)$, s.t. $\exists$ distinct uv-path P_1, P_2.

$\because$ $P_1 \neq P_2$. $\therefore$ $\exists$ $e = xy \in E(G)$ s.t. $e \in E(P_1)$ and $e \notin E(P_2)$.

$\therefore$ $(P_1 \cup P_2) - e$ is connected and contains a xy-path P.

$\Rightarrow$ $P + e$ is a cycle in G. $\rightarrow\leftarrow$

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Proof. (2/7)

(b) $\Rightarrow$ (c):

$\because$ $\forall\ x, y \in V(G)$, $\exists!$ xy-path. $\therefore$ G is connected.

$\forall\ e = xy \in E(G)$, G is connected. $\therefore$ $\omega(G - e) \leq 2$ —— ①

$\because$ xey is a unique xy-path (by (b))

$\therefore$ x, y are in different connected components of $G - e$.

$\Rightarrow$ $\omega(G - e) \geq 2$ —— ②

By ①②, $\forall\ e \in E$(G), $\omega(G - e) = 2$.

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Proof. (3/7)

(c) $\Rightarrow$ (a):

Suppose G is not a tree. $\because$ G is connected. $\therefore$ G contains a cycle C.

Let $e = xy \in E(C)$.

$\because$ G is connected and $\omega(G - e) = 2$, $\therefore$ e is not a loop,

and x, y are in different connected components of $G - e$,

but $C - e$ is an xy-path in $G - e$. $\rightarrow\leftarrow$

$\therefore$ G is a tree.

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  (d) G is connected and $\varepsilon = \nu - 1$.

Proof. (4/7)

(a) $\Rightarrow$ (d): Prove by induction on $\varepsilon \geq 0$.

① $\varepsilon = 0$, its trivial.

② Suppose $\varepsilon = \nu - 1$ for any tree with $\varepsilon < m$.

Now, consider a tree G with $\varepsilon = m \geq 1$.

Choose an edge $e \in E(G)$, $\because$ (a) $\Rightarrow$ (c), $\therefore$ $\omega(G - e) = 2$.

Let G_1, G_2 be two components of $G - e$.

$\because$ G_1, G_2 both are trees and $\varepsilon(G_1) < m$, $\varepsilon(G_2) < m$.

$\therefore$ By I.H, $\varepsilon(G_i) = \nu(G_i) - 1$ $\forall\ i = 1, 2$.

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  (d) G is connected and $\varepsilon = \nu - 1$.

Proof. (5/7)

(a) $\Rightarrow$ (d): ② Choose an edge $e \in E(G)$, $\because$ (a) $\Rightarrow$ (c), $\therefore$ $\omega(G - e) = 2$.

Let G_1, G_2 be two components of $G - e$.

$\because$ G_1, G_2 both are trees and $\varepsilon(G_1) < m$, $\varepsilon(G_2) < m$.

$\therefore$ By I.H, $\varepsilon(G_i) = \nu(G_i) - 1\ \ \forall\ i = 1, 2$.

$\Rightarrow \varepsilon(G) = \varepsilon(G_1) + \varepsilon(G_2) + |\{e\}| = (\nu(G_1) - 1) + (\nu(G_2) - 1) + 1$

$= \nu(G_1) + \nu(G_2) - 1 = \nu(G) - 1$

$\therefore$ By the principle of mathematical induction, (d) is necessary.

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Proof. (6/7)

(d) $\Rightarrow$ (a): Prove by induction on $\nu \geq 1$.

① $\nu = 1$, then $\varepsilon = 0$. $\therefore$ G is a trivial and hence no cycle.

② Suppose any connected graph of order n and $\varepsilon = n - 1$ contains no cycle.

Now, consider a connected graph G with order $n + 1 \geq 2$, and size $\varepsilon = (n + 1) - 1 = n$.

$\because$ G is connected. $\therefore$ $\delta(G) \geq 1$.

If $\delta(G) \geq 2$, then by Coro. 1.1:

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Proof. (7/7)

(d) $\Rightarrow$ (a): If $\delta(G) \geq 2$, then by Coro. 1.1:

$$2n = 2\varepsilon = \sum_{x \in V(G)} d_G(x) \geq 2(n+1) \;\rightarrow\leftarrow$$

$\therefore$ $\delta(G) = 1$, i.e. $\exists\, x \in V(G)$, s.t. $d_G(x) = 1$,

then $G - x$ is connected graph of order n, $\varepsilon = n - 1$.

$\therefore$ By I.H., $G - x$ contains no cycle.

$\Rightarrow$ G contains no cycle.

$\Rightarrow$ G is connected and contains no cycle.

$\therefore$ G is a tree.

2.1 Trees and Spanning Trees

- **Corollary 2.1: A graph G is a forest $\Leftrightarrow \varepsilon = \nu - \omega$**

- **Example 2.1.1: G: a forest and $\delta(G) \geq 1$, G contains $\geq 2\omega$ 1-degree vertices.**

  Proof.

  Let $R = \{x \mid d_G(x) = 1, \forall\, x \in V(G)\}$, and $|R| = r$.

  By Coro. 2.1, and Coro. 1.1:

$$2(\nu - \omega) = 2\varepsilon = \sum_{v \in V(G)} d_G(x) = \sum_{v \in V \setminus R} d_G(x) + r$$

$$\geq 2(\nu - r) + r = 2\nu - r$$

$$\Rightarrow r \geq 2\omega$$