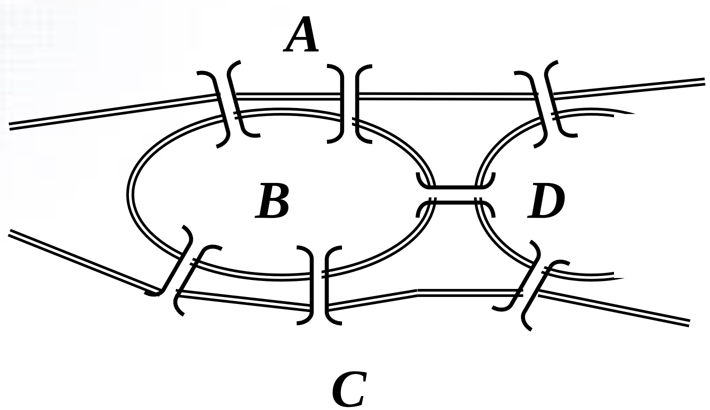
Chapter 1
Basic Concepts of Graphs

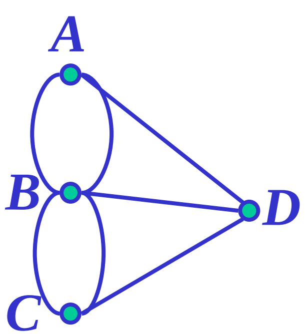
§ 1.8 Eulerian Graph

1.8 Eulerian Graph

- **Def:**
  - **Euler (directed) trail: traversed every edge**
  - **Euler (directed) circuit**
  - **eulerian; noneulerian**

- **History: Königsberg seven bridges problem: (1736), Euler(1707~1783)**





1.8 Eulerian Graph

- Euler Leonhard（1707～1783）生於 Basel，卒於聖彼得堡。瑞士數學家，貢獻遍及數學各領域，是數學史上最偉大的數學家之一，也是最多產的數學家。
- Euler 生於公元1707年4月15日，隨即其家庭就搬到 Basel 近郊的 Riehen。Euler 的父親 Paul Euler 是一名加爾文教派的教師，但他在大學求學期間與 John Bernoulli 的哥哥 Jacob Bernoulli 家住過並從 Jacob 身上學了不少數學。
- Paul 希望 Euler 讀神學，但他卻犯了最大的錯誤，在 Euler 很小的時候便教他數學，挑動了他內心中的數學靈魂。Paul 計劃 Euler 將來成為牧師傳道、宣揚聖經真理，但 Euler 讀大學時所接觸卻是 Bernoulli 家族這個宣傳數學真理的家族。Paul 只是希望其兒子成為 Riehen 的牧師。但 John Bernoulli 卻跟他勸說「Euler 註定要成為大數學家，而非 Riehen 的牧師。」我們感謝上帝，因為 Paul 的信仰並沒有使他走火入魔，把自己的旨意當作上帝的旨意。最後 Paul 終於在 John Bernoulli 之勸說同意 Euler 攻讀數學。從此展開他燦爛的學術生涯，並成為數學史上最偉大的數學家之一。
- 歐拉在約翰·伯努里的指導下學習，並跟隨約翰的長子尼古拉在聖彼得堡學習;在1741年(34)，受腓特烈大帝之邀，至1766年(59)住在柏林學院，而從1766年(59)至1783年(76)住在聖彼得堡;歐拉結過兩次婚，並有十三個孩子;雖然他在1735年(28)和1766年(59)雙眼先後失明，但藉助其驚人的記憶力，仍以口授的方式留下他的發現，並繼續為科學有所貢獻。
- Euler 的數學生涯開始於牛頓去世的那一年。這實在是一個不可多得的時代，解析幾何、微積分的發展已達到某種程度，並被應用到不同領域的問題。更重要的是牛頓的萬有引力定律已經是天文、物理學的基礎。並進而是研究各類物理問題不可或缺的工具。Euler躬逢其時，再加上自身的才華，逐一對整個數學 ─純數學與應用數學─進行有系統的研究。
- Euler對於數學的貢獻是全面性的，從數論到分析，無論抽象或應用，基本上我們可以稱他是一個百科全書型的數學家。
- 「他是有史以來瑞士最多產的科學家，也是一個不可思議的數學幻想家，他在任何領域都能發現數學，在任何情況都能進行研究。…」

1.8 Eulerian Graph

- **Thm 1.7: A digraph is eulerian ⇔ it is connected and balanced**

Proof. (1/2)

(⇒) Suppose *G* is an Euler digraph.

Let *C* be an Euler directed circuit of *G*.

① By definition, *G* is connected.

② $\forall\ x \in V(C)$, $\exists$ 2 edges are incident with *x*:

one out-going and one in-coming edge

$\therefore\ d_G^+(x) = d_G^-(x),\ \forall\ x \in V(G)$

1.8 Eulerian Graph

- **Thm 1.7: A digraph is eulerian $\Leftrightarrow$ it is connected and balanced**

  Proof. (2/2)

  ($\Leftarrow$) Suppose G is a connected and balanced digraph is not eulerian,

  and the number of edges of G are as few as possible.

  By exercise 1.7.3, $\because$ $\delta^+ \neq 0$, $\delta^- \neq 0$, $\therefore$ $\exists$ directed cycle in G.

  Let C be a maximum directed circuit in G.

  $\because$ G is not eulerian. $\therefore$ $G - E(C) \neq \phi$.

  Let G' be a connected component of $G - E(C)$ with $\varepsilon(G') \neq 0$.

  $\because$ G, C are balanced. $\therefore$ G' is balanced.

  ($\because$ $\forall\, x \in V(G')$, $d_{G'}^+(x) = d_G^+(x) - d_C^+(x) = d_G^-(x) - d_C^-(x) = d_{G'}^-(x)$)

  $\because$ G' is a connected and balanced digraph with $\varepsilon(G') < \varepsilon(G)$

  $\therefore$ G' is eulerian, say C' is a directed circuit of G'

  $\because$ G is connected , $\therefore$ $V(C) \cap V(C') \neq \phi$.

  i.e. $C \oplus C'$ is a directed circuit of G with length > $\varepsilon(C)$, $\rightarrow\leftarrow$ to choice of C.

1.8 Eulerian Graph

Thm 1.7: A digraph is eulerian ⇔ it is connected and balanced

- **Corollary 1.7.1: A connected digraph G contains an Euler directed trail from x to y**

  $\Leftrightarrow$ $d_G^+(x) - d_G^-(x) = 1 = d_G^-(y) - d_G^+(y)$; and
  $d_G^+(u) = d_G^-(u), \forall u \in V \backslash \{x, y\}$ (✷)

Proof. Let $G' = G + \{(y, x)\}$

(⇒) Let T be an Euler directed trail from x to y in G.

The $T + (y, x)$ is an Euler directed circuit of G'.

By Thm 1.7, G' is balanced.

$$\Rightarrow \begin{cases} d_G^+(u) = d_{G'}^+(u) = d_{G'}^-(u) = d_G^-(u), \forall u \in V \backslash \{x, y) \\ d_G^+(x) = d_{G'}^+(x) = d_{G'}^-(x) = d_G^-(x) + 1, \\ d_G^-(y) = d_{G'}^-(y) = d_{G'}^+(y) = d_G^+(y) + 1. \end{cases} \Rightarrow (✷)$$

(⇐) Suppose G saticfies (✷)

Then G' is balanced.

By Thm 1.7, G' is eulerian, let C be an Euler directed circuit of G'.

⇒ $C - (y, x)$ is an Euler directed trail from x to y in G.

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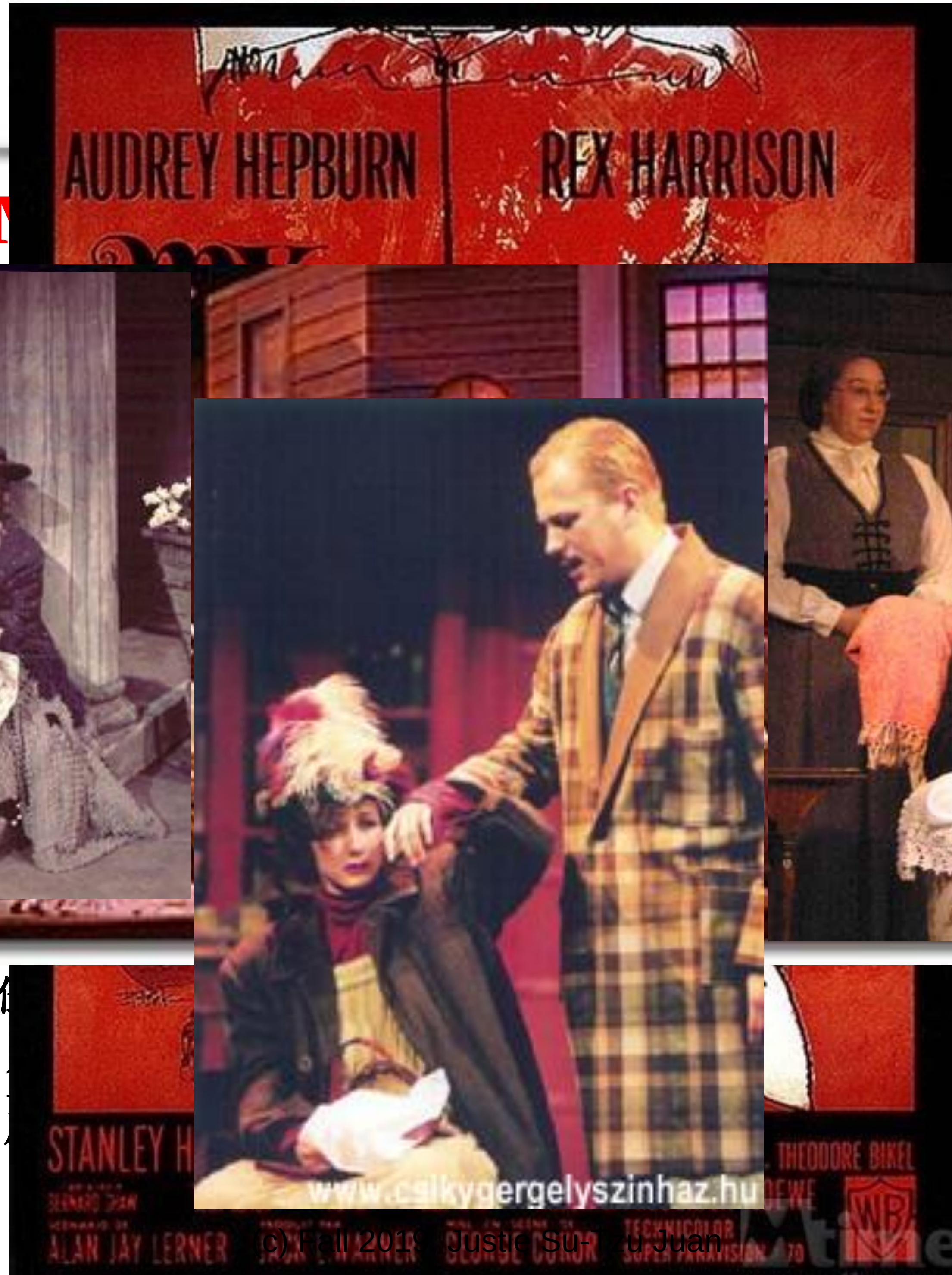

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1.8 Eulerian Graph

Example 1.7.3: *G*: an undirected graph,
G has a balanced oriented graph ⇔ *G* contains no vertex of odd degree.

- **Corollary 1.7.2**: An undirected graph *G* is eulerian ⇔ *G* is connected and has no vertex of odd degree.

  Proof.

  Thm 1.7: A digraph is eulerian ⇔ it is connected and balanced

  (⇒) Let *C* be an Euler circuit of *G*.
  Let *D* be an oriented graph of *G* by assigning every edge of *G* a direction in order of *C*. ∴ *D* is eulerian.
  By Thm 1.7, *D* is connected and balanced.
  $\Rightarrow \forall x \in V(G) = V(D)$, $d_G(x) = d_D^+(x) + d_D^-(x) = 2d_D^+(x)$ is even.
  i.e. *G* contains no odd vertex.

  (⇐) By Example 1.7.3, ∃ a balanced oriented graph *D* of *G*.
  By Thm 1.7, *D* is eulerian.
  Let *C* be an Euler directed circuit of *D*.
  ∴ the underlying graph of *C* is an Euler circuit of *G*.
  i.e. *G* is eulerian.

1.8 Eulerian Graph

Corollary 1.7.2: An undirected graph G is eulerian $\Leftrightarrow$
G is connected an has no vertex of odd degree.

- **Corollary 1.7.3: G: an undirected graph,**
  G contains an Euler trail $\Leftrightarrow$ G is connected and the number of odd vertices ≤ 2

  Proof. (2/2)

  ($\Leftarrow$) If G contains no odd vertex, then by Corollary 1.7.2,
  G contains an Euler circuit $\Rightarrow$ G contains an Euler trail
  If G contains odd vertex, then G contains exactly 2 odd vertices.
  (By Corollary 1.1.2)
  Let x, y be two odd vertices of G.
  Let $G + xy = G'$, then G' is connected and no odd vertex.
  $\therefore$ By Corollary 1.7.2, G' contains an Euler circuit C.
  $\Rightarrow$ $C - xy$ is an Euler trail of G.

1.8 Eulerian Graph

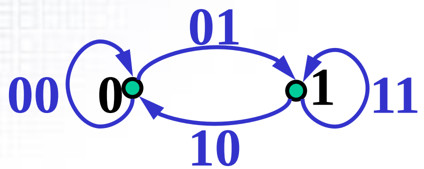
- **<u>Example 1.8.1</u>: *N*-dimensional *d*-ary de Bruijn digraph *B*(*d*, *n*) = (*V*, *E*),** where $n \geq 1$, $d \geq 2$.

  $V = \{x_1x_2\ldots x_n : x_i \in \{0, 1, \ldots, d-1\}, i = 1, 2, \ldots, n\}$

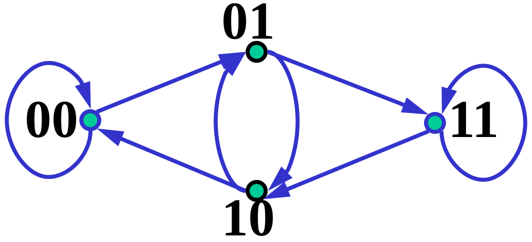
  $E = \{(x_1x_2\ldots x_n, y_1y_2\ldots y_n) : y_i = x_{i+1}, i = 1, 2, \ldots, n-1; y_n \in \{0, 1, \ldots, d-1\}\}$

- **<u>Note</u>:** $B(d, n) = L^{n-1}(K_d^{+})$

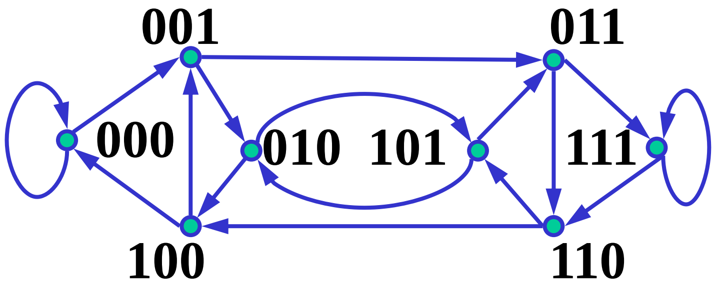
- **ex:**



B(2, 1)

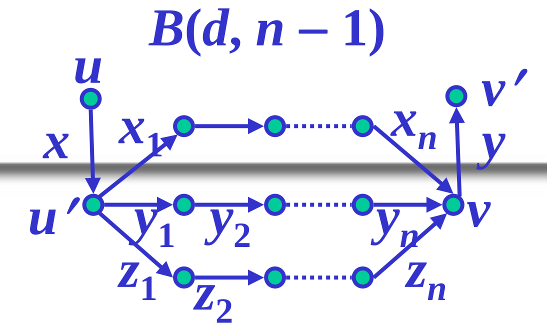


B(2, 2)



B(2, 3)

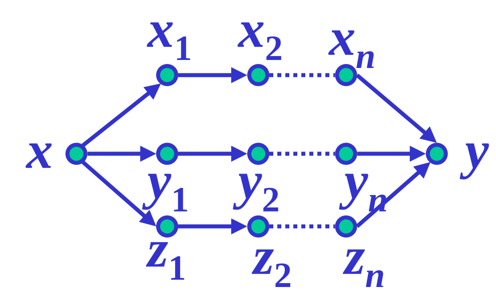
1.8 Eulerian Graph



- **<u>Example 1.8.2</u>: $\forall\, x \neq y \in V(B(d, n))$, $\exists\, d - 1$ internally disjoint (x, y)-paths of length at most $n + 1$.**

Proof. (1/5)

Proof by induction on $n \geq 1$.



① $\because B(d, 1) = K_d^+$, $\therefore$ the theorem is true for $n = 1$

② Suppose $n \geq 2$ and the theorem holds for $n - 1$.

Assume $x \neq y \in V(B(d, n))$:

<u>case 1</u> $(x, y) \notin E(B(d, n))$:

$\because B(d, n) = L(B(d, n - 1))$. $\therefore x, y$ correspond to 2 edge in $B(d, n - 1)$.

let such 2 edge be $x = (u, u')$, $y = (v, v')$

$\because (x, y) \notin E(B(d, n))$ $\therefore u' \neq v$ in $B(d, n - 1)$

By I.H., $\exists\, d - 1$ internally disjoint (u', v)-paths of length at most n in $B(d, n - 1)$

$\Rightarrow \exists\, d - 1$ internally disjoint (x, y)-paths of length at most $n + 1$ in $B(d, n)$

1.8 Eulerian Graph

- **<u>Example 1.8.2</u>: $\forall\ x \neq y \in V(B(d, n))$, $\exists\ d-1$ internally disjoint (x, y)-paths of length at most $n + 1$.**

  Proof. (2/5)

  Proof by induction on $n \geq 1$.

  ① $\because B(d, 1) = K_d^+$, $\therefore$ the theorem is true for $n = 1$

  ② Suppose $n \geq 2$ and the theorem holds for $n - 1$.

  Assume $x \neq y \in V(B(d, n))$:

  <u>case 2</u> $(x, y) \in E(B(d, n))$:

  say $x = x_1x_2\ldots x_{n-1}x_n$, $y = x_2x_3\ldots x_ny_n$

  Construct $d - 1$ (x, y)-walks $P_1, P_2, \ldots, P_{d-1}$ as:

  <u>case 2.1</u>: $x_1 \neq y_n$: $P_1 = (x_1x_2\ldots x_{n-1}x_n) - (x_2x_3\ldots x_ny_n)$

  $P_j = (x_1x_2\ldots x_n) - (x_2x_3\ldots x_nu_j) - (x_3x_4\ldots u_jx_2) - (x_4x_5\ldots u_jx_2x_3) - (u_jx_2\ldots x_{n-1}x_n) - (x_2x_3\ldots x_ny_n)$,

  where $u_2, \ldots, u_{d-1}$ are $d - 2$ distinct elements in $\{0, 1, 2, \ldots, d - 1\} \setminus \{x_1, y_n\}$.

1.8 Eulerian Graph

- **Example 1.8.2**: $\forall\, x \neq y \in V(B(d, n)), \exists\, d - 1$ **internally disjoint** (x, y)**-paths of length at most** $n + 1$.

  Proof. (3/5)

  case 2.1: $x_1 \neq y_n$: $P_1 = (x_1x_2 \ldots x_{n-1}x_n) - (x_2x_3 \ldots x_ny_n)$

  $P_j = (x_1x_2 \ldots x_n) - (x_2x_3 \ldots x_nu_j) - (x_3x_4 \ldots u_jx_2) - (x_4x_5 \ldots u_jx_2x_3) -$
  $(u_jx_2 \ldots x_{n-1}x_n) - (x_2x_3 \ldots x_ny_n)$,

  where $u_2, \ldots, u_{d-1}$ **are** $d - 2$ **distinct elements in** $\{0, 1, 2, \ldots, d - 1\} \backslash \{x_1, y_n\}$

  Note that length(P_1**) = 1, length(**P_j**) =** $n + 1$, $\forall\, j = 2, 3, \ldots, d - 1$

  Claim 1: $P_2, P_3, \ldots, P_{d-1}$ **are internally disjoint.**

  <Proof> If $\exists\, x_2x_3 \ldots x_nu_i = x_{a+1}x_{a+2} \ldots x_nu_jx_2x_3 \ldots x_a$

  then $u_i + \sum_{i=2}^{n} x_i = u_j + \sum_{i=2}^{n} x_i \Rightarrow u_i = u_j \quad \rightarrow\leftarrow$

  same to $\nexists\, x_{b+1}x_{b+2} \ldots x_nu_ix_2x_3 \ldots x_b = x_{a+1}x_{a+2} \ldots x_nu_jx_2x_3 \ldots x_a$

  $\because$ (x, y)**-walk contains a** (x, y)**-path as its subgraph.**

  $\therefore \exists\, P_2', P_3', \ldots, P_{d-1}'$ **are internally disjoint** (x, y)**-paths.**

1.8 Eulerian Graph

- **Example 1.8.2**: $\forall\ x \neq y \in V(B(d, n))$, $\exists\ d - 1$ internally disjoint (x, y)-paths of length at most $n + 1$.

  Proof. (4/5)

  case 2.1: $x_1 \neq y_n$: $P_1 = (x_1x_2\ldots x_{n-1}x_n) - (x_2x_3\ldots x_ny_n)$

  $P_j = (x_1x_2\ldots x_n) - (x_2x_3\ldots x_nu_j) - (x_3x_4\ldots u_jx_2) - (x_4x_5\ldots u_jx_2x_3) - (u_jx_2\ldots x_{n-1}x_n) - (x_2x_3\ldots x_ny_n)$,

  Claim 2: $P_j' \neq P_1$, $\forall\ 2 \leq j \leq d - 1$

  <Proof.> if $\exists\ 2 \leq j \leq d - 1$ s.t. $P_j' = P_1$

  then $\exists$ 2 consecutive vertices on P_j s.t.

  $$\begin{cases} x_cx_{c+1}\ldots x_nu_jx_2\ldots x_{c-1} = x = x_1x_2\ldots x_n \\ x_{c+1}x_{c+2}\ldots x_nu_jx_2\ldots x_c = y = x_2x_3\ldots x_uy_n \end{cases}$$

  $\Rightarrow x_1 = x_c = y_n \quad \rightarrow\leftarrow$

1.8 Eulerian Graph

- **Example 1.8.2**: $\forall\ x \neq y \in V(B(d, n))$, $\exists\ d-1$ internally disjoint (x, y)-paths of length at most $n+1$.

  Proof. (5/5)

  Claim 1: $P_2, P_3, \ldots, P_{d-1}$ are internally disjoint.

  case 2.2: $x_1 = y_n$: let

  $P_j = (x_1x_2\ldots x_n) - (x_2x_3\ldots x_nu_j) - (x_3\ldots x_nu_jx_2) - (u_jx_2\ldots x_n) - (x_2x_3\ldots x_ny_n)$

  where $u_1, u_2, \ldots, u_{d-1}$ are $d-1$ distinct elements in $\{0, 1, 2, \ldots, d-1\}\backslash\{x_1\}$

  Then length$(P_j) = n+1$, $\forall\ j = 1, 2, d-1$

  By Claim 1, we know $P_1, P_2, \ldots, P_{d-1}$ are internally disjoint

  $\therefore\ \exists\ P_1'P_2'\ldots P_{d-1}'$ are internally disjoint (x, y)-paths

1.8 Eulerian Graph

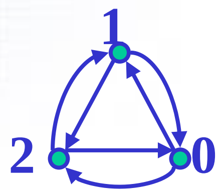
- **Example 1.8.3**: ***n*-dimensional *d*-ary Kautz digraph *K*(*d*, *n*)** = (*V*, *E*), where $n \geq 1$, $d \geq n$.

  $V = \{x_1x_2\ldots x_n: x_i \in \{0, 1, \ldots, d\}, x_{i+1} \neq x_i, i = 1, 2, \ldots, n-1\}$

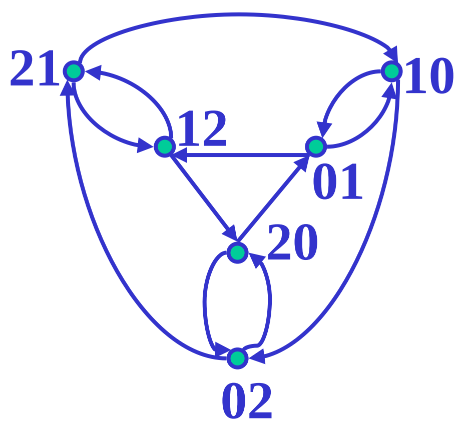
  $E = \{(x_1x_2\ldots x_n, y_1y_2\ldots y_n): y_i = x_{i+1}, \forall\, i = 1, 2, \ldots, n-1, y_n \in \{0, 1, \ldots, d\}\backslash\{x_n\}\}$

- **Note**: $K(d, n) = L^{n-1}(K_{d+1})$

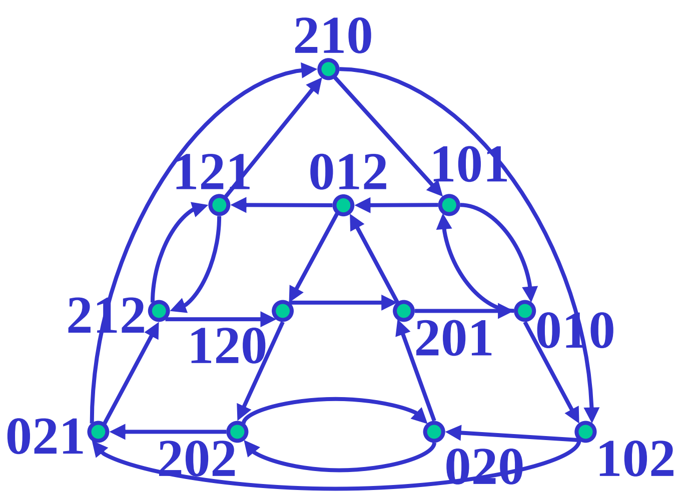
- **ex:**



K(2, 1)



K(2, 2)



K(2, 3)

1.8 Eulerian Graph

- **Remark:** ① $B(d, n)$ and $K(d, n)$ are d-regular and have diameter n, eulerian
  ② $\nu(B(d, n)) = d^n$, $\nu(K(d, n)) = (d + 1)d^{n-1} = d^n + d^{n-1}$

- **exercise: 1.8.3, 1.8.9**

- **加: 1.8.5, 1.8.7**

Chapter 1
Basic Concepts of Graphs

§ 1.9 Hamiltonian Graphs (1)

1.9 Hamiltonian Graphs

- **Def:**

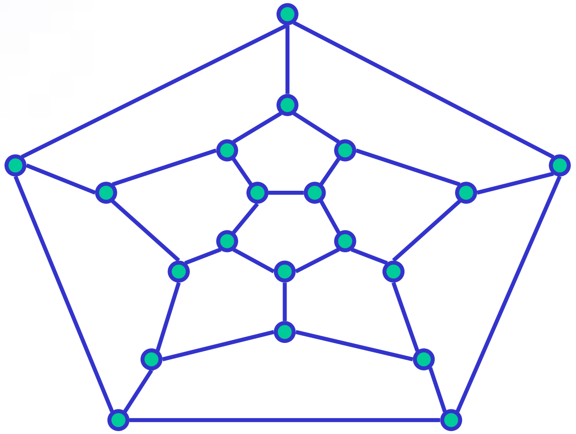
  ① A Hamilton (directed) cycle = a (di)cycle that contains every vertex of a (di)graph.

  ② A (di)graph is called hamiltonian if it contains a Hamilton cycle, otherwise it is called non-hamiltonian

  ③ Hamilton problem

- **History: 1857, Hamilton (1805 – 1865): "The Traveler's Dodecahedron"**

  "A Voyage Round the World"



⇒ 1855, Kirkman

1.9 Hamiltonian Graphs

- **Example 1.9.1: The Petersen graph is non-hamiltonian.**

  Proof.

  Let G is a Petersen graph at the Figure

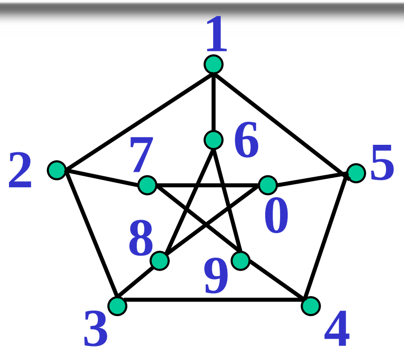
  

  Fig 1.24

  Let $T = \{16, 27, 38, 49, 50\}$.

  If G contains a Hamilton cycle C, then $|E(C) \cap T|$ is even, and $|E(C) \cap T| \neq 2$.

  $\therefore\ |E(C) \cap T| = 4$.

  W.L.O.G. Let $E(C) \cap T = \{27, 38, 49, 50\}$

  $\Rightarrow E(C) \supseteq \{12, 15, 86, 96\}$. $(\because\ 16 \notin E(C))$

  $$\left.\begin{array}{l} \because\ \{12, 27\} \subseteq E(C) \Rightarrow 23 \notin E(C) \\ \because\ \{15, 50\} \subseteq E(C) \Rightarrow 45 \notin E(C) \end{array}\right\} \Rightarrow \{34\} \in E(C)\ \rightarrow\leftarrow$$

  $\therefore$ G is non-hamiltonian.

1.9 Hamiltonian Graphs

- **Note:**
  - G is Eulerian $\Leftrightarrow$ $L(G)$ is hamiltonian (1.9.8(a))
  - No notririal necessary and sufficient condition for a graph to be hamiltonian is known!!
  - G is hamiltonian digraph $\Rightarrow$ G is strongly connected
  - G is hamiltonian undirected graph $\Rightarrow$ G contains no cut-vertex

  } (1.9.1)

- **Theorem 1.8**: If G is hamiltonian, then $\omega(G - S) \le |S|, \forall S \subset V$.

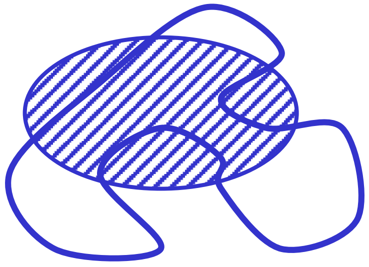
  Proof.

  Let C be a Hamilton cycle of G.

  Then $\omega(C - S) \le |S|, \forall S \subset V(C) = V(G)$

  $\because$ $C - S$ is a spanning subgraph of $G - S$.

  $\therefore$ $\omega(G - S) \le \omega(C - S) \le |S|, \forall S \subset V(G)$.



1.9 Hamiltonian Graphs

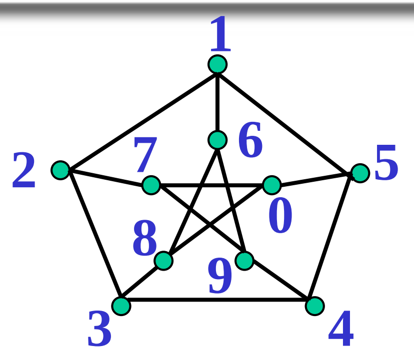


Fig 1.24

- **Example 1.9.2**: $\omega(G - S) \le |S|\ \forall\ S \subset V \Rightarrow G$ **is hamiltonian ?**

  Proof.

  Let G be Petersen graph.

$$\omega(G-S) \begin{cases} = 1 \le |S|, \text{ for } |S| \le 2 \\ \le 2 < |S|, \text{ for } |S| = 3 \\ \le 3 < |S|, \text{ for } |S| = 4 \\ \le 5 \le |S|, \text{ for } |S| \ge 5 \end{cases} \quad \forall\ S \subset V(G)$$

  $\therefore$ **G satisfy $\omega(G - S) \le |S|,\ \forall\ S \subset V$, but G is non-hamiltonian**

- **Note: Parallel edges and loops do not affect whether a graph is hamiltonian.**

  $\therefore$ **We limit our discussion to simple graphs.**

- **Def: Hamilton path, Traceable**