Chapter 1
Basic Concepts of Graphs

§ 1.6 Distances and Diameter (2)

1.6 Distances and Diameter

- **Def:**
  - The **radius** of $G \equiv \mathrm{rad}(G) = \min_{x \in V(G)} \{ \max_{y \in V(G)} \{d_G(x, y)\}\}$
  - A vertex x is called a **central** of G if $\max_{y \in V(G)} \{d_G(x, y)\}$ ($= \min_{x \in V(G)} \{\max_{y \in V(G)} \{d_G(x, y)\}\}$) $= \mathrm{rad}(G)$

- **Note**: $\mathrm{rad}(G) \le d(G) \le 2\mathrm{rad}(G)$

  Proof. exercise 1.6.6

- **Example 1.6.4**: For digraph G, $\mathrm{rad}(G) \le r \Rightarrow \nu(G) \le 1 + r \cdot \Delta^r$.

  Proof.

  Let x be a central vertex of G, and $J_i = \{y \mid d_G(x, y) = i\}$

  $$\Rightarrow \left\{ \begin{array}{l} |J_1| \le \Delta \\ |J_i| \le \Delta \cdot |J_{i-1}| \end{array} \right\} \Rightarrow |J_i| \le \Delta^r$$

  $$\Rightarrow \nu(G) \le 1 + \Delta + \Delta^2 + \ldots + \Delta^r \le 1 + r \cdot \Delta^r$$

1.6 Distances and Diameter

- **Def**: G: connected undirected graph or strongly connected digraph with $\nu \geq 2$.
  - ① The **mean** or **average distance** of $G \equiv$

$$m(G) \equiv \frac{1}{\nu(\nu-1)} \sum_{x,y \in V(G)} (d_G(x,y)),$$

  - ② $\sigma(G) = \sum_{x,y \in V(G)} d_G(x,y)$

- **Note**:
  - ① $m(G) \geq 1$
  - ② $m(G) = 1 \Leftrightarrow G$ is a complete graph
  - ③ For a directed cycle C_n, $n \geq 3$, $\sigma(C_n) = (1/2)n^2(n-1)$, $m(C_n) = n/2$

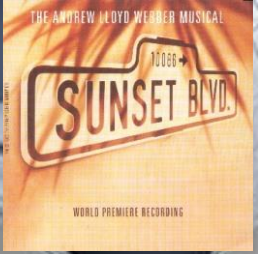
    <sol> $\sigma(C_n) = n(1 + 2 + \ldots + (n-1)) = n \cdot (n(n-1))/2 = (1/2)n^2(n-1)$

    $m(C_n) = (1/(n(n-1))) \cdot \sigma(C_n) = n/2$
  - ④ For an undirected cycle C_n, $m(C_n) = \begin{cases} (n+1)/4 & \text{, if } n \text{ is odd;} \\ n^2/(4(n-1)) & \text{, if } n \text{ is even.} \end{cases}$

    <sol> exercise 1.6.6

1.6 Distances and Diameter

- **Exercise: 1.6.4 (a), 1.6.6**
- **加: 1.6.4 (b), (c)**



音樂劇欣賞6-1

日落大道 Sunset Boulevard

- 原　　著：Billy Wilder一九五〇年的電影「紅樓金粉」。
- 作　　曲：Andrew Lloyd Webber
- 作　詞：Don Black & Christopher Hampton
- 電影原導演：Billy Wilder
- 首　　演：倫敦首演: 1993年7月12日 - 1997年4
  洛杉磯首演: 1993年12月9日 叔伯
  百老匯首演: 1994年11月17日 - 199
  加拿大首演: 1995年10月15日 – (N
  電影首映: 1964年10月21日

http://www.math.ucdavis.edu/~cchsiao/interest/musical/sunset/musicmain.htm

Chapter 1
Basic Concepts of Graphs

§ 1.7 Circuits and Cycles (1)

1.7 Circuits and Cycles

- **<u>Def</u>: ① A cycle of length *k* is called a *k*-cycle, denoted by C_k.**

  ② odd cycle (even cycle)

  ③ 3-cycle ≡ triangle

  ④ "cycle": graph or subgraph; undirecred or directed

1.7 Circuits and Cycles

- **Example 1.7.1: G: undirected graph with $\delta \geq 2$.**

  ① G contains a cycle C_n.

  ② If G is simple, then $n \geq \delta + 1$

  Proof.

  If G contains loops or parallel edges, then the conclusion holds clearly.

  Suppose G is simple and let $P = (x_0, x_1, \ldots, x_k)$ be a longest path.

  $\because N_G(x_0) \subseteq \{x_1, x_2, \ldots, x_k\}$

  and $|N_G(x_0)| = d_G(x_0) \geq \delta(G) \geq 2$

  $\therefore \exists\, x_i \in N_G(x_0),\ \delta \leq i \leq k$

  i.e. $(x_0, x_1, \ldots, x_{i-1}, x_i, x_0)$ is a cycle of length $i + 1$

  $\Rightarrow G$ contains a cycle of length $\geq \delta + 1$

- **Def: The girth of G, $g(G) \equiv$ the length of a shortest (directed) cycle in a (di)graph G.**

1.7 Circuits and Cycles

- <u>Example 1.7.2</u>: G: k-regular undirected graph with $g(G) = g \geq 3$, then

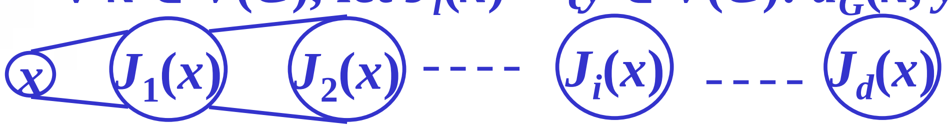
$$\nu(G) \geq \begin{cases} 1 + k + k(k-1) + \ldots + k(k-1)^{\frac{1}{2}(g-3)}, & \text{if } g \text{ is odd,} \\ 2(1 + (k-1) + \ldots + (k-1)^{\frac{1}{2}(g-2)}), & \text{if } g \text{ is even.} \end{cases}$$

Proof. (1/2)

$\because g \geq 3$, $\therefore G$ is simple.

<u>case 1</u>: g is odd: let $g = 2d + 1$, $d \geq 1$.

$\forall\, x \in V(G)$, let $J_i(x) = \{y \in V(G): d_G(x, y) = i, 0 \leq i \leq d.\}$



$\Rightarrow |J_0(x)| = 1$, $|J_1(x)| = k$,

$\because g = 2d + 1$, $\therefore \begin{cases} ① \ \forall\, y \in J_i(x), 0 \leq i \leq d, \exists!\ xy\text{-path in } G. \\ ② \ \forall\, y, z \in J_i(x), 0 \leq i \leq d-1, \nexists\ yz \in E(G) \end{cases}$

$\Rightarrow |J_i(x)| = k(k-1)^{i-1}$, $i = 1, 2, \ldots, d$

$\therefore\ \nu(G) \geq 1 + k + k(k-1) + \ldots + k(k-1)^{d-1}$

1.7 Circuits and Cycles

- **<u>Example 1.7.2</u>: *G*: *k*-regular undirected graph with $g(G) = g \geq 3$, then**

$$\nu(G) \geq \begin{cases} 1 + k + k(k-1) + \ldots + k(k-1)^{\frac{1}{2}(g-3)}, & \text{if } g \text{ is odd,} \\ 2(1 + (k-1) + \ldots + (k-1)^{\frac{1}{2}(g-2)}) & \text{, if } g \text{ is even.} \end{cases}$$

Proof. (2/2)

$\because g \geq 3$, $\therefore G$ is simple.

<u>case 1</u>: g is odd: let $g = 2d + 1$, $d \geq 1$.

$\therefore \nu(G) \geq 1 + k + k(k-1) + \ldots + k(k-1)^{d-1}$

$\because d = (g-1)/2$

$\therefore \nu(G) \geq 1 + k + k(k-1) + \ldots + k(k-1)^{\frac{1}{2}(g-3)}$

<u>case 2</u>: exercise 1.7.10(b)

- **<u>Def</u>: A (di)graph *G* of order ν (≥ 3) is vertex-pancyclic if $\forall x \in V(G)$, *x* is contained in (directed) cycle of length *k*, $\forall\, 3 \leq k \leq \nu$.**

1.7 Circuits and Cycles

- **Thm 1.5: Every strongly connected tournament of order $\nu (\geq 3)$ is vertex-pancyclic.**

  Proof. (1/3)

  Prove by induction on k (≥ 3):

  Let G be a strongly connected tournament.

  $\forall u \in V(G)$:

  ① For $k = 3$, let $S = N_G^+(u)$, $T = N_G^-(u)$.

  $\because$ G is strongly connected, $\therefore$ $S \neq \phi$, $T \neq \phi$.

  $\because$ G is tournament. $\therefore$ $S \cup T \cup \{u\} = V(G)$

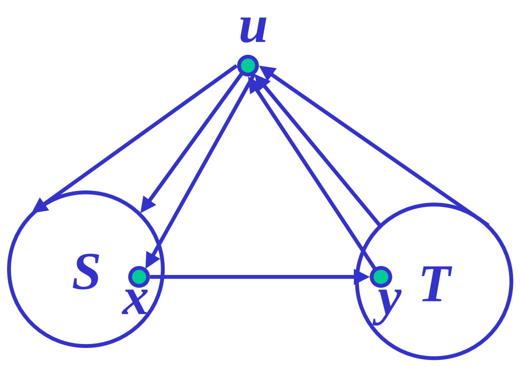
  i.e. $T \cup \{u\} = \overline{S}$

  $\because$ G is strongly connected. $\therefore$ $(S, T) = (S, \overline{S}) \neq \phi$

  (o.w. $\forall x \in S, y \in T, \nexists$ (x, y)-path $\rightarrow\leftarrow$)

  i.e. $\exists$ $x \in S, y \in T$ s.t. $(x, y) \in E(G)$

  $\therefore$ (u, x, y, u) is a directed 3-cycle containing u.



1.7 Circuits and Cycles

- **Thm 1.5**: **Every strongly connected tournament of order $\nu\,(\geq 3)$ is vertex-pancyclic.**

  Proof. (2/3)

  ② **Suppose that u is contained in directed cycles of all lengths between 3 and n, where $n < \nu$:**

  Let $C = (u = u_0, u_1, \ldots, u_{n-1}, u_0)$ be a directed n-cycle.

  $\forall\, x \in V(G) \backslash V(C)$, if $N_G^+(x) \cap V(C) \neq \phi$ and $N_G^-(x) \cap V(C) \neq \phi$,

  then $\exists\, u_i \in V(C)$, s.t. $(u_i, x), (x, u_{i+1}) \in E(G)$.

  $\Rightarrow (u_0, u_1, u_2, \ldots, u_i, x, u_{i+1}, \ldots, u_{n-1}, u_0)$ be a directed $(n+1)$-cycle containing u.

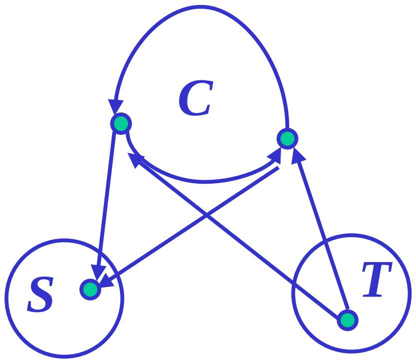
  otherwise, let $S = \{x \in V(G) \backslash V(C)\colon N_G^+(x) \cap V(C) = \phi\}$

  $T = \{x \in V(G) \backslash V(C)\colon N_G^-(x) \cap V(C) = \phi\}$

  either $x \in S$ or $x \in T$.

  $\because$ G is strongly connected and $n < \nu$.

  $\therefore$ $S \neq \phi$, $T \neq \phi$, and $(S, T) \neq \phi$.



1.7 Circuits and Cycles

- **Thm 1.5: Every strongly connected tournament of order $\nu (\geq 3)$ is vertex-pancyclic.**

  Proof. (3/3)

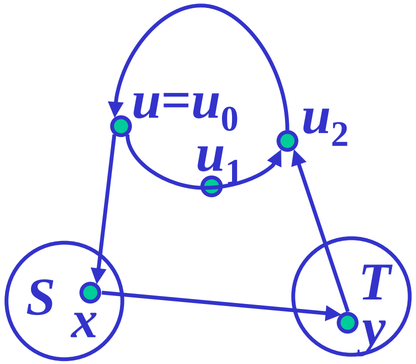
  ② $\because$ G is strongly connected and $n < \nu$.

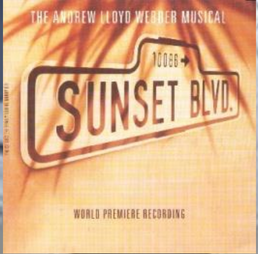
  $\therefore$ $S \neq \phi$, $T \neq \phi$, and $(S, T) \neq \phi$.

  let $x \in S$, $y \in T$ s.t. $(x, y) \in E(G)$

  Thus u is contained in the directed $(n + 1)$-cycle

  $(u_0, x, y, u_2, u_3, \ldots, u_{n-1}, u_0)$





音樂劇欣賞6-2

荒廢的豪宅，
這具屍體展開。
星若瑪也斯的
治了他的劇作
利斯對大導演
要把這本劇本
到攝影場。這
和伊利斯都不
作的貝蒂一見
生氣，將所有
開這間大屋的
的，他叫巨名
妄執人心醜陋

也裏發現
付款所逼
委託，修
管家侍
洛比德導
老爺車感
馬。
這件事後
露了。若
死了他。
母和大明
具有獨特

1.7 Circuits and Cycles

Thm 1.5: Every strongly connected tournament of order $\nu\,(\geq 3)$ is vertex-pancyclic.

- **Corollary 1.5: G: a strongly connected tournament of order $\nu \geq 5$. $\forall\, x, y \in V(G)$, $\exists$ (x, y)-walk of length $= d + 3$, where d = the diameter of G.**

Proof.

Let P be a shortest (x, y)-path in G.

$\because\ 0 \leq d_G(x, y) \leq d \leq \nu - 1$, $\therefore\ 0 \leq d - d_G(x, y) \leq \nu - 1$

$\therefore\ 3 \leq d - d_G(x, y) + 3 \leq \nu + 2$

$\because\ \nu \geq 5$, $\therefore$ By Thm 1.5, $\exists$ 3, 4, 5-cycle containing y.

case 1: if $d - d_G(x, y) = 0$ or 1 or 2:

$\because\ \nu \geq 5$, $\therefore$ By Thm 1.5, $\exists$ $(d - d_G(x, y) + 3)$-cycle C containing y,

Thus, $P \oplus C$ is an (x, y)-walk of length

$d_G(x, y) + (d - d_G(x, y) + 3) = d + 3$

case 2: If $3 \leq d - d_G(x, y) \leq \nu - 1$

By Thm 1.5: $\exists$ $(d - d_G(x, y))$-cycle C containing y and

$\exists$ 3-cycle C_3 containing y.

Thus $P \oplus C \oplus C_3$ is an (x, y)-walk of length $d_G(x, y) + (d - d_G(x, y)) + 3$

$= d + 3$

1.7 Circuits and Cycles

- **Thm 1.6**: A strongly connected digraph G is bipartite $\Leftrightarrow$
  G contains no odd directed circuit

  Proof. (1/4)

  ($\Rightarrow$) Let $\{X, Y\}$ be the bipartition of G.

  Suppose $C = x_0 e_1 x_1 \ldots x_{k-1} e_k x_0$ is a directed k-circuit in G.

  W.L.O.G. say $x_0 \in X$, then $x_1 \in Y$

  $x_2 \in X$, then $x_3 \in Y$,

  … $x_{k-1} \in Y$

  In general, $x_{2i} \in X$ and $x_{2i+1} \in Y$

  $\therefore k - 1 = 2i + 1$ for some integer i.

  $\Rightarrow k = 2i + 2$ i.e. C is even.

1.7 Circuits and Cycles

- **<u>Thm 1.6</u>: A strongly connected digraph G is bipartite $\Leftrightarrow$ G contains no odd directed circuit**

Proof. (2/4)

($\Leftarrow$) $\because$ G contains no odd directed circuit. $\therefore$ G has no loop.

Choose $u \in V(G)$, define $X = \{x \in V(G): d_G(u, x) \text{ is even}\}$

$Y = \{y \in V(G): d_G(u, y) \text{ is odd}\}$.

$\because$ G is strongly connected. $\therefore$ $X \cup Y = V(G)$

i.e. $\{X, Y\}$ is a partition of $V(G)$

Consider $G[Y]$, $\forall\, y, z \in Y$, by definition $\exists \begin{matrix} P_1 \\ P_2 \end{matrix} =$ a shortest $\begin{matrix} (u, y) \\ (y, u) \end{matrix}$-path and $\exists \begin{matrix} Q_1 \\ Q_2 \end{matrix} =$ a shortest $\begin{matrix} (u, z) \\ (z, u) \end{matrix}$-path in G and length(P_1), length(Q_1) are odd.

$\because$ $P_1 \cup P_2$ and $Q_1 \cup Q_2$ both are directed close walks

$\therefore$ length(P_2), length(Q_2) are odd. (by following <u>Note</u>)

1.7 Circuits and Cycles

- **Thm 1.6: A strongly connected digraph G is bipartite $\Leftrightarrow$ G contains no odd directed circuit**

Proof. (3/4)

Note: length(P_2) is odd :(length(Q_2) in the same)

By exercise 1.5.1, $P_1 \cup P_2$ = the union of several directed closed trail.

let $E_1 = \{e: e \in E(P_1)$ and $e \notin E(P_2)\}$

$E_2 = \{e: e \in E(P_2)$ and $e \notin E(P_1)\}$

$E_{12} = \{e: e \in E(P_1) \cap E(P_2)\}$

$\therefore 2|E_{12}| + |E_1| + |E_2|$ = length($P_1 \oplus P_2$) = Σ|directed closed trail| is even number.

$\Rightarrow |E_1| + |E_2|$ is a even number

$\therefore |E_1|$ and $|E_2|$ are either odd or even in the same time and $|E_1| + |E_{12}|$ = length(P_1) is odd.

$\therefore |E_2| + |E_{12}|$ = length (P_2) is odd, too.

1.7 Circuits and Cycles

- **<u>Thm 1.6</u>: A strongly connected digraph *G* is bipartite ⇔ *G* contains no odd directed circuit**

Proof. (4/4)

If $\exists\ (y, z) \in E(G)$, then $P_1 \oplus (y, z) \oplus Q_2$ contains an odd dicircuit. →←

If $\exists\ (z, y) \in E(G)$, then $Q_1 \oplus (z, y) \oplus P_2$ contains an odd dicircuit. →←

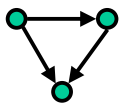
∴ $G[Y]$ is empty.

By the same argument, we can prove that $G[X]$ is empty too.

∴ $\{X, Y\}$ is a bipartition of *G* and *G* is a bipartite graph.

1.7 Circuits and Cycles

Thm 1.6: A strongly connected digraph G is bipartite $\Leftrightarrow$ G contains no odd directed circuit

- **Note**: The strongly connectedness is no necessary for "$\Rightarrow$", but is necessary for "$\Leftarrow$." ex: 

- **Corollary 1.6.1**: G: a strongly connected digraph; G is bipartite $\Leftrightarrow$ G contains no odd directed cycle.

Proof. ($\Rightarrow$) By Thm 1.6, G contains no odd circuit.

$\therefore$ G contains no odd cycle.

($\because$ an odd directed cycle is a special odd directed circuit)

($\Leftarrow$) Suppose C is an odd directed circuit in G.

Then C is not a directed cycle (by assumption) and

let $C = C_1 \oplus C_2 \oplus \ldots \oplus C_k$, $k \geq 2$ where C_i is a cycle for $1 \leq i \leq k$.

$\because$ C is odd, $\therefore$ $\exists\, 1 \leq i \leq k$, C_i is odd $\rightarrow\leftarrow$

$\therefore$ G contains no odd directed circuit.

By Thm 1.6, G is a bipartite graph.

1.7 Circuits and Cycles

- **<u>Corollary 1.6.2</u>: *G*: an undirected graph, *G* is bipartite ⇔ *G* contains no odd cycle.**

- **<u>Corollary 1.6.3</u>: *G*: a digraph: *G* is bipartite ⇔ *G* contains no odd cycle**

- **<u>Example 1.7.3</u>: *G*: an undirected graph,**

  G has a balanced oriented graph ⇔ G contains no vertex of odd degree.

Proof. (1/2)

(⇒) It is clearly.

(⇐) Prove by induction on ε.

① $\varepsilon = 0$: trivial

② If the assertion holds $\forall\ \varepsilon \le m$, now consider G be an undirected graph without vertices of odd degree and $\varepsilon(G) = m + 1$

Let $S = \{x \in V(G): d_G(x) = 0\}$, and let $G_1 = G - S$.

1.7 Circuits and Cycles

Example 1.7.1: G: undirected graph with $\delta \geq 2$.

① G contains a cycle C_n. ② If G is simple, then $n \geq \delta + 1$

- **Example 1.7.3**: G: an undirected graph,

  G has a balanced oriented graph $\Leftrightarrow$ G contains no vertex of odd degree.

  Proof. (2/2)

  ($\Leftarrow$) Prove by induction on ε.

  $\therefore \forall v \in V(G_1), d_{G_1}(v) \geq 2$, i.e. $\delta(G_1) \geq 2$

  By Ex 1.7.1, G_1 contains a cycle C.

  Given each edge in C an orientation to get a directed cycle C'.

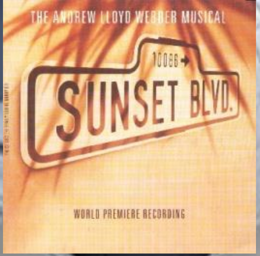
  Let $G_2 = G - E(C) \Rightarrow \varepsilon(G_2) = \varepsilon(G) - \varepsilon(C) < \varepsilon(G) = m + 1$

  and G_2 contains no odd vertices.

  $\therefore$ By I. H., $\exists$ a balanced oriented graph D' of G_2.

  $\Rightarrow$ Let $D = D' \oplus C'$, D is a balanced oriented graph of G.

音樂劇欣賞


THE ANDREW LLOYD WEBBER MUSICAL
SUNSET BLVD.
WORLD PREMIERE RECORDING

Exercise 1.5.1(c): Prove that any directed closed walk can be expressed as the union of several edge-disjoint closed trails, and construct an example to show that the term "directed" can not be deleted.

- **Example 1.7.4: *G*: a strongly connected digraph: *G* contains an odd circuit ⇒ *G* contains an odd directed circuit (⇒ *G* contains an odd directed cycle.)**

Proof. ⟨method 1⟩ By Corollary 1.6.3 → reader.

⟨method 2⟩ Let $C = x_1e_1x_2e_2\ldots x_ie_ix_{i+1}\ldots x_{2k+1}e_{2k+1}x_1$ be an odd circuit in G, where $x_i \in V(G)$, $e_i \in E(G)$.

∵ G is strongly connected.

∴ let $P_i \equiv$ a shortest (x_i, x_{i+1})-path, $\forall\ 1 \leq i \leq 2k$

$P_{2k+1} \equiv$ a shortest (x_{2k+1}, x_1)-path.

If $\exists\ 1 \leq i \leq 2k + 1$, length($P_i$) is even,

then $\psi_G(e_i) = (x_{i+1}, x_i)$ and $P_i + e_i$ is an odd directed cycle in G.

o.w. $\forall\ 1 \leq i \leq 2k + 1$, length($P_i$) is odd.

Let $W = P_1 \oplus P_2 \oplus \ldots \oplus P_{2k+1}$, W is a odd closed directed walk.

By exercise 1.5.1(c), W = the union of several directed circuit.

∴ ∃ at least one is odd.

1.7 Circuits and Cycles

- **Example 1.7.5: *G*: a non-bipartite undirected graph**

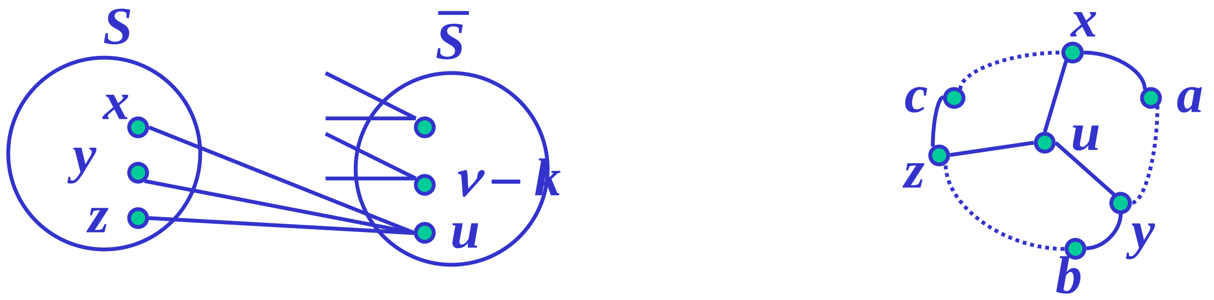
  G is simple and $\varepsilon > (1/4)(\nu - 1)^2 + 1 \Rightarrow G$ contains a triangle.

  Proof. (1/2)

  $\because$ *G* is non-bipartite, by Corollary 1.6.2, $\therefore$ *G* contains odd cycle.

  Let *C* be a shortest odd cycle, where $V(C) = S$, and $|S| = k$. Suppose $k \geq 5$:

  If $|(S, \overline{S})| > 2(\nu - k)$, then $\exists\ u \in \overline{S}$, s.t. $|N_G(u) \cap S| \geq 3$. Say $x, y, z \in N_G(u) \cap S$ but $\because$ the length of a shortest odd cycle ≥ 5, $\therefore \exists\ a, b, c \in S \backslash \{x, y, z\}$ and $C_1 = (u, x, \ldots, a, \ldots, y, u)$; $C_2 = (u, y, \ldots, b, \ldots, z, u)$; $C_3 = (u, z, \ldots, c, \ldots, x, u)$ is three cycles on *G*.



1.7 Circuits and Cycles

Example 1.3.1: If G is a simple undirected graph without triangles, then $\varepsilon(G) \leq (1/4)\nu^2$.

- **Example 1.7.5: G: a non-bipartite undirected graph**

  G is simple and $\varepsilon > (1/4)(\nu - 1)^2 + 1 \Rightarrow G$ contains a triangle.

  Proof. (2/2)

  $\because$ length (C_1), length(C_2), length(C_3) < k and
  length(C_1) – 2 + length(C_2) – 2 + length(C_3) – 2 = k, is odd.
  $\therefore \exists\ 1 \leq i \leq 3$, s.t. C_i is a odd cycle with length < k $\rightarrow\leftarrow$

  $\therefore |(S, \bar{S})| \leq 2(\nu - k)$ …①

  By Example 1.3.1, $\because \bar{S}$ without triangle $\Rightarrow \varepsilon(G[\bar{S}]) \leq (1/4)(\nu - k)^2$ … ②

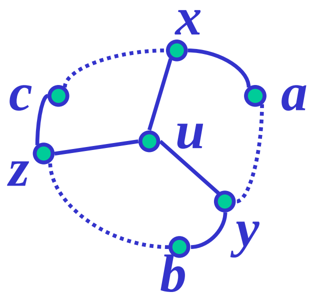
  $\varepsilon(G) = \varepsilon(G[S]) + |(S, \bar{S})| + \varepsilon(G[\bar{S}])$

  $\leq k + 2(\nu - k) + (1/4)(\nu - k)^2$ (by ①,②)

  $\leq 2\nu - 5 + (1/4)(\nu - 5)^2$

  $= (1/4)(\nu - 1)^2 + 1 \rightarrow\leftarrow$

  $\therefore G$ contains a triangle.



1.7 Circuits and Cycles

- **Exercise: 1.7.3(a)(c)**
- **加: 1.7.4(c), 1.7.6, 1.7.8**