# Chapter 1
# Basic Concepts of Graphs

## § 1.5 Walks, Paths and Connection (2)

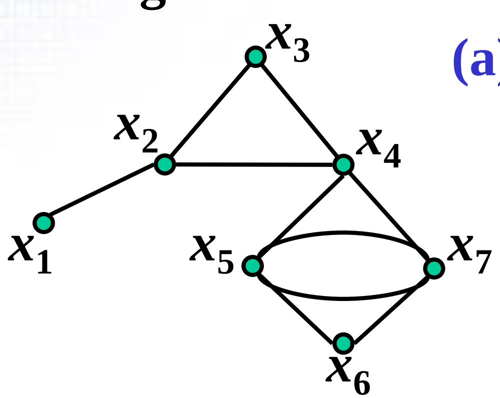
# 1.5 Walks, Paths and Connection

- **<u>Def</u>: Let $G$ be a loopless graph, $x \in V(G)$ and $e \in E(G)$:**
  - **If $\omega(G - x) > \omega(G)$, then $x$ is called a cut-vertex.**
  - **If $\omega(G - e) > \omega(G)$, then $e$ is called a cut-edge.**
  - **A connected graph is called a block if it contains no cut-vertex.**

- **<u>Note</u>: ① If $|\nu(G)| \geq 3$, $G$ contains a cut-edge $\Rightarrow$ $G$ contains a cut-vertex.**

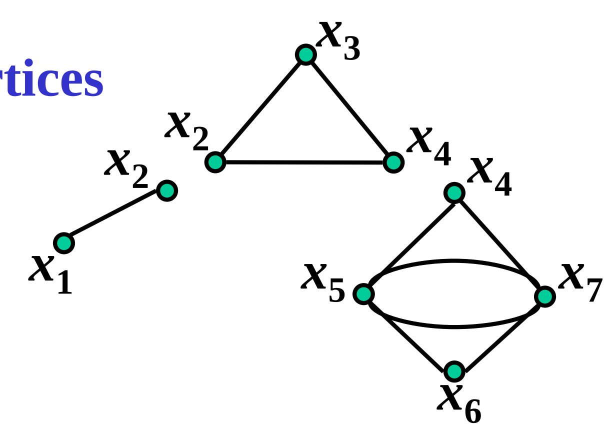
  **② Every graph can be expressed as the union of several blocks.**

- **ex: Fig 1.17**




**(a) $x_2$, $x_4$: cute-vertices**

**$x_1x_2$: cut-edge**




**(b) the blocks of (a)**

# 1.5 Walks, Paths and Connection

- **Example 1.5.3**: $G$: graph with $\nu(G) \geq 2$,

  $\exists$ 2 vertices that are not cut-vertices in $G$.

**Proof.**

Let $P = x_0e_1x_1e_2x_2\ldots x_{k-1}e_kx_k$ be a longest path in $G$.

Then $k \geq 1$. (If $G$ is empty, then all vertices are not cut vertices)

Suppose $x_0$ is a cut-vertex. $\Rightarrow \omega(G - x_0) > \omega(G)$.

Let $G_0$, $G_1$ be two connected components of $G - x_0$, where

$G_1$ contains $x_1$. (i.e. $x_1, x_2, \ldots, x_k$ all in $G_1$.)

Choose $y \in N_G(x_0) \cap V(G_0)$, i.e. $\exists\ e \in E(G)$ with end-vertices $x_0$, $y$.

$\because y \in V(G_0)$, $\therefore y \neq x_i$, $\forall\ 1 \leq i \leq k$

$\therefore Q = yex_0e_1x_1e_2x_2\ldots x_{k-1}e_kx_k$ is a path in $G$

and length($P$) < length($Q$) $\rightarrow\leftarrow$

**Def**: length($P$) $\equiv$ the length of $P$

$\therefore x_0$ is not a cut-vertex of $G$.

Similarly, $x_k$ is not a cut-vertex of $G$, too.

# 1.5 Walks, Paths and Connection

- **Def**: Let $G$ be a digraph,
  - $x, y \in V(G)$ are said to be **strongly connected** if $\exists$ $(x, y)$-path and $(y, x)$-path in $G$.
  - "to be strongly connected" is an equivalence relation on $V(G)$.
  - The subgraph induced by an equivalence class is called a **strongly connected component** of $G$.
  - $G$ is called to be **strongly connected** if it has one strongly connected component $\Leftrightarrow \forall\, x, y \in V(G)$, $x, y$ are strongly connected.

- **Note**: ① For undirected graph, the definition are the same.

  ② For a digraph $G$, $G$ is strongly connected $\Rightarrow$ $G$ is connected.

  ③ For a digraph $G$, $G$ is strongly connected $\Leftrightarrow$

  both $(S, \overline{S}) \neq \phi$ and $(\overline{S}, S) \neq \phi$, $\forall\, S \neq \phi \subseteq V(G)$.

# 1.5 Walks, Paths and Connection

- **Example 1.5.4: A simple digraph $G$ with $\varepsilon > (\nu - 1)^2$ is strongly connected.**

  **Proof.**

  If not, i.e. $G$ is not strongly connected.

  By note, $\exists\, S \neq \phi \subseteq V(G)$ s.t. $(S, \overline{S}) = \phi$.

  Let $|S| = k$. $\because |(\overline{S}, S)| \leq k(\nu - k)$

  $$\therefore\ \varepsilon \leq 2\binom{k}{2} + 2\binom{\nu-k}{2} + k(\nu - k)$$
  $$= k(k-1) + (\nu - k)(\nu - k - 1) + k(\nu - k)$$
  $$= k(k-1) + (\nu - k)(\nu - 1)$$
  $$= k(k-1) + [\nu - 1 - (k-1)](\nu - 1)$$
  $$= k(k-1) + (\nu - 1)^2 - (\nu - 1)(k-1)$$
  $$= (\nu - 1)^2 - (k-1)(\nu - k - 1) \leq (\nu - 1)^2 \rightarrow\leftarrow$$

  $\therefore$ $G$ is strongly connected.

# 1.5 Walks, Paths and Connection

**Thm 1.2: Every tournament contains Hamilton directed path.**

- **Def: A digraph $G$ is called be unilateral connected if $\exists$ either $(x, y)$-path or $(y, x)$-path for any $x, y \in V(G)$.**

- **Example 1.5.5: $G$ is unilateral connected $\Leftrightarrow$ $G$ contain a directed walk going through all vertices of $G$.**

**Proof.**

**($\Leftarrow$) trivial (By Ex 1.5.1 (a) + (b))**

**($\Rightarrow$) Construct a simple digraph $G'$ where**

$$\begin{cases} V(G') = V(G) \\ E(G') = \{(x, y): \exists\ (x, y)\text{-path } P_{xy} \text{ in } G\} \end{cases}$$

**By hypothesis, $G'$ contains a tournament as its spanning subgraph.**

**By Thm 1.2, $G'$ contains a Hamilton directed path $P'$.**

**$\Rightarrow$ let $W$ = replacing an edge $(x, y)$ in $P'$ with $P_{xy}$ in $G$.**

**Then $W$ is a directed walk going through all vertices of $G$.**

# 1.5 Walks, Paths and Connection

- **<u>Exercises</u>: 1.5.11 (a)**
- **加: 1.5.1(b), 1.5.8**

# Chapter 1
# Basic Concepts of Graphs

## § 1.6 Distances and Diameter

# 1.6 Distances and Diameter

- **Def**: $\forall\, x, y \in V(G)$
  ① $d_G(x, y) \equiv \min\{\text{length}(P): P$ is a $(x, y)$ path in $G\}$ = the **distance** from $x$ to $y$.
  ② $P$ is a **shortest** $(x, y)$-path if $\text{length}(P) = d_G(x, y)$ for $(x, y)$-path $P$.

- **Note**: ① In undirected graph $G$, $d_G(x, y) = d_G(y, x)$
  ② ① is not always true for digraph.

- **Def**:
  - The **diameter** of $G \equiv d(G) \equiv \max\{d_G(x, y): \forall\, x, y \in V(G)\}$
  - A graph $G$ is called **path**, denoted by $P_n$ if $G$ is a path of $n$ vertices.

# 1.6 Distances and Diameter

- **ex:**

  ① $d(P_n) = \begin{cases} n-1, & \text{if } P_n \text{ is undirected;} \\ \infty, & \text{if } P_n \text{ is directed.} \end{cases}$

  ② $d(K_n) = 1$

  ③ $d(K_{m,n}) = 2 \quad \text{if } \max\{m, n\} > 1$

  ④ **For Petersen graph** $G$, $d(G) = 2$

  ⑤ $d(Q_n) = n$

- **Note:** ① $d(G) = 1 \Leftrightarrow K_v \subseteq G$.

  ② $d(G)$ **is well-defined if** $G$ **is** $\begin{cases} \text{connected undirected graph or} \\ \text{a strongly connected digraph.} \end{cases}$

# 1.6 Distances and Diameter

- <u>Example 1.6.1</u>: $G$: a connected undirected graph with $K_2^C \subseteq G$

  $\exists\, x, y \in V(G)$ s.t. $d_G(x, y) = 2$

  **Proof.**

  Let $x$, $z$ be two nonadjacent vertices in $G$.

  $\because$ $G$ is connected.

  $\therefore$ $\exists$ a shortest $xz$-path $P = xe_1x_1e_2x_2e_3\ldots x_{k-1}e_kz$, where $k \geq 2$.

  Let $y = x_2$, then

  $\because$ $xe_1x_1e_2x_2$ is a $xy$-path.

  $\therefore d_G(x, y) \leq 2$.

  If $d_G(x, y) = 1$, then $\exists\, e \in E(G)$ s.t. $\psi_G(e) = xy$ $\rightarrow\leftarrow$ (shortest)

  $\therefore$ $d_G(x, y) = 2$

# 1.6 Distances and Diameter

- **Example 1.6.2**: $G$: a connected simple undirected graph of order $\nu$, and $\delta(G) = \delta$, then $d(G) \le 3\nu / (\delta + 1)$.

**Proof.**

Let $x$, $y$ be two vertices of $G$, s.t. $d_G(x, y) = d(G) = d$ and $P = (x_0, x_1, \ldots, x_{d-1}, x_d)$ be a shortest $xy$-path in $G$.

$\because N_G(x_{3i}) \cap N_G(x_{3j}) = \phi, \forall\ 0 \le i < j \le \lfloor d/3 \rfloor$.

(o.w. $P$ is not shortest. $\because (x_0, x_1, \ldots, x_{3i}, y, x_{3j}, \ldots, x_d)$ is shorten then $P$, where $y \in N(x_{3i}) \cap N(x_{3j})$.)

$\therefore\ \nu \ge \delta(\lfloor d/3 \rfloor + 1) + (\lfloor d/3 \rfloor + 1) \ge \delta \cdot (d/3) + d/3 = (d/3)(\delta+1)$

$\Rightarrow d(G) = d \le 3\nu/(\delta + 1)$

# 1.6 Distances and Diameter

- **<u>Example 1.6.3</u>: $G$: a strongly connected digraph of order $\nu$ and $\Delta(G) = \Delta$,**

  **then** $d(G) \begin{cases} = \nu - 1 & \text{, for } \Delta = 1; \\ \geq \lceil \log_{\Delta}(\nu(\Delta - 1) + 1) \rceil - 1, & \text{for } \Delta \geq 2. \end{cases}$

**Proof.**

$\because$ $G$ is a strongly connected digraph. $\therefore$ $G$ is well-defined.

Let $d(G) = k$, and $x \in V(G)$.

Let $R_i = \{y \mid d_G(x, y) = i\}$, then $|R_i| \leq \Delta^i, \forall\ 1 \leq i \leq k$.

$\therefore\ \nu \leq 1 + \Delta + \Delta^2 + \ldots + \Delta^{k-1} + \Delta^k = \begin{cases} k + 1 & \text{, for } \Delta = 1 \\ (\Delta^{k+1} - 1)/(\Delta - 1), & \text{for } \Delta \geq 2. \end{cases}$ (1.6)

<u>Case 1</u>: for $\Delta = 1$: $\nu \leq k + 1 \Rightarrow k \geq \nu - 1$

but $k = d(G) \leq \nu - 1$. $\therefore$ $k = d(G) = \nu - 1$

<u>Case 2</u>: for $\Delta \geq 2$: $(\Delta - 1)\nu \leq \Delta^{k+1} - 1$

$(\Delta - 1)\nu + 1 \leq \Delta^{k+1}$

$\because\ \Delta \geq 2$, $\therefore$ $d(G) = k \geq \lceil \log_{\Delta}\{(\Delta - 1)\nu + 1\} \rceil - 1$ ($\because\ k \in N$)

# 1.6 Distances and Diameter

$$\nu \le 1 + \Delta + \Delta^2 + \ldots + \Delta^{k-1} + \Delta^k = \begin{cases} k+1 & , \quad \text{for } \Delta = 1 \\ (\Delta^{k+1} - 1)/(\Delta - 1), & \quad \text{for } \Delta \ge 2. \end{cases}$$

- **Def:**
  - The upper bounds in (1.6) is called **($\Delta$, $k$)-Moore bounds** for digraphs of maximum degree $\Delta$ and diameter $k$.
  - The digraphs whose order = Moore bounds is called a **($\Delta$, $k$)-Moore digraph**.

- **Note:**
  - A directed cycle of length $k + 1$ is the unique (1, $k$)-Moore digraph.
  - No ($\Delta$, $k$)-Moore digraph for $\Delta \ge 2$ and $k \ge 2$. (Example 1.10.1)
- The Moore-bounds for undirected graph are given in Exercise 1.6.5.

# 1.6 Distances and Diameter

- **Def:**
  - If $P = (x_1, v_1, v_2, \ldots, v_m, y_1)$ is an $(x_1, y_1)$-path in $G_1$, then $\forall\, b \in V(G_2)$, say $Pb = (x_1b, y_1b)$-path $= (x_1b, v_1b, v_2b, \ldots, v_mb, y_1b)$ in $G_1 \times G_2$.
  - If $W = (x_2, u_1, u_2, \ldots, u_l, y_2)$ is an $(x_2, y_2)$-path in $G_2$, then $\forall\, a \in V(G_1)$, say $aW = (ax_2, ay_2)$-path in $G_1 \times G_2 = (ax_2, au_1, au_2, \ldots, au_l, ay_2)$.

- **Note:** Let $Q = Px_2 \cup y_1W$ is an $(x_1x_2, y_1y_2)$-path in $G_1 \times G_2$ with length $\varepsilon(P) + \varepsilon(W)$.

# 1.6 Distances and Diameter

- **<u>Theorem 1.3</u>: ① $d(G_1 \times G_2 \times \ldots \times G_n) = d(G_1) + d(G_2) + \ldots + d(G_n)$**

  **② $d(Q_n) = n$, where $Q_n$ is an $n$-cube.**

  **Proof. (1/3)**

  **① By associative law and the induction on $n \geq 2$, we need to only prove $d(G_1 \times G_2) = d(G_1) + d(G_2)$.**

  **(i) $\forall\, x = x_1x_2, y = y_1y_2 \in V(G_1 \times G_2)$, where $x_1, y_1 \in V(G_1)$, $x_2, y_2 \in V(G_2)$.**

  **Let $P$ be a shortest $(x_1, y_1)$-path in $G_1$,**

  **$W$ be a shortest $(x_2, y_2)$-path in $G_2$.**

  **<u>case 1</u>: $x_1 = y_1$; $x_1W$ is a (shortest) $(x, y)$-path in $G_1 \times G_2$.**

  **$\therefore d_{G_1 \times G_2}(x, y) \leq \varepsilon(x_1W) = \varepsilon(W) \leq d(G_2) \leq d(G_1) + d(G_2)$**

  **<u>case2</u>: $x_2 = y_2$: $Px_2$ is a (shortest) $(x, y)$-path in $G_1 \times G_2$**

  **$\therefore d_{G_1 \times G_2}(x, y) \leq \varepsilon(Px_2) = \varepsilon(P) \leq d(G_1) \leq d(G_1) + d(G_2)$**

  **<u>case 3</u>: $x_1 \neq y_1 \wedge x_2 \neq y_2$: $Px_2 \cup y_1W$ is an $(x, y)$-path in $G_1 \times G_2$.**

  **$\therefore d_{G_1 \times G_2}(x, y) \leq \varepsilon(Px_2) + \varepsilon(y_1W) \leq d(G_1) + d(G_2)$**

# 1.6 Distances and Diameter

- **Theorem 1.3**: ① $d(G_1 \times G_2 \times \ldots \times G_n) = d(G_1) + d(G_2) + \ldots + d(G_n)$
  ② $d(Q_n) = n$, where $Q_n$ is an $n$-cube.

  Proof. (2/3)

  (ii) Let $x_1, y_1 \in V(G_1)$, $x_2, y_2 \in V(G_2)$ such that
  $d_{G_1}(x_1, y_1) = d(G_1)$, $d_{G_2}(x_2, y_2) = d(G_2)$.
  Let $P$ be a shortest $(x_1, y_1)$-path in $G_1$ and
  $W$ be a shortest $(x_2, y_2)$-path in $G_2$.
  If $\exists$ $Q'$ is an $(x_1x_2, y_1y_2)$-path in $G_1 \times G_2$ shorter than $Px_2 \cup y_1W$, say
  $Q' = ((x_1x_2) =)c_1d_1, c_2d_2, \ldots, c_nd_n\ (= y_1y_2))$,
  then: let $P'$ be an $(x_1, y_1)$-walk in $G_1$ such that $P'$ determined by distinct vertices in the first coordinates of vertices of $Q'$ in the original order

# 1.6 Distances and Diameter

- **Theorem 1.3**: ① $d(G_1 \times G_2 \times \ldots \times G_n) = d(G_1) + d(G_2) + \ldots + d(G_n)$

  ② $d(Q_n) = n$, where $Q_n$ is an $n$-cube.

**Proof. (3/3)**

let $W'$ be an $(x_2, y_2)$-walk in $G_2$ such that $W'$ determined by distinct vertices in the second coordinates of vertices of $Q'$ in the original order.

$\because \varepsilon(P') + \varepsilon(W') = \varepsilon(Q') < \varepsilon(Px_2 \cup y_1W) = \varepsilon(P) + \varepsilon(W)$

$\therefore$ either $\varepsilon(P') < \varepsilon(P)$ or $\varepsilon(W') < \varepsilon(W)$ $\rightarrow\leftarrow$

$\Rightarrow d(G_1 \times G_2) \geq d(x_1x_2, y_1y_2) \geq \varepsilon(Px_2 \cup y_1W) = \varepsilon(P) + \varepsilon(W) = d(G_1) + d(G_2)$.

$\therefore$ By (i) and (ii), $d(G_1 \times G_2) = d(G_1) + d(G_2)$

② $\because Q_n = \overbrace{K_2 \times K_2 \times \ldots \times K_2}^{n}$, and $d(K_2) = 1$

$\therefore d(Q_n) = \underbrace{1 + 1 + \ldots + 1}_{n} = n$.

# 1.6 Distances and Diameter

- **Theorem 1.4: Let $G$ be a strongly connected digraph with $\nu \geq 2$, and $L$ be the line digraph of $G$. Then ① $d(G) \leq d(L) \leq d(G) + 1$.**
  **② $d(G) = d(L) \Leftrightarrow G$ is a directed cycle.**
  **③ $d(K(d, n)) = d(B(d, n)) = n$.**

**Proof. (1/5) (略)**

① By ex 1.6.3, $L$ is also strongly connected, $\therefore$ $d(G)$, $d(L)$ are well-defined.

(i) Let $x, y \in V(G)$, such that $d_G(x, y) = d(G)$.

Let $P$ = a shortest $(x, y)$-path in $G$.

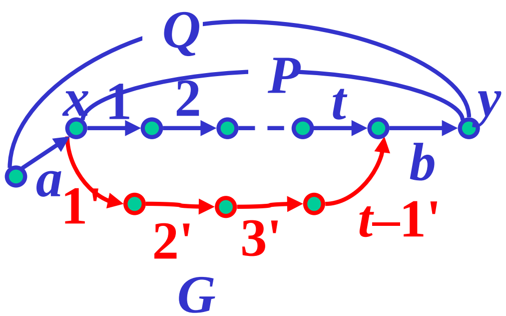
$\because$ $G$ is strongly connected. $\therefore$ $\exists\, a \in E_G^-(x)$.

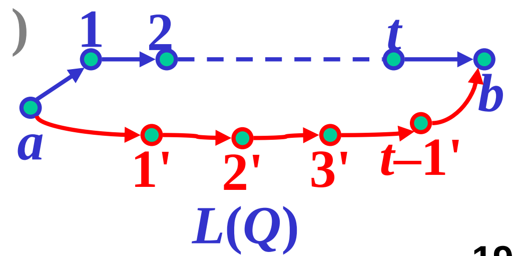
Let $b \in E_G^-(y) \cap E(P)$, and $Q = a + P$

$\Rightarrow$ $L(Q)$ is a shortest $(a, b)$-path in $L$.

( o.w. $P$ is not a shortest $(x, y)$-path in $G$ $\rightarrow\leftarrow$ )

$\therefore$ $d(L) \geq \varepsilon(L(Q)) = \varepsilon(P) = d(G)$



# 1.6 Distances and Diameter

- **<u>Theorem 1.4</u>: Let $G$ be a strongly connected digraph with $\nu \geq 2$, and $L$ be the line digraph of $G$. Then ① $d(G) \leq d(L) \leq d(G) + 1$.**
  **② $d(G) = d(L) \Leftrightarrow G$ is a directed cycle.**
  **③ $d(K(d, n)) = d(B(d, n)) = n$.**

**Proof. (2/5)**

①

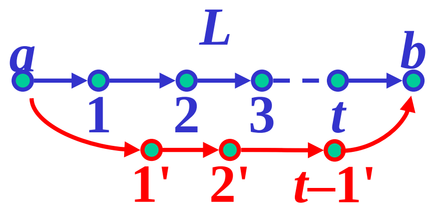
(ii) Let $a, b \in V(L)$ s.t. $d_L(a, b) = d(L)$

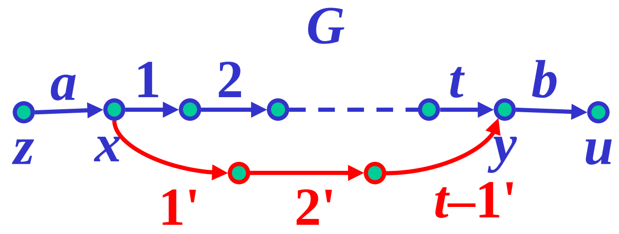
$\Rightarrow \exists\, x, y, z, u \in V(G)$ s.t. $a = (z, x)$, $b = (y, u)$.

$\Rightarrow d_G(x, y) = d_L(a, b) - 1$. (o.w. $d_L(a, b) \leq d(L) - 1 \rightarrow\leftarrow$)

$\Rightarrow d(G) \geq d_G(x, y) = d_L(a, b) - 1 = d(L) - 1$

$\therefore$ By (i)(ii), $d(G) \leq d(L) \leq d(G) + 1$



# 1.6 Distances and Diameter

- **Theorem 1.4: Let $G$ be a strongly connected digraph with $\nu \geq 2$, and $L$ be the line digraph of $G$. Then ① $d(G) \leq d(L) \leq d(G) + 1$.**
  **② $d(G) = d(L) \Leftrightarrow G$ is a directed cycle.**
  **③ $d(K(d, n)) = d(B(d, n)) = n$.**

**Proof. (3/5)**

② (⇐) ∵ $L(C_n) = C_n$. ∴ $d(G) = d(L)$.

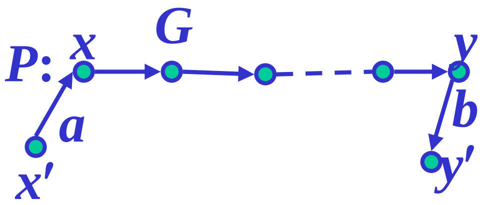
(⇒) Let $d(G) = d$, and $x, y \in V(G)$ s.t. $d_G(x, y) = d$.

Let $P = (x, y)$-path of length $d$ in $G$.

∵ $G$ is strongly connected, ∴ $d_G^-(x) \geq 1$ and $d_G^+(y) \geq 1$

i.e. $\exists\, x', y' \in V(G)$, s.t. $a = (x', x)$, $b = (y, y') \in E(G)$.

(i) If $a \neq b$, then $d_L(a, b) = d + 1$ →← $(d(L) \geq d_L(a, b) = d + 1 = d(G) + 1)$



# 1.6 Distances and Diameter

- **Theorem 1.4: Let $G$ be a strongly connected digraph with $\nu \geq 2$, and $L$ be the line digraph of $G$. Then ① $d(G) \leq d(L) \leq d(G) + 1$.**
  **② $d(G) = d(L) \Leftrightarrow G$ is a directed cycle.**
  **③ $d(K(d, n)) = d(B(d, n)) = n$.**

**Proof. (4/5)**

(ii) If $a = b$, i.e. $\exists\ c = (y, x) \in E(G)$, i.e. $P \cup \{c\}$ is a dicycle in $G$.

Let $P \cup \{c\} = (x_0, x_1, x_2, \ldots, x_d, x_0) = C$, where $x_0 = x$, $x_d = y$.

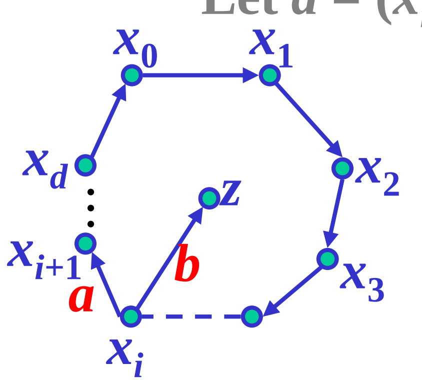
If $G \neq$ dicycle, then $\exists\ x_i \in E(G)$ and $z \in V(G)$ s.t. $(x_i, z)$ or $(z, x_i) \in E(G)$.

Choose such $x_i$ s.t. $i$ is as large as possible. W.L.O.G., say $(x_i, z) \in E(G)$.

$\Rightarrow d_G(x_{i+1}, x_i) = d$ (o.w. $P$ is not shortest)

Let $a = (x_i, x_{i+1})$ and $b = (x_i, z) \Rightarrow d_L(a, b) = d + 1$ $\rightarrow\leftarrow$

$((z, x_i) \in E(G)$ the same)



# 1.6 Distances and Diameter

- **Theorem 1.4: Let $G$ be a strongly connected digraph with $\nu \geq 2$, and $L$ be the line digraph of $G$. Then ① $d(G) \leq d(L) \leq d(G) + 1$.**
  **② $d(G) = d(L) \Leftrightarrow G$ is a directed cycle.**
  **③ $d(K(d, n)) = d(B(d, n)) = n$.**

**Proof. (5/5)**

③ ∵ $K_{d+1}$ and $K_d^+$ ($d \geq 2$) not directed cycle and

$d(K_{d+1}) = d(K_d^+) = 1$

∴ $d(K(d, n)) = d(L^{n-1}(K_{d+1})) = 1 + n - 1 = n$

$d(B(d, n)) = d(L^{n-1}(K_d^+)) = 1 + n - 1 = n$

# 1.6 Distances and Diameter

- **Def:**
  - The radius of $G \equiv \text{rad}(G) = \min_{x \in V(G)} \{ \max_{y \in V(G)} \{d_G(x, y)\}\}$
  - A vertex $x$ is called a central of $G$ if $\max_{y \in V(G)} \{d_G(x, y)\}$ ( $= \min_{x \in V(G)} \{\max_{y \in V(G)} \{d_G(x, y)\}\}$ )
    $= \text{rad}(G)$

- **Note:** $\text{rad}(G) \leq d(G) \leq 2\text{rad}(G)$

  **Proof.** exercise 1.6.6

- **Example 1.6.4:** For digraph $G$, $\text{rad}(G) \leq r \Rightarrow \nu(G) \leq 1 + r \cdot \Delta^r$.

  **Proof.**

  Let $x$ be a central vertex of $G$, and $J_i = \{y \mid d_G(x, y) = i\}$

  $$\Rightarrow \left\{ \begin{array}{l} |J_1| \leq \Delta \\ |J_i| \leq \Delta \cdot |J_{i-1}| \end{array} \right\} \Rightarrow |J_i| \leq \Delta^r$$

  $$\Rightarrow \nu(G) \leq 1 + \Delta + \Delta^2 + \ldots + \Delta^r \leq 1 + r \cdot \Delta^r$$

# 1.6 Distances and Diameter

- **Def**: $G$: connected undirected graph or strongly connected digraph with $\nu \geq 2$.
  - ① The mean or average distance of $G \equiv$

$$m(G) \equiv \frac{1}{\nu(\nu-1)} \sum_{x,y \in V(G)} (d_G(x,y)),$$

  - ② $\sigma(G) = \sum_{x,y \in V(G)} d_G(x,y)$

- **Note**:
  - ① $m(G) \geq 1$
  - ② $m(G) = 1 \Leftrightarrow G$ is a complete graph
  - ③ For a directed cycle $C_n$, $n \geq 3$, $\sigma(C_n) = (1/2)n^2(n-1)$, $m(C_n) = n/2$

    <sol> $\sigma(C_n) = n(1 + 2 + \ldots + (n-1)) = n \cdot (n(n-1))/2 = (1/2)n^2(n-1)$

    $m(C_n) = (1/(n(n-1))) \cdot \sigma(C_n) = n/2$
  - ④ For an undirected cycle $C_n$, $m(C_n) = \begin{cases} (n+1)/4 & \text{, if } n \text{ is odd;} \\ n^2/(4(n-1)), & \text{if } n \text{ is even.} \end{cases}$

    <sol> exercise 1.6.6

# 1.6 Distances and Diameter

- **<u>Exercise</u>: 1.6.6**
- **加: 1.6.4 (b), (c)**