Chapter 1
Basic Concepts of Graphs

§ 1.4 Subgraphs and Operations

1.4 Subgraphs and Operations

- **Def:**
  - G, H are two graphs such that $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and $\psi_H = \psi_G|_{E(H)}$, then — H is called a **subgraph** of G, denoted by $H \subseteq G$.
    — G is called a **supergraph** of H.
  - **spanning subgraph**: if $V(H) = V(G)$
  - $S \subseteq V(G)$,
    ① The **induced subgraph** by S. (or subgraph induced by S), $G[S]$:
    $$\begin{cases} V(G[S]) = S \\ E(G[S]) = \{e \in E(G) \mid \exists\, x, y \in S, \text{ s.t. } \psi_G(e) = (x, y)\} \end{cases}$$
    ② $G - S \equiv G[V \setminus S]$
    ③ If $S = \{v\}$, $G - v \equiv G - \{v\}$

1.4 Subgraphs and Operations

- **Def:**
  - $B \subseteq E(G)$,

    ① The **edge-induced subgraph** by B (or subgraph induced by B) $G[B]$:

    $$\begin{cases} V(G[B]) = \{x \in V(G) \mid \exists\, e \in B \text{ s.t. } \psi_G(e) = (x, y) \text{ or } \psi_G(e) = (y, x)\} \\ E(G[B]) = B. \quad (\psi_{G[B]} = \psi_G|_B) \end{cases}$$

    ② $G - B$: $\begin{cases} V(G - B) = V(G) \\ E(G - B) = E(G) \setminus B. \quad (\psi_{G-B} = \psi_G|_{E(G-B)}) \end{cases}$

    ③ If $B = \{e\}$, $G - e \equiv G - \{e\}$

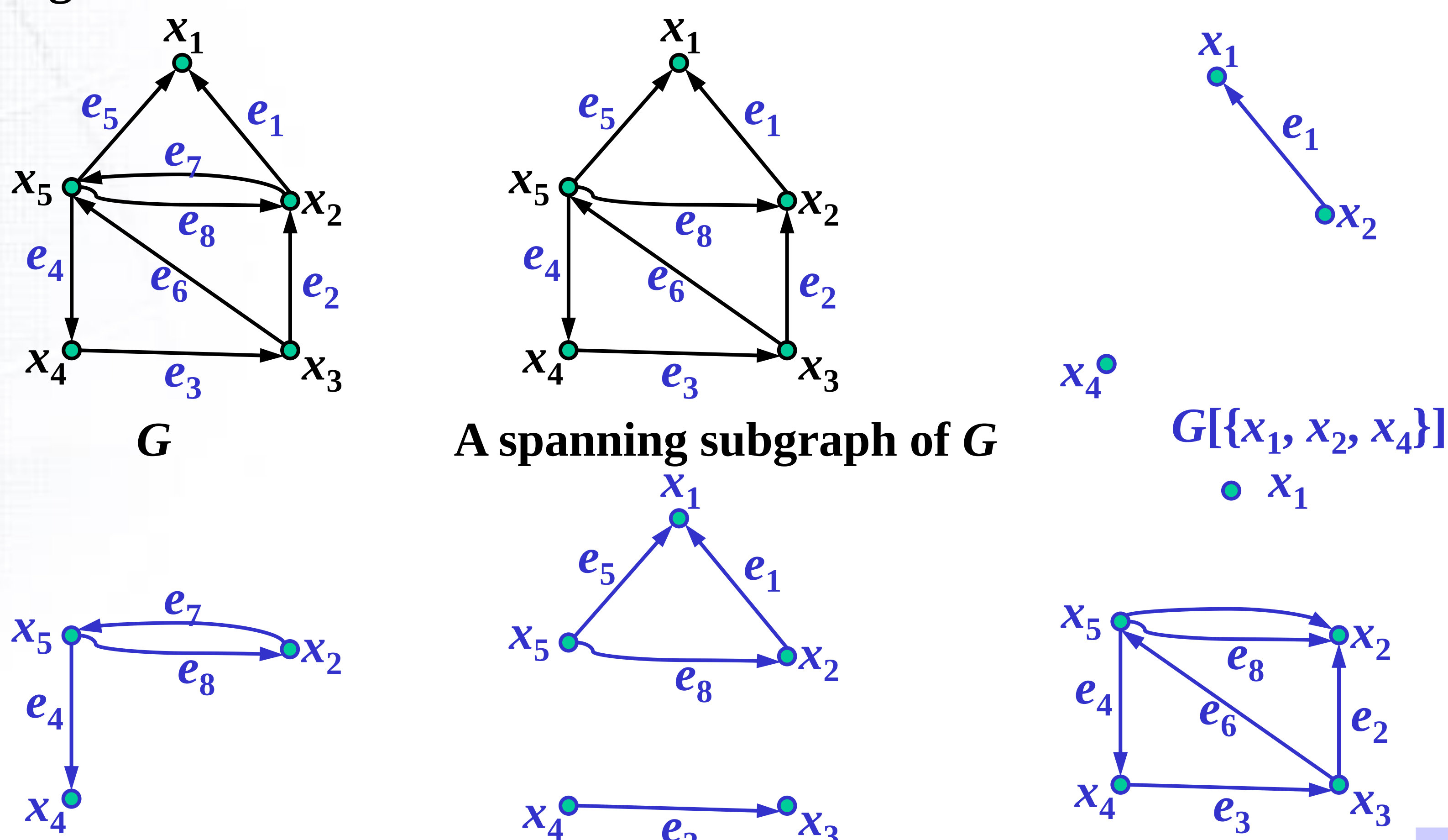
  - F: extra edge set,

    ① $G + F$: $\begin{cases} V(G + F) = V(G). \\ E(G + F) = E(G) \cup F. \end{cases}$

    ② If $F = \{e\}$, $G + e \equiv G + \{e\}$

1.4 Subgraphs and Operations

- <u>ex</u>: **Fig 1.10:**



G — A spanning subgraph of G — $G[\{x_1, x_2, x_4\}]$

$G - \{x_1, x_3\}$ — $G[\{e_1, e_3, e_5, e_8\}]$ — $G - \{e_1, e_5\}$

1.4 Subgraphs and Operations

- **Def**: $G_1, G_2 \subseteq G$,

  ① say G_1, G_2 are:
  - **disjoint** if $V(G_1) \cap V(G_2) = \phi$.
  - **edge-disjoint** if $E(G_1) \cap E(G_2) = \phi$

  ② The **union** $G_1 \cup G_2$ of G_1 and G_2: $\begin{cases} V = V(G_1) \cup V(G_2) \\ E = E(G_1) \cup E(G_2) \end{cases}$
  - when G_1 and G_2 are disjoint; write $G_1 + G_2$
  - when G_1 and G_2 are edge-disjoint; write $G_1 \oplus G_2$

  ③ If $V(G_1) \cap V(G_2) \neq \phi$, define the **intersection** $G_1 \cap G_2$ of G_1 and G_2:

  $$\begin{cases} V = V(G_1) \cap V(G_2) \\ E = E(G_1) \cap E(G_2) \end{cases}$$

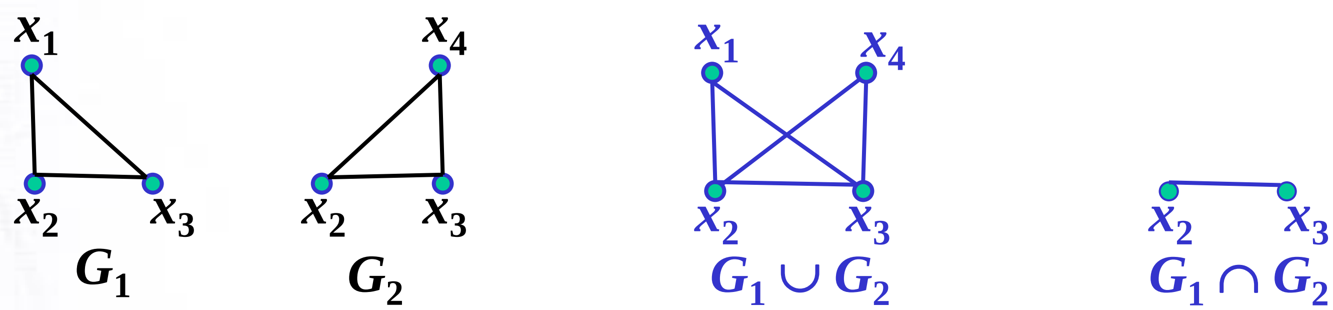
  ④ If $G_i \cong H$ for each $i = 1, 2, \ldots, n$, then write $nH \equiv G_1 + G_2 + \ldots + G_n$

  ⑤ An edge e of G is said to be **contracted**

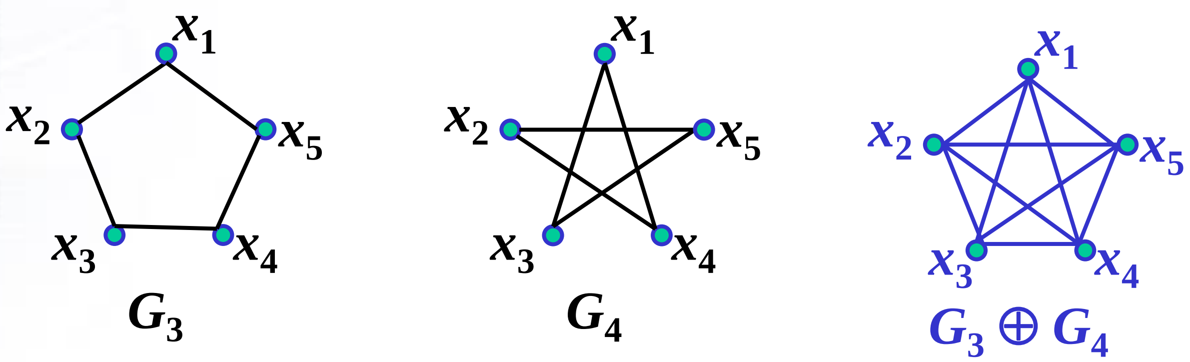
  $\equiv G \cdot e \equiv$ delete e and identity its end-vertices.

1.4 Subgraphs and Operations

- ex: ①

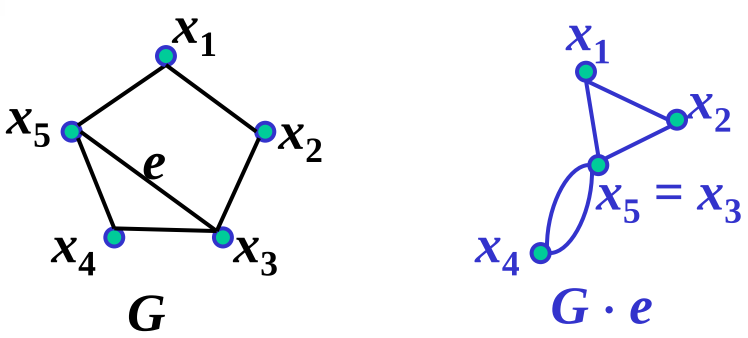


②



: $K_5 = C_5 \oplus C_5$

③



1.4 Subgraphs and Operations

Theorem 1.1: For any digraph D, $\varepsilon(D) = \sum_{x\in X} d_D^+(x) = \sum_{x\in X} d_D^-(x)$.

- **Example 1.4.1: Let G be a balanced digraph. Then $d_G^+(X) = d_G^-(X)\ \forall\ X \subset V(G)$.**

Proof.

Let $H = G[X]$.

$\because$ G is balanced.

$\therefore d_G^+(x) = d_G^-(x),\ \forall\ x \in V(G)$ — ①

By Thm 1.1, $\sum_{x\in X} d_H^+(x) = \sum_{x\in X} d_H^-(x)$ — ②

$$\Rightarrow d_G^+(X) = \sum_{x\in X} (d_G^+(x) - d_H^+(x)) \overset{\text{by ①}}{=} \sum_{x\in X} (d_G^-(x) - d_H^+(x))$$

$$= \sum_{x\in X} d_G^-(x) - \sum_{x\in X} d_H^+(x)$$

$$\overset{\text{by ②}}{=} \sum_{x\in X} d_G^-(x) - \sum_{x\in X} d_H^-(x)$$

$$= \sum_{x\in X} (d_G^-(x) - d_H^-(x)) = d_G^-(X)$$

1.4 Subgraphs and Operations

Corollary 1.1: For any undirected graph G, ① $2\varepsilon(G) = \sum_{x \in V} d_G(x)$
② the number of odd vertices is even.

- **Example 1.4.2**: Let G be an undirected graph without loops. Then G contains a bipartite spanning subgraph H s.t. $d_G(x) \le 2d_H(x)$, $\forall\, x \in V(G)$. Hence, $\varepsilon(G) \le 2\varepsilon(H)$.

Proof.

Let H be a bipartite spanning subgraph of with edges as many as possible, and let $\{X, Y\}$ be a bipartition.

① If $\exists\, x \in V(G)$ s.t. $d_G(x) > 2d_H(x)$, W.L.O.G., say $x \in X$, then

let $d = d_G(x) - d_H(x) > d_H(x)$.

Let $X' = X \backslash \{x\}$, $Y' = Y \cup \{x\}$ and $H' \subseteq G$ s.t.

$$\begin{cases} V(H') = V(H) = V(G), \\ E(H') = E(H) \backslash \{xy: xy \in E(H)\} \cup \{xy \in E(G): xy \notin E(H)\} \end{cases}$$

Then, $\varepsilon(H) \ge \varepsilon(H') = \varepsilon(H) - d_H(x) + d > \varepsilon(H)$ $\rightarrow\leftarrow$

$\therefore\ d_G(x) \le 2d_H(x)$, $\forall\, x \in V(G)$.

② By Corollary 1.1: $\varepsilon(G) = (1/2) \sum_{x \in V(G)} d_G(x) \le (1/2) \sum_{x \in V(H)} 2d_H(x) = 2\varepsilon(H)$.

1.4 Subgraphs and Operations

- **Def: The cartesian product $G_1 \times G_2$ of two simple graphs G_1, G_2:**

$$V(G_1 \times G_2) = V(G_1) \times V(G_2)$$

$$E(G_1 \times G_2) = \{(x_1x_2, y_1y_2): x_1 = y_1 \text{ and } (x_2, y_2) \in E(G_2), \text{ or } x_2 = y_2 \text{ and } (x_1, y_1) \in E(G_1)\}$$

- **ex:** $Q_2 = K_2 \times K_2$ $\quad Q_3 = K_2 \times Q_2$ $\quad Q_4 = K_2 \times Q_3$

  $Q_n = K_2 \times Q_{n-1}$

- **Note: The cartesian product satisfies commutative and associative labs,**

  ① $G_1 \times G_2 = G_2 \times G_1$, $\forall\, G_1, G_2$: simple graphs

  ② $(G_1 \times G_2) \times G_3 = G_1 \times (G_2 \times G_3)$, $\forall\, G_1, G_2, G_3$: simple graphs.

1.4 Subgraphs and Operations

- **Def: In general, let $G_i = (V_i, E_i)$ be a graph $\forall\ i = 1, 2, \ldots, n$. Write $G_1 \times G_2 \times \ldots \times G_n$ for the cartesian product of $G_1, G_2, \ldots, G_n$, where**

$$\begin{cases} V(G_1 \times G_2 \times \ldots \times G_n) = V_1 \times V_2 \times \ldots \times V_n \\ E(G_1 \times G_2 \times \ldots \times G_n) = \{(x_1x_2\ldots x_n, y_1y_2\ldots y_n): x_1x_2\ldots x_n \text{ and } y_1y_2\ldots y_n \\ \qquad \text{differ exactly in } i\text{th coordinate, and } (x_i, y_i) \in E_i\}. \end{cases}$$

- **Example 1.4.3: $Q_n = \underbrace{K_2 \times K_2 \times \ldots \times K_2}_{n \text{ terms}}$**

1.4 Subgraphs and Operations

- **Def:**
  - The line graph of G, denoted by $L(G)$, is a graph that
    $$\begin{cases} V(L(G)) = E(G) \\ E(L(G)) = \{(a, b): \exists\, x, y, z \in V(G), \text{ s.t. } \Psi_G(a) = (x, y),\ \Psi_G(b) = (y, z)\} \end{cases}$$
  - If $L(G)$ is non-empty and has no isolated vertices, then $L(L(G))$ exists.
  - $$\begin{cases} L^0(G) \equiv G \\ L^1(G) \equiv L(G) \\ L^n(G) \equiv L(L^{n-1}(G)), \text{ called the } n\text{th iterated line graph of } G. \end{cases}$$

1.4 Subgraphs and Operations

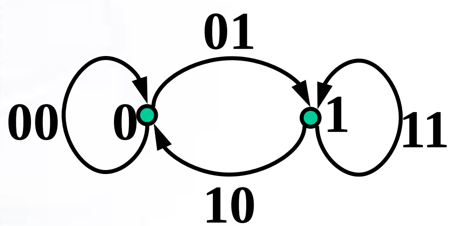
- **Example 1.4.4:**

  ① **The *n*-dimensional *d*-ary Kautz digraph, $K(d, n) \equiv L^{n-1}(K_{d+1})$ (§ 1.8).**

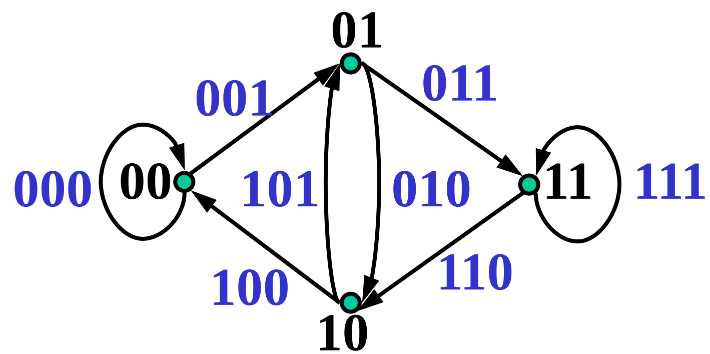
  ② **$K_d^+ \equiv$ a complete digraph K_d adding one loop at each vertex. ($d \geq 2$)**

  ③ **The de Bruijn digraphs, $B(d, n) \equiv L^{n-1}(K_d^+)$.**

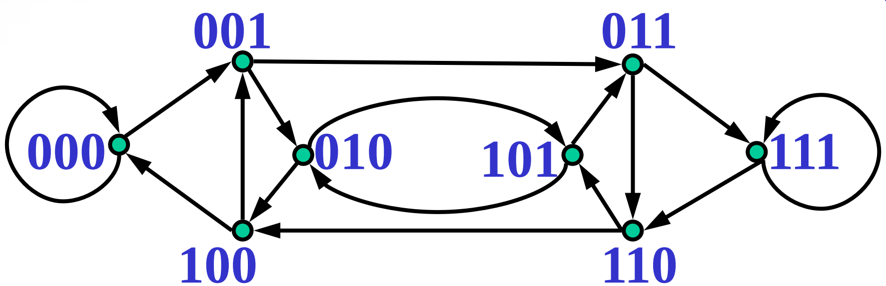
- **ex:**



$B(2, 1) = K_2^+$



$B(2, 2) = L(B(2, 1))$



$B(2, 3) = L(B(2, 2))$

1.4 Subgraphs and Operations

- **Exercise: 1.4.5**

- **加: 1.4.2, 1.4.7**

Chapter 1
Basic Concepts of Graphs

§ 1.5 Walks, Paths and Connection

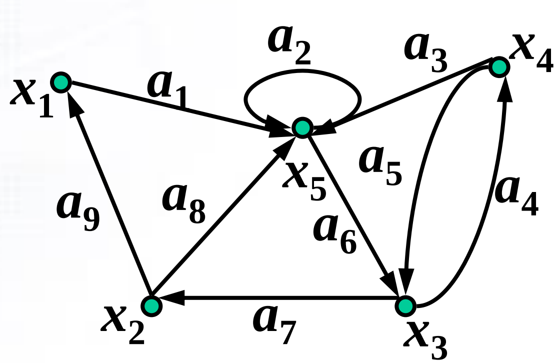
1.5 Walks, Paths and Connection

- **<u>Def</u>:**
  - Let $x, y \in V(G)$. An **xy-walk** of **length** k in G is a sequence
    $W = x_0e_1x_1e_2\ldots e_kx_k$, where $x_0 = x$, $x_k = y$, where x_i are vertices,
    e_j are edges, and x_{j-1}, x_j are end-vertices of e_j $\forall\ 0 \le i \le k$, $1 \le j \le k$.
  - If G is simple, then write $W = (x, x_1, \ldots, x_{k-1}, y)$ for short.
  - x and y are called the **origin** and the **terminus** of W, other vertices are **internal vertices** of W.
  - If edges are distinct, W is called a **trail**.
  - If vertices are distinct, W is called a **path**.
  - It is **closed** if $x = y$.
  - A closed trail is called a **circuit**.
  - A circuit is called a **cycle** if its vertices are distinct except $x = y$.

1.5 Walks, Paths and Connection

- **Def: In digraph G,**
  - **xy-walk is called directed xy-walk, (x, y)-walk if $\forall\ e_i \in E(W)$, $\Psi_G(e_i) = (x_{i-1}, x_i)$.**
  - **directed trail, directed path, directed circuit, directed cycle**

- **ex:**



x_1x_3-walk $W = x_1a_1x_5a_2x_5a_3x_4a_3x_5a_8x_2a_7x_3$

x_1x_3-trail $T = x_1a_1x_5a_2x_5a_8x_2a_7x_3$

x_1x_3-path $P = x_1a_1x_5a_8x_2a_7x_3$

(x_1, x_3)-walk $W' = x_1a_1x_5a_2x_5a_6x_3a_4x_4a_3x_5a_6x_3$

(x_1, x_3)-trail $T' = x_1a_1x_5a_2x_5a_6x_3$

directed circuit $C = x_1a_1x_5a_2x_5a_6x_3a_7x_2a_9x_1$

directed cycle $C' = x_1a_1x_5a_6x_3a_7x_2a_9x_1$

1.5 Walks, Paths and Connection

- **Def:**
  - A **longest** path ≡ it has the maximum length over all paths
  - A path is called a **Hamilton path** ≡ it contains all vertices

- **補充: Def:**
  - Let $S \subseteq V(G)$, G: undirected graph.
    $[S, \overline{S}] \equiv \{xy \in E(G): x \in S, y \in \overline{S}, \text{ or } x \in \overline{S}, y \in S\}$.

1.5 Walks, Paths and Connection

- **Example 1.5.1: Every simple graph *G* must contain a path of length ≥ $\delta = \delta(G)$.**

  Proof.

  W.L.O.G., say *G* is undirected graph.

  Let $P = (x_0, x_1, \ldots, x_k)$ be a longest path in *G*.

  $\because$ *P* is a longest path.

  $\therefore N_G(x_0) \subseteq \{x_1, x_2, \ldots, x_k\}$. i.e. $|N_G(x_0)| \leq k$

  $\because d_G(x_0) \geq \delta(G)$

  $\therefore k \geq |N_G(x_0)| = d_G(x_0) \geq \delta$

1.5 Walks, Paths and Connection

- **<u>Thm 1.2</u>: Every tournament contains Hamilton directed path.**

  Proof.

  ① It is true for $\nu \le 2$.

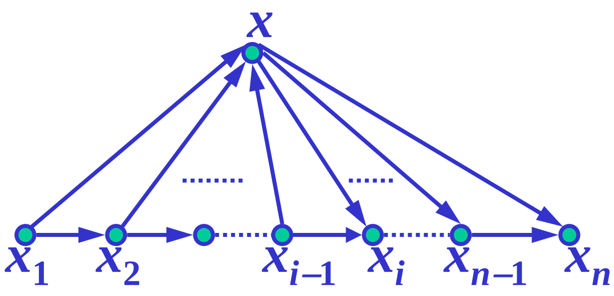
  ② When $\nu \ge 3$. Suppose to the contrary that T is a tournament and let $P = (x_1, x_2, \ldots, x_n)$ be a longest dipath in G and $n < \nu$,

  $\therefore \exists\, x \in V(T)\backslash V(P)$ s.t. $(x, x_n), (x_1, x) \in E(T)$.

  $\Rightarrow \exists\, x_i$ be the first vertices from x_2 to x_n where $(x_{i-1}, x), (x, x_i) \in E(T)$.

  $\therefore (x_1, x_2, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)$ is a dipath that length $> |P|$ $\rightarrow\leftarrow$

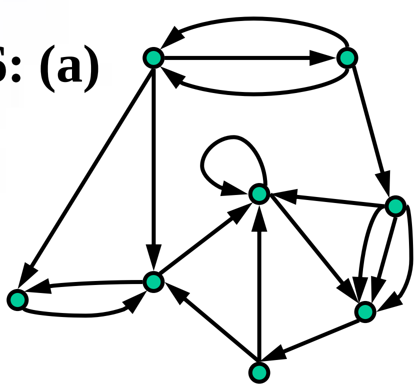
  $\therefore \exists$ a Hamilton directed path in T.



1.5 Walks, Paths and Connection

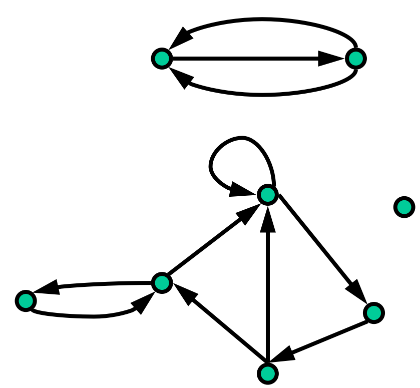
- <u>**Def**</u>**:**
  - **$x, y \in V(G)$, x, y are said to be connected if $\exists$ xy-path in G.**
  - **"to be connected" is an equivalence relation of $V(G)$.**
  - **Let $\{V_1, V_2, \ldots, V_\omega\}$ be the equivalence partition of $V(G)$, then $G[V_i]$ is called a connected component of G.**
  - **$\omega = \omega(G)$ is called the number of connected components of G.**
  - **If $\omega = 1$, then G is a connected graph, otherwise disconnected graph.**

- **ex: Fig 1.16: (a)**



connected graph

(b)



disconnected graph

- <u>**Note**</u>**: A graph is connected $\Leftrightarrow$ $[S, \bar{S}] \neq \phi$, $\forall$ S $(\neq \phi) \subseteq V(G)$.**

1.5 Walks, Paths and Connection

- **Example 1.5.2: Let G be a simple undirected graph with $V = \{x_1, x_2, \ldots, x_\nu\}$ satisfying $d_G(x_1) \le d_G(x_2) \le \ldots \le d_G(x_\nu)$. If $d_G(x_k) \ge k \ \forall\ 1 \le k \le \nu - d_G(x_\nu) - 1$, then G is connected.**

Proof.

Suppose to the contrary that G is disconnected.

$\Rightarrow \exists\ S \ne \phi \subseteq V(G)$ s.t. $[S, \overline{S}] = \phi$

W.L.O.G. let $x_\nu \in \overline{S}$, then $|\overline{S}| \ge d_G(x_\nu) + 1$. ($\because$ G is simple)

Let $|S| = k$, then $k = |S| = \nu - |\overline{S}| \le \nu - d_G(x_\nu) - 1$

$\therefore d_G(x_k) \ge k$ by the hypothesis.

$\Rightarrow d_G(x_i) \ge k,\ \forall\ i = k, k+1, \ldots, \nu$

$\therefore d_G(x_i) \in \overline{S},\ \forall\ i = k, k+1, \ldots, \nu$. i.e. $|\overline{S}| \ge \nu - k + 1$.

$\Rightarrow k = |S| = \nu - |\overline{S}| \le \nu - (\nu - k + 1) = k - 1 \quad \rightarrow\leftarrow$

$\therefore$ G is connected.

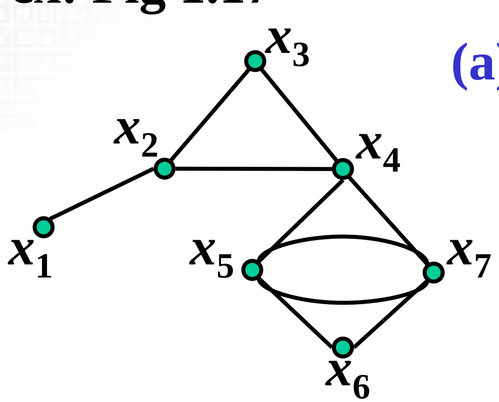
1.5 Walks, Paths and Connection

- **<u>Def</u>: Let G be a loopless graph, $x \in V(G)$ and $e \in E(G)$:**
  - **If $\omega(G - x) > \omega(G)$, then x is called a cut-vertex.**
  - **If $\omega(G - e) > \omega(G)$, then e is called a cut-edge.**
  - **A connected graph is called a block if it contains no cut-vertex.**

- **<u>Note</u>: ① If $|\nu(G)| \geq 3$, G contains a cut-edge $\Rightarrow$ G contains a cut-vertex.**

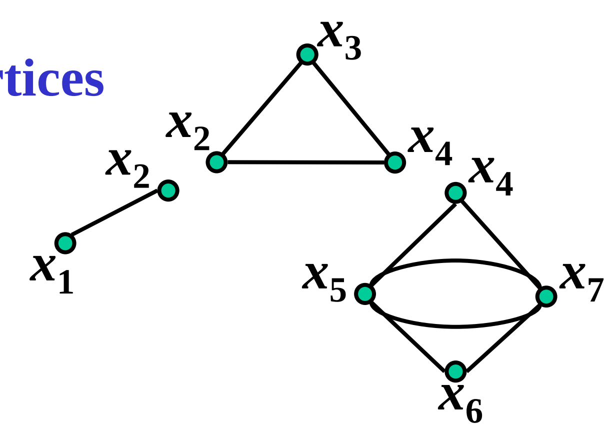
  ② Every graph can be expressed as the union of several blocks.

- **ex: Fig 1.17**



(a) x_2, x_4: cute-vertices

x_1x_2: cut-edge



(b) the blocks of (a)

1.5 Walks, Paths and Connection

- **Example 1.5.3**: G: graph with $\nu(G) \geq 2$,
  $\exists$ 2 vertices that are not cut-vertices in G.

Proof.

Let $P = x_0e_1x_1e_2x_2\ldots x_{k-1}e_kx_k$ be a longest path in G.
Then $k \geq 1$. (If G is empty, then all vertices are not cut vertices)
Suppose x_0 is a cut-vertex. $\Rightarrow$ $\omega(G - x_0) > \omega(G)$.
Let G_0, G_1 be two connected components of $G - x_0$, where
G_1 contains x_1. (i.e. $x_1, x_2, \ldots, x_k$ all in G_1.)
Choose $y \in N_G(x_0) \cap V(G_0)$, i.e. $\exists$ $e \in E(G)$ with end-vertices x_0, y.
$\because$ $y \in V(G_0)$, $\therefore$ $y \neq x_i$, $\forall$ $1 \leq i \leq k$
$\therefore$ $Q = yex_0e_1x_1e_2x_2\ldots x_{k-1}e_kx_k$ is a path in G
and length(P) < length(Q) $\rightarrow\leftarrow$

Def: length(P) $\equiv$ the length of P

$\therefore$ x_0 is not a cut-vertex of G.
Similarly, x_k is not a cut-vertex of G, too.

1.5 Walks, Paths and Connection

- **Def**: Let G be a digraph,
  - $x, y \in V(G)$ are said to be **strongly connected** if $\exists$ (x, y)-path and (y, x)-path in G.
  - "to be strongly connected" is an equivalence relation on $V(G)$.
  - The subgraph induced by an equivalence class is called a **strongly connected component** of G.
  - G is called to be **strongly connected** if it has one strongly connected component $\Leftrightarrow \forall\, x, y \in V(G)$, x, y are strongly connected.

- **Note**: ① For undirected graph, the definition are the same.

  ② For a digraph G, G is strongly connected $\Rightarrow$ G is connected.

  ③ For a digraph G, G is strongly connected $\Leftrightarrow$

  both $(S, \overline{S}) \neq \phi$ and $(\overline{S}, S) \neq \phi$, $\forall\, S \neq \phi \subseteq V(G)$.

1.5 Walks, Paths and Connection

- **<u>Example 1.5.4</u>: A simple digraph G with $\varepsilon > (\nu - 1)^2$ is strongly connected.**

  Proof.

  If not, i.e. G is not strongly connected.

  By note, $\exists\ S \neq \phi \subseteq V(G)$ s.t. $(S, \overline{S}) = \phi$.

  Let $|S| = k$. $\because |(\overline{S}, S)| \leq k(\nu - k)$

  $$\therefore\ \varepsilon \leq 2\binom{k}{2} + 2\binom{\nu-k}{2} + k(\nu - k)$$
  $$= k(k-1) + (\nu - k)(\nu - k - 1) + k(\nu - k)$$
  $$= k(k-1) + (\nu - k)(\nu - 1)$$
  $$= k(k-1) + [\nu - 1 - (k-1)](\nu - 1)$$
  $$= k(k-1) + (\nu - 1)^2 - (\nu - 1)(k-1)$$
  $$= (\nu - 1)^2 - (k-1)(\nu - k - 1) \leq (\nu - 1)^2 \rightarrow\leftarrow$$

  $\therefore$ G is strongly connected.

1.5 Walks, Paths and Connection

Thm 1.2: Every tournament contains Hamilton directed path.

- **Def: A digraph *G* is called be unilateral connected if ∃ either (x, y)-path or (y, x)-path for any $x, y \in V(G)$.**

- **Example 1.5.5: *G* is unilateral connected ⇔ *G* contain a directed walk going through all vertices of *G*.**

Proof.

(⇐) trivial (By Ex 1.5.1 (a) + (b))

(⇒) Construct a simple digraph G' where

$$\begin{cases} V(G') = V(G) \\ E(G') = \{(x, y): \exists\ (x, y)\text{-path } P_{xy} \text{ in } G\} \end{cases}$$

By hypothesis, G' contains a tournament as its spanning subgraph.

By Thm 1.2, G' contains a Hamilton directed path P'.

⇒ let W = replacing an edge (x, y) in P' with P_{xy} in G.

Then W is a directed walk going through all vertices of G.

1.5 Walks, Paths and Connection

- **Exercises: 1.5.1 (a), 1.5.11 (a)**
- **加: 1.5.8, 1.5.10**