Chapter 1
Basic Concepts of Graphs

§ 1.2 Graph Isomorphic

1.2 Graph Isomorphic

- **Def:**
  - A graph $G = (V(G), E(G), \psi_G)$ is **isomorphic** to a graph $H = (V(H), E(H), \psi_H)$ if $\exists$ 2 bijective mappings $\theta: V(G) \to V(H)$ and $\phi: E(G) \to E(H)$
    s.t. $\forall\, e \in E(G)$,
    $$\psi_G(e) = (x, y) \Leftrightarrow \psi_H(\phi(e)) = (\theta(x), \theta(y)) \in E(H). \quad (\star)$$
  - (θ, ϕ): **isomorphic mapping** from G to H.
  - G and H are **isomorphic**, write $G \cong H$ (or $G = H$)
  - (θ, ϕ): an **isomorphism** between G and H.

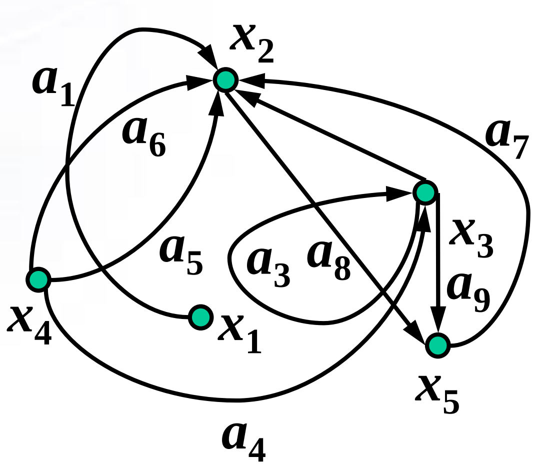
1.2 Graph Isomorphic

- ex: In example 1.1.1 and 1.1.2, $D \cong H$.

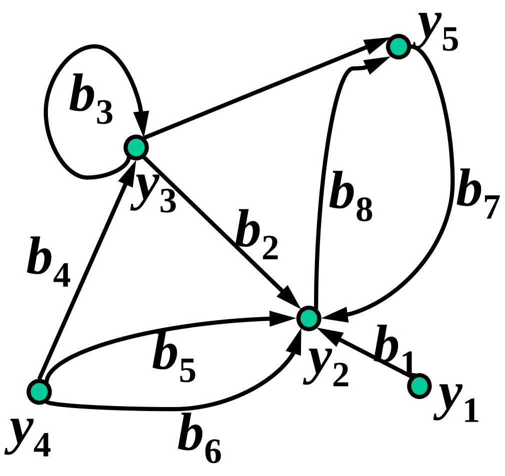
  Let θ: $V(D) \to V(H)$ and ϕ: $E(D) \to E(H)$ be

  $\theta(x_i) = y_i$, $\forall\, i = 1, 2, \ldots, 5$,

  $\phi(a_j) = b_j$, $\forall\, j = 1, 2, \ldots, 9$.



D



H

1.2 Graph Isomorphic

$\psi_G(e) = (x, y) \Leftrightarrow \psi_H(\phi(e)) = (\theta(x), \theta(y)) \in E(H)$. (☆)

- **Def**: For simple graphs G, H, G and H are isomorphic
  $\Leftrightarrow \exists$ a bijection $\theta: V(G) \to V(H)$ s.t.
  $(x, y) \in E(G) \Leftrightarrow (\theta(x), \theta(y)) \in E(H)$.
  ((☆) is called the adjacency-preserving condition)

- **Note**:
  - $G \cong H \Rightarrow \nu(G) = \nu(H), \varepsilon(G) = \varepsilon(H)$. (反之不成立!!)

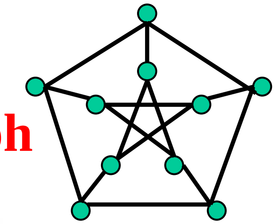
"to be isomorphic" is an equivalence relation.

(反身, 對稱, 遞移)

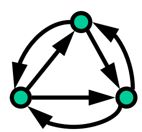
∴ divide all graph into equivalence classes.

1.2 Graph Isomorphic

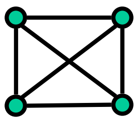
- **Def:**
  - **Petersen graph**



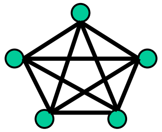
  - **complete graph, K_ν. ex:**



K_3



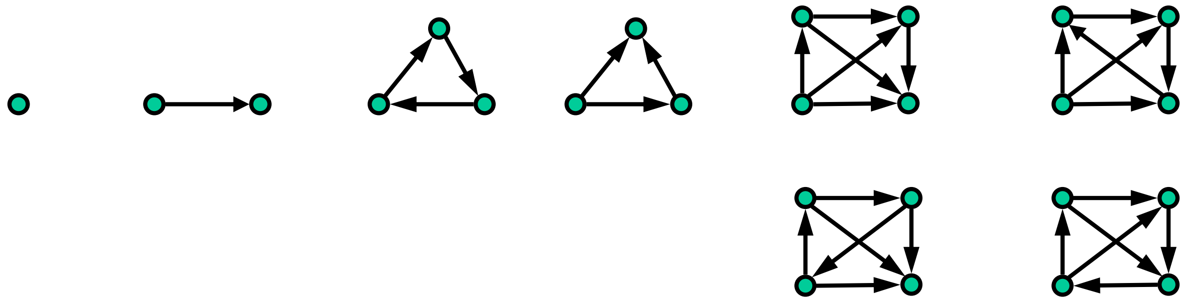
K_4



K_5

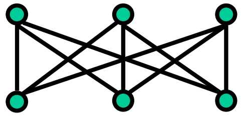
  - **tournament**

    ex:



- **Note:** $\varepsilon(K_\nu) = \begin{cases} \nu(\nu-1), & \text{if } K_\nu \text{ is directed,} \\ (1/2)\nu(\nu-1), & \text{if } K_\nu \text{ is undirected.} \end{cases}$

1.2 Graph Isomorphic

- **Def:**
  - **bipartite graph: vertex-set can be partitioned into X and Y, so that each edge has one end-vertex in both.**
  - **$\{X, Y\}$ is called a bipartition of the graph.**
  - **If ∃ a bipartition $\{X, Y\}$ where $|X| = |Y|$, then called equally bipartite.**
  - **$G(X \cup Y, E)$**
  - **k-partite graph**
  - **equally k-partite graph**
  - **complete bipartite graph, $K_{m,n}$**   **ex:**  **$K_{3,3}$**
  - **star $\equiv K_{1,n}$**
  - **$K_n(2) = K_{n,n}$**
  - **Complete k-partite graph**
  - **$K_n(k)$**

1.2 Graph Isomorphic

- **Note: 1.** $\varepsilon(K_{m,n}) = mn$

  2. $\varepsilon(K_n(k)) = (1/2)k(k-1)n^2$

  3. For any bipartite simple graph G of order n,

$$\varepsilon(G) \leq \begin{cases} (1/4)n^2, & \text{if } n \text{ is even;} \\ (1/4)(n^2-1), & \text{if } n \text{ is odd.} \end{cases}$$

1.2 Graph Isomorphic

- **Def**: **G is called an associated bipartite graph with the digraph D, where**

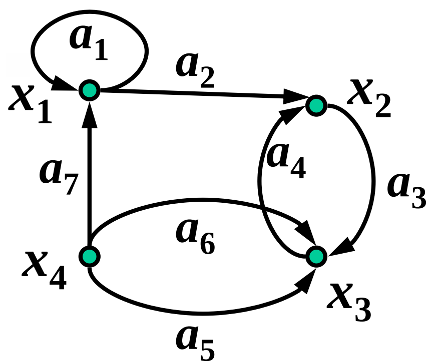
  if $V(D) = \{x_1, x_2, \ldots, x_\nu\}$ and $E(D) = \{a_1, a_2, \ldots, a_\varepsilon\}$,

  then $G = (X \cup Y, E(G), \psi_G)$ with $X = (x_1', x_2', \ldots, x_\nu'\}$, $Y = \{x_1'', x_2'', \ldots, x_\nu''\}$

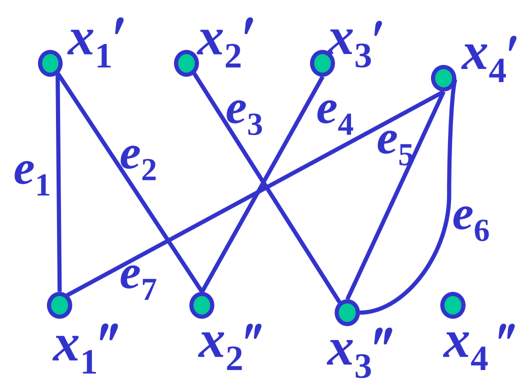
  $E(G) = \{e_1, e_2, \ldots, e_\varepsilon\}$, where $\psi_G(e_l) = x_i' x_j''$

  $\Leftrightarrow \exists\, a_l \in E(D)$ s.t. $\psi_D(a_l) = (x_i, x_j)$, $l = 1, 2, \ldots, \varepsilon$.

- **ex: Fig 1.7: D**



G:



- **Note**: **If G is an associated bipartite graph of D, $\nu(G) = 2\nu(D)$ and $\varepsilon(G) = \varepsilon(D)$.**

1.2 Graph Isomorphic

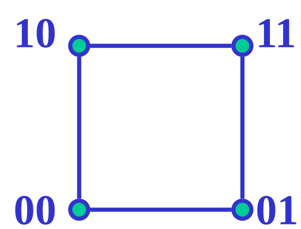
- **Def**: ***n*-cube** (or **hypercube**), $Q_n = (V(Q_n), E(Q_n))$ is defined as:

  $V(Q_n) = \{x_1x_2\ldots x_n : x_i \in \{0, 1\}, i = 1, 2, \ldots, n\}$.

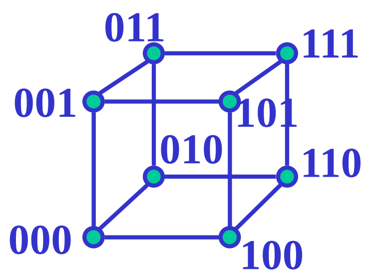
  $E(Q_n) = \{xy : x = x_1x_2\ldots x_n, y = y_1y_2\ldots y_n \in V(Q_n), \sum_{i=1}^{n} |x_i - y_i| = 1\}$

- **ex:**

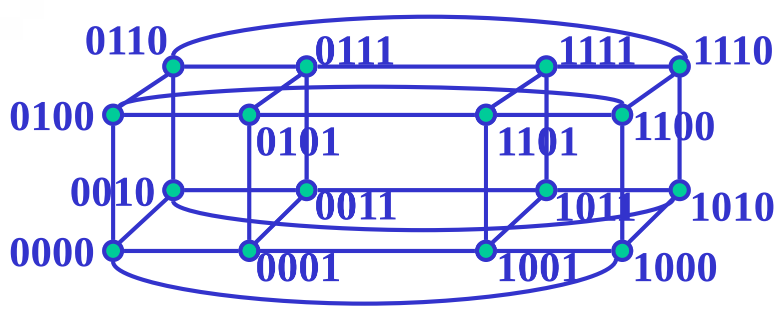
Q_1



Q_2



Q_3



Q_4

1.2 Graph Isomorphic

- **Example 1.2.1**: Q_n is an equally bipartite simple graph.

Sol. (1/2)

① Q_n is simple by definition with $\nu(Q_n) = 2^n$

② Let $X = \{x_1x_2\ldots x_n : x_1 + x_2 + \ldots + x_n \equiv 0 \pmod 2\}$

$Y = \{x_1x_2\ldots x_n : x_1 + x_2 + \ldots + x_n \equiv 1 \pmod 2\}$

By definition, $X \cup Y = V(Q_n)$, $X \cap Y = \phi$.

$\therefore$ $\{X, Y\}$ is a bipartition of $V(Q_n)$.

Suppose $\exists\, x = x_1x_2\ldots x_n$, $x' = x_1'x_2'\ldots x_n' \in X$ s.t. $xx' \in E(Q_n)$.

$\Rightarrow \sum_{i=1}^{n} |x_i - x_i'| = 1$

$\Rightarrow |(x_1 + x_2 + \ldots + x_n) - (x_1' + x_2' + \ldots + x_n')| = 1$

$\Rightarrow \rightarrow\leftarrow$ ($\because$ $x, x' \in X$. $\therefore$ $x_1 + x_2 + \ldots + x_n \equiv 0 \pmod 2$,

$x_1' + x_2' + \ldots + x_n' \equiv 0 \pmod 2$.)

$\therefore$ There is no edge between any two vertices in X.

Similarly, there is no edge between any two vertices in Y.

1.2 Graph Isomorphic

- **Example 1.2.1: Q_n is an equally bipartite simple graph.**

  Sol. (2/2)

  ③ $\forall x \in X$, let $N(x) = \{y \in Y: xy \in E(Q_n)\}$

  $\therefore |N(x)| = n$ by definition.

  Similarly, $|N(y)| = n$.

  Let $E_X \equiv$ the set of edges incident with vertices in X.

  $E_Y \equiv$ the set of edges incident with vertices in Y.

  $\Rightarrow n|X| = |E_X| = \varepsilon(Q_n) = |E_Y| = n|Y|$

  $\Rightarrow \begin{cases} |X| = |Y| = (1/2)\nu(Q_n) = 2^{n-1} \\ \varepsilon(Q_n) = n \cdot 2^{n-1}. \end{cases}$

1.2 Graph Isomorphic

- **Def**: $T_{k,\nu}$ ≡ complete k-partite graph of order ν in which each part has either $m = \lfloor \nu/k \rfloor$ or $n = \lceil \nu/k \rceil$ vertices.

- **Example**: (a) $\varepsilon(T_{3,13}) = ?$

  $13 = 3 \times 4 + 1,\ m = 4,\ n = 5 = m + 1.$

  $\varepsilon(T_{3,13}) = (4(4 + 5) + 4(4 + 5) + 5(4 + 4)) / 2 = 56$

  $\binom{\nu-m}{2} + (k - 1)\binom{m+1}{2} = \binom{13-4}{2} + (3 - 1)\binom{4+1}{2}$

  $= \binom{9}{2} + 2\binom{5}{2}$

  $= 36 + 20 = 56$

  (b) $\varepsilon(G) \le 56$ for any complete 3-partite graph G with order 13?

  and $\varepsilon(G) = 56$ iff $G \cong T_{3,13}$?

1.2 Graph Isomorphic

- **Example 1.2.2**: (a) $\varepsilon(T_{k,\nu}) = \binom{\nu-m}{2} + (k-1)\binom{m+1}{2}$;

  (b) $\varepsilon(G) \leq \varepsilon(T_{k,\nu})$ for any complete k-partite graph G with order ν and the equality holds iff $G \cong T_{k,\nu}$.

Proof. (1/2) (略)

(a) Let $\nu = km + r$, $0 \leq r < k$. Then $r = \nu - km$.

$\because (-r(m^2 + m) + r(m^2 - m)) = 0$

$$\varepsilon(T_{k,\nu}) = \binom{\nu}{2} - r\binom{m+1}{2} - (k-r)\binom{m}{2}$$
$$= (1/2)\{\nu(\nu-1) - r\,m(m+1) - (k-r)m(m-1)\}$$
$$= (1/2)\{\nu(\nu-1) - 2m(\nu-km) - km(m-1)\}$$
$$= (1/2)\{(\nu^2 - \nu - 2\nu m + m^2 + m) + km(m+1) - m(m+1)\}$$
$$= (1/2)(\nu-m)(\nu-m-1) + (1/2)(k-1)m(m+1)$$
$$= \binom{\nu-m}{2} + (k-1)\binom{m+1}{2}$$

1.2 Graph Isomorphic

- **Example 1.2.2**: (a) $\varepsilon(T_{k,\nu}) = \binom{\nu-m}{2} + (k-1)\binom{m+1}{2}$;

  (b) $\varepsilon(G) \le \varepsilon(T_{k,\nu})$ for any complete k-partite graph G with order ν and the equality holds iff $G \cong T_{k,\nu}$.

Proof. (2/2) (略)

(b) Suppose $G = K_{n_1,n_2,\ldots,n_k}$ is a complete k-partite graph with order ν and the largest number of edges where $n_1 \ge n_2 \ge \ldots \ge n_k$. Then

$$\varepsilon(G) = \binom{\nu}{2} - \sum_{l=1}^{k}\binom{n_l}{2}$$

If $G \ncong T_{k,\nu}$, then $\exists\ 1 \le i < j \le k$ s.t. $n_i - n_j > 1$

Let G' be a complete k-partite graph, that the number of vertices in its k-partition are: $n_1, n_2, \ldots, n_{i-1}, (n_i - 1), n_{i+1}, \ldots, n_{j-1}, (n_j + 1), n_{j+1}, \ldots, n_k$.

Then $\varepsilon(G') = \binom{\nu}{2} - \sum_{l=1\neq i,j}^{k}\binom{n_l}{2} - \binom{n_i-1}{2} - \binom{n_j+1}{2}$

$$= \binom{\nu}{2} - \sum_{l=1}^{k}\binom{n_l}{2} + (n_i - 1) - n_j$$

$$= \binom{\nu}{2} - \sum_{l=1}^{k}\binom{n_l}{2} + (n_i - n_j - 1) > \binom{\nu}{2} - \sum_{l=1}^{k}\binom{n_l}{2} = \varepsilon(G). \rightarrow\leftarrow$$

$\therefore G \cong T_{k,\nu}$

1.2 Graph Isomorphic

- **Exercises: 1.2.6**

- **加: Construct a self-complementary undirected graph of order nine.**

- **Def:**
  - **Complement, G^c, of $G \equiv \begin{cases} V(G^c) = V(G) \\ E(G^c) = \{(x, y): (x, y) \notin E(G), x, y \in V(G)\} \end{cases}$**
  - **self-complementary: $G \cong G^c$.**

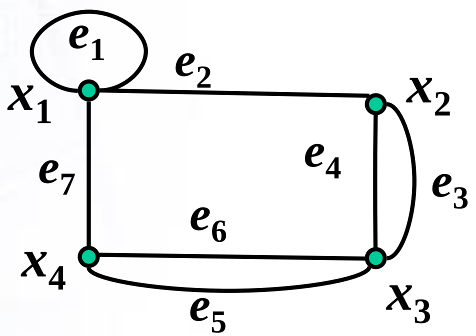
Chapter 1
Basic Concepts of Graphs

§ 1.3 Vertex Degrees

1.3 Vertex Degrees

- **Def**: In an undirected graph G, $x \in V(G)$.
  - The **degree** of x, $d_G(x) \equiv$ the # of edges incident to x, loop counting as 2 edges.
  - ***d*-degree vertex**

ex:



$d_G(x_1) = d_G(x_3) = 4$

$d_G(x_2) = d_G(x_4) = 3$

x_1 is a 4-degree vertex

  - The **open neighbors** of x, $N(x) = N_G(x) \equiv \{ y \mid xy \in E(G)\}$.
  - The **close neighbors** of x, $N[x] = N_G[x] \equiv N(x) \cup x$.
  - **isolated vertex** $\equiv$ 0-degree vertex
  - **odd** (**even**) vertex: degree is odd (even).

1.3 Vertex Degrees

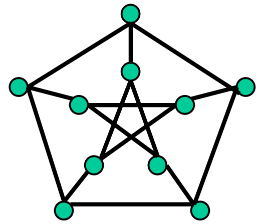
- **Def: In an undirected graph G, $x \in V(G)$.**
  - **A graph is k-regular $\equiv \forall\, x \in V$, $d_G(x) = k$.**
  - **A graph is regular $\equiv \exists\, k$, s.t. G is k-regular.**
  - **k is called the regularity of G.**
  - **maximum degree of G, $\Delta(G) \equiv \max \{d_G(x): x \in V(G)\}$.**
  - **minimum degree of G, $\delta(G) \equiv \min \{d_G(x): x \in V(G)\}$.**

    ex: K_n is $(n-1)$-regular,

    $K_{n,n}$ is n-regular.

    Petersen graph is 3-regular

    Q_n is n-regular.

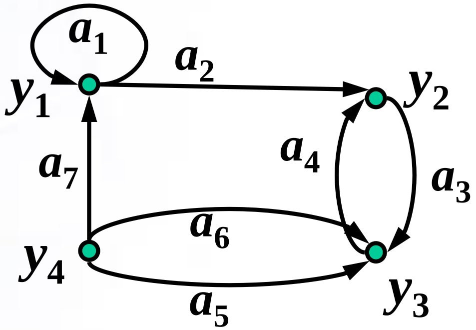


- **Note: If G is k-regular, then $\Delta(G) = \delta(G) = k$.**

1.3 Vertex Degrees

- **Def**: In digraph D, $y \in V(D)$.
  - $E_D^+(y)$ ($E_D^-(y)$): a set of out-going (in-coming) edges of y.
  - **out-degree** of y, $d_D^+(y) \equiv |E_D^+(y)|$
    in-degree of y, $d_D^-(y) \equiv |E_D^-(y)|$

ex: D



$d_D^+(y_1) = 2$, $d_D^+(y_2) = 1$, $d_D^+(y_3) = 1$, $d_D^+(y_4) = 3$

$d_D^-(y_1) = 2$, $d_D^-(y_2) = 2$, $d_D^-(y_3) = 3$, $d_D^-(y_4) = 0$

  - The **out-neighbors** of x, $N^+(x) = N_D^+(x) \equiv \{ y \mid (x, y) \in E(D)\}$.
  - The **in-neighbors** of x, $N^-(x) = N_D^-(x) \equiv \{ y \mid (y, x) \in E(D)\}$.
  - y is **balanced** if $d_D^+(y) = d_D^-(y)$. ex: y_1
    D is **balanced** if each of its vertices is balanced.

1.3 Vertex Degrees

- **Def**: In digraph D, $y \in V(D)$.
  - $\Delta^{+}(D) = \max\{d_D^{+}(y) : y \in V(D)\}$. **maximum out-degree**

    $\Delta^{-}(D) = \max\{d_D^{-}(y) : y \in V(D)\}$. **maximum in-degree**
  - $\delta^{+}(D) = \min\{d_D^{+}(y) : y \in V(D)\}$. **minimum out-degree**

    $\delta^{-}(D) = \min\{d_D^{-}(y) : y \in V(D)\}$. **minimum in-degree**
  - **maximum degree**, $\Delta(D) = \max\{\Delta^{+}(D), \Delta^{-}(D)\}$

    minimum degree, $\delta(D) = \min\{\delta^{+}(D), \delta^{-}(D)\}$
  - A digraph D is ***k*-regular** if $\Delta(D) = \delta(D) = k$.

- Note: Let $G = (X \cup Y, E)$ be a bipartite undirected graph,

  ① $\sum_{x \in X} d_G(x) = \varepsilon(G) = \sum_{y \in Y} d_G(y)$

  ② $2\varepsilon(G) = \sum_{x \in V(G)} d_G(x)$

1.3 Vertex Degrees

- **Theorem 1.1: For any digraph D, $\varepsilon(D) = \sum_{x\in V} d_D^+(x) = \sum_{x\in V} d_D^-(x)$.**

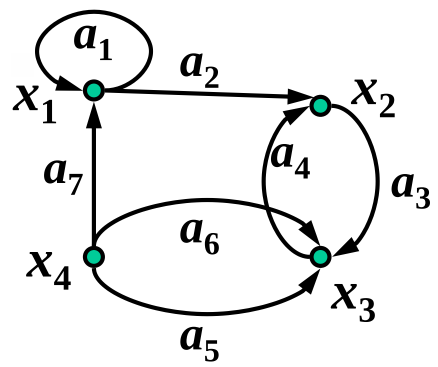
  Proof.

  Let G be the associated bipartite graph with D of bipartition $\{X, Y\}$.

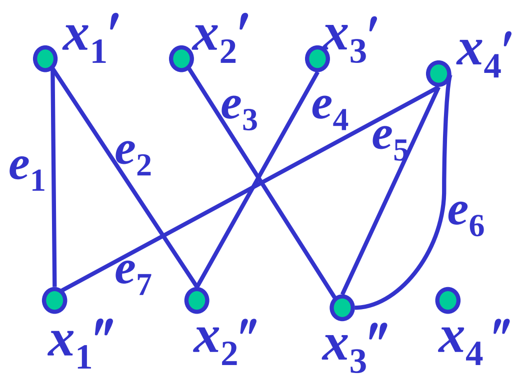
  $\therefore\ d_G(x') = d_D^+(x),\ d_G(x'') = d_D^-(x),\ \forall\ x \in V(D)$.

  $\Rightarrow \sum_{x\in V} d_D^+(x) = \sum_{x'\in X} d_G(x') = \varepsilon(G) = \sum_{x''\in Y} d_G(x'') = \sum_{x\in V} d_D^-(x)$

- **ex: Fig 1.7: D**



G:



1.3 Vertex Degrees

Theorem 1.1: For any digraph D, $\varepsilon(D) = \sum_{x\in V} d_D^+(x) = \sum_{x\in V} d_D^-(x)$.

- **Corollary 1.1: For any undirected graph G, ① $2\varepsilon(G) = \sum_{x\in V} d_G(x)$**

  ② the number of odd vertices is even.

  Proof.

  ① Let D be the symmetric digraph of G.

  $\Rightarrow \varepsilon(D) = 2\varepsilon(G)$.

  Note that $d_G(x) = d_D^+(x) = d_D^-(x)$, $\forall\, x \in V$.

  $\therefore$ **By Theorem 1.1,** $\sum_{x\in V} d_G(x) = \sum_{x\in V} d_D^+(x) = \sum_{x\in V} d_D^-(x) = \varepsilon(D) = 2\varepsilon(G)$.

  ② Let V_o be the set of odd vertices, let V_e be the set of even vertices.

  $\Rightarrow \sum_{x\in V_o} d_G(x) + \sum_{x\in V_e} d_G(x) = \sum_{x\in V} d_G(x) = 2\varepsilon(G)$

  $\because \sum_{x\in V} d_G(x), \sum_{x\in V_e} d_G(x)$ **both are even,**

  $\therefore \sum_{x\in V_o} d_G(x)$ **is also even.**

  $\because d_G(x)$ **is odd** $\forall\, x \in V_o$.

  $\therefore |V_o|$ **is even.**

1.3 Vertex Degrees

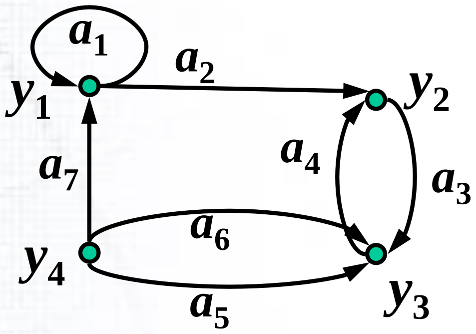
- **Def**: In digraph D, let $S, T \subseteq V(D)$.
  - $E_D(S, T) \equiv \{(x, y) \in E(D): x \in S, y \in T\}$ (= (S, T))
  - $\mu_D(S, T) \equiv |E_D(S, T)|$ (= $\mu(S, T)$)
  - $[S, T] \equiv (S, T) \cup (T, S)$
  - If $T = \overline{S} = V(D)\backslash S$: $E_D^+(S) \equiv (S, \overline{S})$ & $E_D^-(S) \equiv (\overline{S}, S)$
    $d_D^+(S) \equiv |E_D^+(S)|$ & $d_D^-(S) \equiv |E_D^-(S)|$
  - **out-neighbors** of S in D, $N_D^+(S) = \{y \in \overline{S}: (x, y) \in E(D), \forall x \in S\}$.
  - **in-neighbors** of S in D, $N_D^-(S) = \{x \in \overline{S}: (x, y) \in E(D), \forall y \in S\}$.

In undirected graph G, let $S \subseteq V(G)$.

- $E_G(S) \equiv$ the edges incident with vertices in S in G.
- **neighbors** of S in G, $N_G(S)$
- $d_G(S) = |E_G(S)|$

1.3 Vertex Degrees

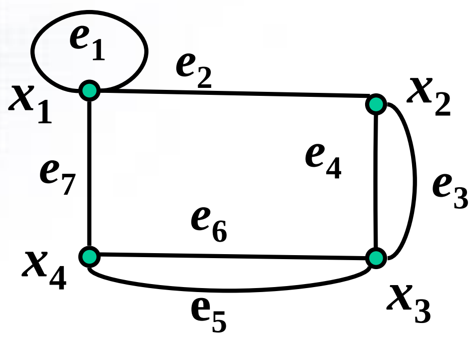
- ex:



Let $S = \{y_1, y_2\}$

$E_D^+(S) = \{a_3\}$, $d_D^+(S) = 1$, $N_D^+(S) = \{y_3\}$.

$E_D^-(S) = \{a_4, a_7\}$, $d_D^-(S) = 2$, $N_D^-(S) = \{y_3, y_4\}$.



Let $S = \{x_1, x_2\}$

$E_G(S) = \{e_1, e_2, e_3, e_4, e_7\}$,

$N_G(S) = \{x_1, x_2, x_3, x_4\}$,

$d_G(S) = 5$.

1.3 Vertex Degrees

Corollary 1.1: For any undirected graph G, ① $2\varepsilon(G) = \sum_{x\in V} d_G(x)$
② the number of odd vertices is even.

- **Example 1.3.1: If G is a simple undirected graph without triangles, then $\varepsilon(G) \le (1/4)\nu^2$.**

  Proof.

  $\forall\, xy \in E(G)$, $\because$ G is simple and no triangle.

  $\therefore\ [d_G(x) - 1] + [\, d_G(y) - 1] \le \nu - 2$,

  i.e. $d_G(x) + d_G(y) \le \nu$

  $\therefore\ \sum_{xy\in E(G)} (d_G(x) + d_G(y)) \le \varepsilon \cdot \nu$

  $\Rightarrow \sum_{x\in V(G)} d_G{}^2(x) \le \varepsilon \cdot \nu$

  By Cauchy's inequality and Corollary 1.1:

  $$\varepsilon \cdot \nu \ge \sum_{x\in V} d_G{}^2(x) \ge (1/\nu)\left(\sum_{x\in V} d_G(x)\right)^2 = (4/\nu)\varepsilon^2.$$

  $\Rightarrow \varepsilon \le (1/4)\nu^2$.

$$(x_1{}^2 + \ldots + x_n{}^2)(1^2 + \ldots + 1^2) \ge (x_1 \cdot 1 + x_2 \cdot 1 + \ldots + x_n \cdot 1)^2$$

1.3 Vertex Degrees

- **Example 1.3.2: Let G is a self-complementary simple undirected graph with $\nu \equiv 1 \pmod 4$. Prove that the number of $(1/2)(\nu - 1)$-degree vertices in G is odd.**

Proof. (1/2)

Let $\begin{cases} V_o \text{ be the set of odd vertices,} \\ V_e \text{ be the set of even vertices.} \end{cases}$

$|V_o|$ is even by Corollary 1.1.

$\because$ $\nu \equiv 1 \pmod 4$ is odd,

$\therefore$ $|V_e|$ is odd and $(1/2)(\nu - 1)$ is even.

Let V_e' be the set of vertices in V_e whose degree $\neq (1/2)(\nu - 1)$.

1.3 Vertex Degrees

- **<u>Example 1.3.2</u>: Let G is a self-complementary simple undirected graph with $\nu \equiv 1 \pmod 4$. Prove that the number of $(1/2)(\nu - 1)$-degree vertices in G is odd.**

Proof. (2/2)

Let $x \in V_e'$. $\because G \cong G^c$.

$\therefore \exists\, y_x \in V(G)$ s.t. $d_G(y_x) = d_{G^c}(x)$.

$\Rightarrow d_G(y_x) = d_{G^c}(x) = (\nu - 1) - d_G(x)$ is even.

$\therefore y_x \in V_e$.

$\because d_G(x) \neq (1/2)(\nu - 1)$. $\therefore d_G(y_x) \neq (1/2)(\nu - 1) \Rightarrow y_x \neq x$.

$\therefore y_x \in V_e'$.

and if $x, z \in V_e'$ and $x \neq z \Rightarrow y_x \neq y_z$.

$\Rightarrow$ the vertices in V_e' occur in pairs, i.e. $|V_e'|$ is even.

$\Rightarrow |V_e| - |V_e'|$ is odd.

i.e. the number of $(1/2)(\nu - 1)$-degree vertices is odd.

1.3 Vertex Degrees

- **Exercises: 1.3.2, 1.3.6(a)**
- **加: 1.3.5, 1.3.8**