**Computer Science and Information Engineering**
**National Chi Nan University**

# Combinatorial Mathematics

**Dr. Justie Su-Tzu Juan**

# Chapter 12 Trees

## § 12.1 Definition, Properties, and Examples

**Slides for a Course Based on the Text**
***Discrete & Combinatorial Mathematics* ($5^{th}$ Edition)**
**by Ralph P. Grimaldi**

# § 12.1 Definitions, Properties, and Examples

**Outline:**

1. **Definitions**
2. **Theorems**

# § 12.1 Definitions, Properties, and Examples

**Def 11.1:**

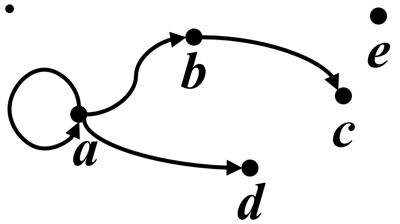
① $G = (V, E)$ is a **directed graph** (or **digraph**) ≡

$V(G) = V$: finite nonempty set: **vertex set**: a set of **vertices** (or **nodes**)

$E(G) = E \subseteq V \times V$: **edge set**: a set of **edges** (or **arcs**)

② If $E$ is a set of unordered pairs of $V$: $G$ is called an **undirected graph** (or **graph**).

ex:




① $(b, c)$ is **incident** with $b$, $c$

② $b$ is **adjacent to** $c$

③ $c$ is **adjacent from** $b$

④ $b$ is the **origin** (or **source**) of $(b, c)$

$c$ is the **terminus** (or **terminating vertex**) of $(b, c)$

⑤ $(a, a)$ is a **loop**

⑥ $e$ is an **isolated vertex**

# § 12.1 Definitions, Properties, and Examples

**Def: A graph contains no loop is called loop-free**

**Def 11.2: ① *x*-*y* walk in an graph *G* is a loop-free finite alternating sequence:**

$x = x_0, e_1, x_1, e_2, x_2, \ldots, e_{n-1}, x_{n-1}, e_n, x_n = y$

**where** $x_i \in V$, $e_j \in E$, $\forall\ i = 0, 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$

**and** $e_i = \{x_{i-1}, x_i\}$, $\forall\ 1 \le i \le n$.

**② the length of *x*-*y* walk is the number of edges in it (*n*)**

**③ if $n = 0$, the walk is called trivial**

**④ if $x = y$: the walk is called a closed walk,**
**otherwise it is called open walk**

**Def 11.3: ① *x*-*y* trail ≡ an *x*-*y* walk with no edge is repeated**

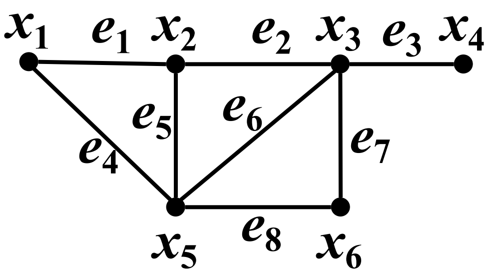
**② *x*-*y* path ≡ an *x*-*y* walk with no vertex is repeated**

**③ circuit ≡ a closed trail**

**④ cycle ≡ a closed *x*-*y* walk with no vertex is repeated except $x = y$.**

# § 12.1 Definitions, Properties, and Examples

**ex:**




① $x_1e_1x_2e_5x_5e_5x_2e_2x_3$: a walk

② $x_1e_1x_2e_2x_3e_6x_5e_5x_2e_1x_1$: a close walk

③ $x_2e_2x_3e_6x_5e_8x_6e_7x_3e_3x_4$: a trail

④ $x_2e_2x_3e_6x_5e_8x_6$: path

⑤ $x_2e_2x_3e_6x_5e_5x_2$: cycle

**Def 11.4:**

① $G = (V, E)$ be a graph, $G$ is **connected** $\equiv \forall\, x, y \in V, \exists\; x$-$y$ path in $G$

② otherwise, $G$ is called **disconnected**

**Def 12.1:** ① $G = (V, E)$ be a loop-free undirected graph, is called a **tree** if $G$ is connected and contains no cycle

② **forest**: contains no cycle

# § 12.1 Definitions, Properties, and Examples

**Def**: $G = (V, E)$ is a graph (or digraph), then

(**Def 11.7**) ① Graph $G_1 = (V_1, E_1)$ is called a **subgraph** of $G$ ($G_1 \subseteq G$), if $\phi \neq V_1 \subseteq V$ and $E_1 \subseteq E$.

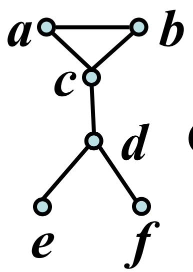
(**Def 11.8**) ② If $V_1 = V$, $G_1$ is called a **spanning subgraph** of $G$.

(**Def 11.9**) ③ If $E_1 = \{\{x, y\} \in E: \forall\, x \in V_1, y \in V_1\}$, $G_1$ is called the **induced subgraph** (or **subgraph of $G$ induced by $V_1$**).

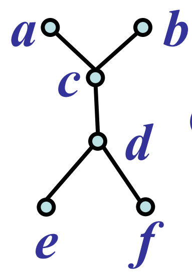
**Def**: ① **spanning tree** for a connected graph is a spanning subgraph that is also a tree.

② **spanning forest** for a connected graph is a spanning subgraph that is also a forest.

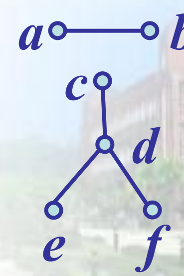
ex:




$G$




$G_1$: a spanning tree of $G$




$G_2$: a spanning forest of $G$

# § 12.1 Definitions, Properties, and Examples

**Thm 12.1**: $T = (V, E)$: tree, $\forall\, a \neq b \in V$, $\exists!$ $a$-$b$ path in $T$.

**Proof.**

$\because$ $T$ is connected and no cycle.

$\therefore$ $\exists$ $a$-$b$ path $P_1$

If $\exists$ $a$-$b$ path $P_2 \neq P_1$

then some edges of $P_1 \cup P_2$ would form a cycle. $\rightarrow\leftarrow$

$\therefore$ there is a unique path that connects $a$ and $b$.

**Def 11.10**: $G$: an undirected graph $G = (V, E)$

① $v \in V(G)$, $G - v \equiv$ the subgraph of $G$ induced by $V - \{v\}$

② $e \in E(G)$, $G - e \equiv V(G - e) = V(G)$; $E(G - e) = E(G) - \{e\}$

**Note**: $\forall$ simple graph $G$, $\exists$ $u$-$v$ walk $\Rightarrow$ $\exists$ $u$-$v$ path

# § 12.1 Definitions, Properties, and Examples

<u>Thm 12.2</u>: $G = (V, E)$ is an undirected graph:

$G$ is connected $\Leftrightarrow$ $G$ has a spanning tree.

**Proof. (1/2)**

($\Leftarrow$) $G$ has a spanning tree $T$.

$\therefore \forall\ a, b \in V(T) = V(G)$, $\exists$ $a$-$b$ path in $T \subseteq G$.

$\Rightarrow$ $G$ is connected.

($\Rightarrow$) If $G$ is connected and $G$ is not a tree:

Let $G'$ be a connected spanning subgraph of $G$ with minimal edge $E'$.

If $G'$ is not a tree, then $\exists$ cycle $C_1$ in $G'$.

take $e = uv \in C_1$ and let $G'' = G' - e$

$\forall\ x, y \in V(G) = V(G') = V(G'')$, $\because$ $G'$ is connected $\therefore \exists$ $x$-$y$ path $P$

① if $e \notin P$, then $P$ in $G''$

② if $e \in P$, then $x$-…-$u$-$(C_1 - e)$-$v$-…-$y$ is an $x$-$y$ walk in $G''$

$\therefore \exists$ $x$-$y$ path in $G''$

# § 12.1 Definitions, Properties, and Examples

**Thm 12.2**: $G = (V, E)$ is an undirected graph:

$G$ is connected $\Leftrightarrow$ $G$ has a spanning tree.

**Proof. (2/2)**

($\Rightarrow$) $\forall\, x, y \in V(G) = V(G') = V(G'')$, $\because$ $G'$ is connected $\therefore$ $\exists$ $x$-$y$ path $P$

① if $e \notin P$, then $P \in G''$

② if $e \in P$, then $x$-…-$u$-$(C_1 - e)$-$v$-…-$y$ is a $x$-$y$ walk in $G''$

$\therefore$ $\exists$ $x$-$y$ path in $G''$

then $V(G'') = V(G') = V(G)$, and $G''$ is connected

with $E(G'') = E(G') - 1 < E(G')$ $\rightarrow\leftarrow$

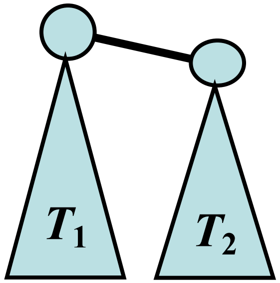
$\therefore$ $G'$ is a tree.

i.e. $G$ has a spanning tree

# § 12.1 Definitions, Properties, and Examples

**Thm 12.3**: $\forall$ tree $T = (V, E)$, $|V| = |E| + 1$

**Proof.**




Prove by induction on $|E|$

① If $|E| = 0$, $T$ = a single isolated vertex

Hence $|V| = 1 = |E| + 1$

② Assume the theorem is true for every tree $T$, with $E(T) \leq k$, where $k \geq 0$.

Now, consider a tree $T = (V, E)$, with $|E| = k + 1$.

let $e \in E(T)$

$T - e = T_1 \cup T_2$, where $T_1 = (V_1, E_1)$, $T_2 = (V_2, E_2)$

($T_1$ and $T_2$ both are tree, O.W. $T$ is not a tree)

and $|V| = |V_1| + |V_2|$, $|E| = |E_1| + |E_2| + 1$

$\because 0 \leq |E_1| \leq k$, $0 \leq |E_2| \leq k$, $\therefore$ By I.H.:

$|V| = |V_1| + |V_2| = |E_1| + 1 + |E_2| + 1 = |E| + 1$, it's true.

$\therefore$ By induction and ①, ②: $\forall$ tree $T$, $|V(T)| = |E(T)| + 1$

# § 12.1 Definitions, Properties, and Examples

**Def:** A graph $G$ is called a **simple** graph if

1. $\nexists$ loop.
2. $\nexists$ parallel edges.

**Def:** $G = (V, E)$ is a simple graph,

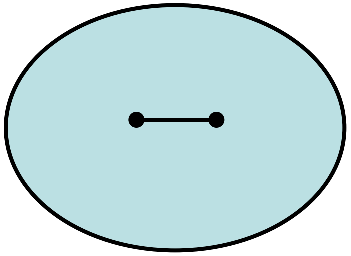
① **degree** of vertex $v$ $\equiv$ $\deg_G(v)$ (or $d_G(v)$) (or $d(v)$)

$= |\{uv \in E(G): \forall\, u \in V(G)\}|$

② vertex of **degree** $k$ = some vertex in $\{v: \deg_G(v) = k\}$

# § 12.1 Definitions, Properties, and Examples

**Thm 11.2:** $\forall$ graph $G = (V, E)$, $2|E| = \sum_{v \in V} \deg(v)$



**Proof.**

Prove by induction on $|E|$

① $|E| = 0 \Rightarrow \deg(v) = 0\ \forall\ v \in V$, $\therefore \sum_{v \in V} \deg(v) = 0 = |E|$

② Assume $2|E| = \sum_{v \in V} \deg(v)$, $\forall$ tree with $|E| \le k$, where $k \ge 0$.

Now consider a graph $G = (V, E)$ with $|E| = k + 1 \ge 1$

let $e = \{x, y\} \in E$, $G - e = G'(V', E')$

$\therefore V' = V$, $|E'| = |E| - 1 \le k$

and $\deg_{G'}(v) = \begin{cases} \deg_G(v), & \text{if } v \in V \setminus \{x, y\} \\ \deg_G(v) - 1, & \text{if } v = x \text{ or } v = y \end{cases}$

$\therefore$ by I.H.: $2|E| = 2(|E| - 1) + 2$

$$= \sum_{v \in V} \deg_{G'}(v) + 2$$

$$= \sum_{v \in V} \deg_G(v)$$

# § 12.1 Definitions, Properties, and Examples

**Thm 12.4**: $\forall$ tree $T = (V, E)$, if $|V| \geq 2$, then $T$ has at least two **pendant vertices**, (i.e. vertices of degree 1)

**Proof.**

Let $|V| = n \geq 2$, By Thm 12.3, $|V| = |E| + 1$, $|E| = n - 1$

$\therefore$ by Thm 11.2: $2(n-1) = 2|E| = \sum_{v \in V} \deg(v)$

$\because$ $G$ is connected. $\therefore$ $\deg(v) \geq 1$, $\forall\, v \in V(T)$

if $\forall\, v \in V$, $\deg(v) \geq 2$ or $\exists$ only one $v^*$ s.t. $\deg(v^*) = 1$

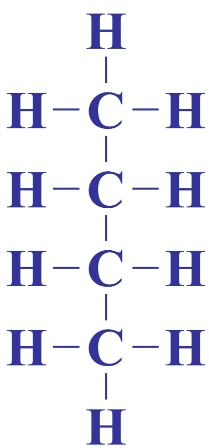
$\Rightarrow \sum_{v \in V} \deg(v) \geq 2(|V| - 1) + 1 = 2n - 1$ $\quad \rightarrow\leftarrow$

$\therefore$ $\exists$ at least 2 pendant vertices.

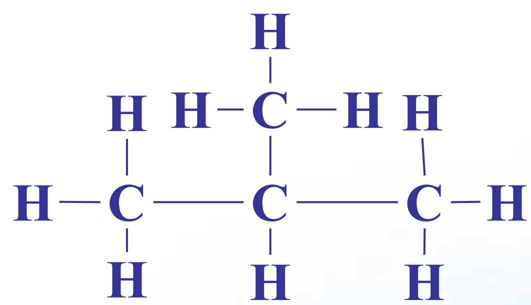
# § 12.1 Definitions, Properties, and Examples

**<u>Ex 12.1</u>: $C_4H_{10}$ : 14 vertices; 13 edges.**

**vertices labeled C has degree 4; labeled H has degree 1**



**(a) butane**
**丁烷**



**(b) 2-methyl propane (<u>iso</u>butane)**
**2甲基丙烷**

# § 12.1 Definitions, Properties, and Examples

飽和烴;飽和碳氫化合物

**Ex 12.2**: In a saturated hydrocarbon (no cycle, single-bond hydrocarbon – called an alkane) has $n$ carbon atoms, show it has $2n + 2$ hydrogen atoms.

鏈烷

**Sol.**

Consider the saturated hydrocarbon as a tree $T = (V, E)$.

$k = |\{v \in V \mid \deg(v) = 1\}|$ = the number of hydrogen atoms

$\Rightarrow |V| = n + k$, and $\because \deg(v) = 4$, $\forall\ v$ is a carbon atoms

$\therefore 4n + k = \sum_{v \in V} \deg(v) = 2|E| = 2(|V| - 1) = 2(n + k - 1)$

$\Rightarrow 4n + k = 2n + 2k - 2$

$\Rightarrow 2n + 2 = k$

# § 12.1 Definitions, Properties, and Examples

**Thm 12.5: ∀ loop-free undirected graph $G = (V, E)$**

**The following statement are equivalent. (T. F. S. E.)**

**(a) $G$ is a tree.**

**(b) $G$ is connected, but ∀ $e \in E$, $G - e$ is disconnected and $G - e$ is two subgraph that are tree (subtree).**

**(c) $G$ contains no cycles, and $|V| = |E| + 1$.**

**(d) $G$ is connected, and $|V| = |E| + 1$.**

**(e) $G$ contains no cycle, and if $a, b \in V$ with $\{a, b\} \notin E$, then $G + \{a, b\}$ $\left(\equiv \begin{array}{l} V(G + \{a, b\}) = V(G) \\ E(G + \{a, b\}) = E(G) \cup \{a, b\} \end{array}\right)$ has one cycle.**

**Proof. (1/4)**

**(a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (e) ⇒ (a)**

**leave for reader**

# § 12.1 Definitions, Properties, and Examples

**Thm 12.5**: ∀ loop-free undirected graph $G = (V, E)$

The following statement are equivalent. (T. F. S. E.)

(a) $G$ is a tree.

(b) $G$ is connected, but $\forall\, e \in E$, $G - e$ is disconnected and $G - e$ is two subgraph that are tree (subtree).

**Proof. (2/4)**

① ((a) ⇒ (b)): $G$ is a tree, then $G$ is connected.

$\forall\, e = \{a, b\} \in E$, if $G - e$ is connected, them $\exists\ a - b$ path $P$ in $G - e$

$\Rightarrow \exists$ cycle ($a$ - $P$ - $b$ - $e$ - $a$) in $G$ →←

∴ $G - e$ is disconnected and $G - e$ may be partition into 2 subsets:

(1) vertex $a$ and $\{v: \exists\ a\text{-}v \text{ path in } G - e\} \equiv G_1$

(2) vertex $b$ and $\{v: \exists\ b\text{-}v \text{ path in } G - e\} \equiv G_2$

These two connected components are trees.

∵ $G_1$, $G_2$ has no loop or cycle.

# § 12.1 Definitions, Properties, and Examples

**Thm 12.5**: $\forall$ loop-free undirected graph $G = (V, E)$

The following statement are equivalent. (T. F. S. E.)

(b) $G$ is connected, but $\forall\ e \in E$, $G - e$ is disconnected and $G - e$ is two subgraph that are tree (subtree).

(c) $G$ contains no cycles, and $|V| = |E| + 1$.

Proof. (3/4)

③ ((b) $\Rightarrow$ (c))

If $G$ contains a cycle $C$, let $e = \{a, b\} \in C$.

$\Rightarrow G - e$ is connected $\rightarrow\leftarrow$

$\therefore G$ has no cycle.

$\therefore G$ is a tree. (since $G$ is connected) $\Rightarrow |V| = |E| + 1$ (Thm 12.3)

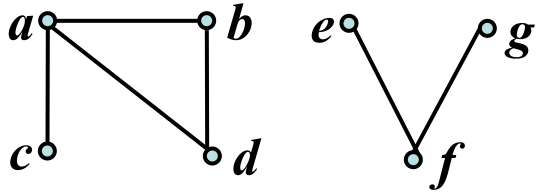
# § 12.1 Definitions, Properties, and Examples

**<u>Def 11.5</u>:** ① A (**connected**) **component** $G_i$ of $G$ is a maximal subgraph of $G$ s.t. $\forall\, x, y \in V(G_i)$, $\exists$ $x$-$y$ path in $G$ ($G_i$ is connected)

(maximal $\equiv \nexists\, G_j \subseteq G$ s.t. $G_i \subseteq G_j$ and $G_j$ is connected.)

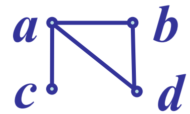
② the number of components of $G \equiv \kappa(G)$

ex:

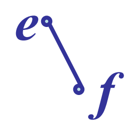



$G$

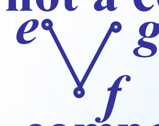
$\kappa(G) = 2$

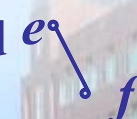



is a connected component of $G$




is not a connected component of $G$

($\because$ 

is a connected component and 

$\subseteq$ 

)

# § 12.1 Definitions, Properties, and Examples

**Thm 12.5**: ∀ loop-free undirected graph $G = (V, E)$

The following statement are equivalent. (T. F. S. E.)

(c) $G$ contains no cycles, and $|V| = |E| + 1$.

(d) $G$ is connected, and $|V| = |E| + 1$.

Proof. (4/4)

④ ((c) ⇒ (d))

Let $\kappa(G) = r$, and let $G_1, G_2, \ldots, G_r$ be the components of $G$.

$\forall\ 1 \le i \le r$, select $v_i \in G_i$

Let $G' = G + \{v_1, v_2\} + \{v_2, v_3\} + \ldots + \{v_{r-1}, v_r\}$

$\Rightarrow \because G'$ is no cycle and loop-free, and connected.

$\therefore G'$ is a tree $\Rightarrow$

$|E| + 1 = |V| = |V'| = |E'| + 1 = (|E| + r - 1) + 1 = |E| + r$

$\Rightarrow r = 1$

i.e. $G$ is connected

習題加入20

# § 12.1 Definitions, Properties, and Examples

**Checklist:**

1. **Definitions**
   - ☐ Digraph, graph, edge, vertex, incident, adjacent, isolated, loop
   - ☐ Walk, trail, path, circuit, cycle, length, close, open
   - ☐ Connected, disconnected, tree, forest
   - ☐ Subgraph, spanning subgraph, induced subgraph, spanning tree
   - ☐ Simple, degree, pendant vertex
2. **Theorems**
   - ☐ Thms 12.1, 12.2, 12.3
   - ☐ Thms 11.2, 12.4, 12,5

# § 12.1 Definitions, Properties, and Examples

<u>補充：</u>

**<u>Def 11.11:</u>** ① **Let $V$ be a set of $n$ vertices. The *complete bipartite* on $V$, denoted by $K_n$, is a loop-free undirected graph, where for all $a$, $b \in V$, $a \neq b$, there is an edge $\{a, b\}$.**

**<u>Def 11.18:</u>** ① **A graph $G$ is called *bipartite* if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \phi$, and every edge of $G$ is of the form $\{a, b\}$ with $a \in V_1$ and $b \in V_2$.**

② **If each vertex in $V_1$ is adjacent to each vertex in $V_2$, we have a *complete bipartite* graph.**

③ **In this case, if $|V_1| = m$, $|V_2| = n$, the graph is denoted by $K_{m,n}$.**

**Let's Kahoot!**