Computer Science and Information Engineering
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Combinatorial Mathematics

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Chapter 10 Recurrence Relation

§ 10.3 The Nonhomogeneous Recurrence Relation (2)

Slides for a Course Based on the Text
***Discrete & Combinatorial Mathematics* (5th Edition)**
by Ralph P. Grimaldi

§ 10.3 The Nonhomogeneous Recurrence Relation

Ex 10.29: loan: S, be paid back: T period time.

the interest rate per period: r

$\Rightarrow$ payment at the end of each period: $P = ?$

Sol. (1/2)

Let a_n = owed after the nth payment.

$$\Rightarrow \begin{cases} a_{n+1} = a_n + ra_n - P,\ 0 \le n \le T-1 \\ a_0 = S;\ a_T = 0. \end{cases}$$

① $a_{n+1}^{(h)} - (1+r)a_n^{(h)} = 0 \Rightarrow a_n^{(h)} = c(1+r)^n$

② Let $a_n^{(p)} = A \Rightarrow A = A + r \cdot A - P \Rightarrow r \cdot A = P \Rightarrow A = P/r$

i.e. $a_n^{(p)} = P/r$

③ $a_n = c(1+r)^n + P/r$

$\because a_0 = S = c + P/r \Rightarrow c = S - P/r$

$\therefore a_n = (S - P/r)(1+r)^n + P/r,\ 0 \le n \le T$

$\because 0 = a_T = (S - P/r)(1+r)^T + P/r$

$\Rightarrow P/r = (P/r - S)(1+r)^T$

§ 10.3 The Nonhomogeneous Recurrence Relation

Ex 10.29: loan: S be paid back: T period time.

the interest rate per period: r

$\Rightarrow$ payment at the end of each period: P = ?

Sol. (2/2)

$$\because\ 0 = a_T = (S - P/r)(1 + r)^T + P/r$$

$$\Rightarrow P/r = (P/r - S)(1 + r)^T$$

$$\therefore\ P = (P - rS)(1 + r)^T \Rightarrow P[1 - (1 + r)^T] = -Sr(1 + r)^T$$

$$\Rightarrow P = -Sr(1 + r)^T[1 - (1 + r)^T]^{-1}$$

$$= Sr[-(1 + r)^{-T}]^{-1}[1 - (1 + r)^T]^{-1}$$

$$= Sr[1 - (1 + r)^{-T}]^{-1}$$

$$\left(\frac{-(1+r)^T}{1-(1+r)^T} = \frac{1}{1-\dfrac{1}{(1+r)^T}}\right)$$

S = 1,000,000 r = 1.5%/12 T = 20 *12 $\rightarrow$ P = 4825.454088819525

S = 1,000,000 r = 3%/12 T = 20 *12 $\rightarrow$ P = 5545.97597853912

S = 1,000,000 r = 8%/12 T = 20 *12 $\rightarrow$ P = 8364.400689934629

§ 10.3 The Nonhomogeneous Recurrence Relation

<u>Ex 10.30</u>: For $n \geq 1$, $S \subset \mathbb{R}$, $|S| = 2^n$.

When determine the maximum and minimum element of S, what's the number of comparisons needed ?

Sol.

Let a_n = the number of needed comparisons:

$a_1 = 1$, when we have a_n :

a_{n+1}: $S = \{x_1, \ldots, x_{2^n}, y_1, \ldots, y_{2^n}\} = S_1 \cup S_2$, $S_1 = \{x_1, \ldots, x_{2^n}\}$, $S_2 = \{y_1, \ldots, y_{2^n}\}$

a) determine the maximum and minimum is S_1, S_2: need $2a_n$

b) $\max\{S_1\} \leftrightarrow \max\{S_2\}$, $\min\{S_1\} \leftrightarrow \{S_2\}$: need 2

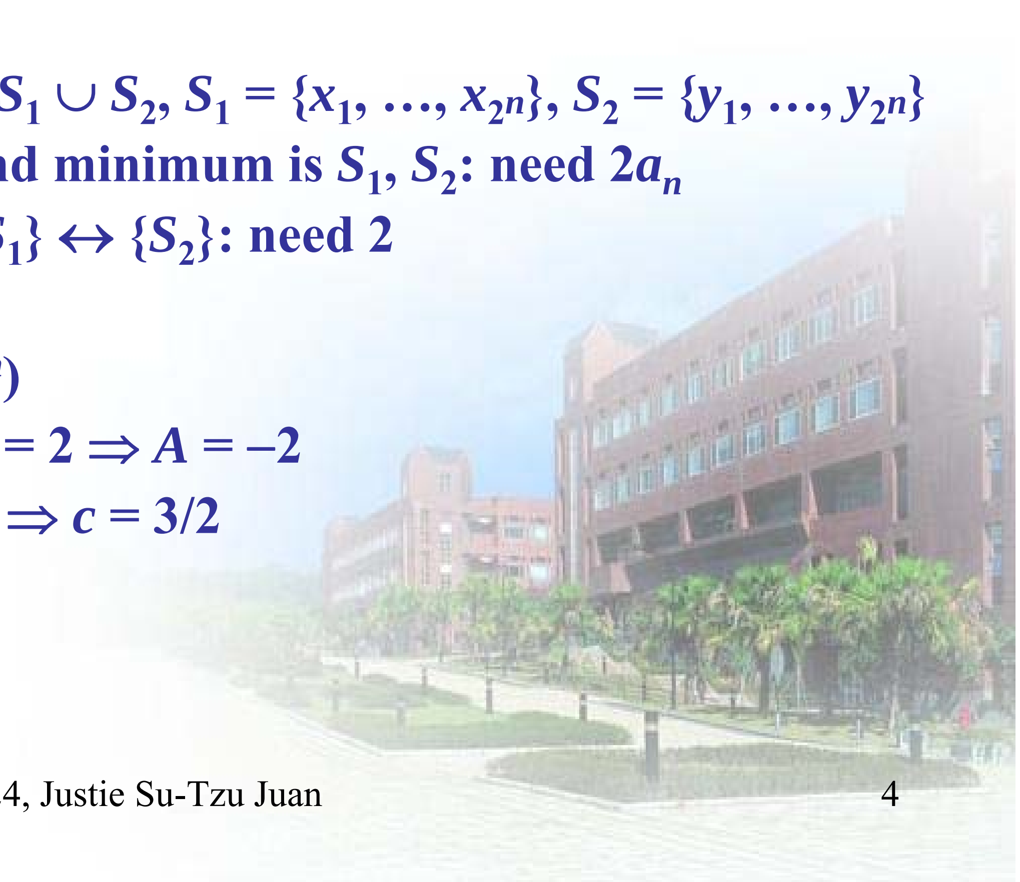
$\therefore a_{n+1} = 2a_n + 2$, $n \geq 1$

① $a_{n+1}^{(h)} - 2a_n^{(h)} = 0 \Rightarrow a_n^{(h)} = c(2^n)$

② $a_n^{(p)} = A$, a constant. $\Rightarrow A - 2A = 2 \Rightarrow A = -2$

③ $a_n = c(2^n) - 2 \Rightarrow a_1 = 1 = 2c - 2 \Rightarrow c = 3/2$

$\therefore a_n = 3 \cdot 2^{n-1} - 2$, $n \geq 1$



§ 10.3 The Nonhomogeneous Recurrence Relation

Note: We might achieve the same results via another method that required fewer comparisons.

Ex 10.31: $\Sigma = \{0, 1, 2, 3\}$,

there are 4^n strings of length n in $\Sigma^n = \{x_1x_2\ldots x_n | x_i \in \Sigma\}$.

a_n = count the strings in Σ^n where there are an even number of 1's. $a_{10} = ?$

Sol. Consider the nth symbol of one of there strings of length n:

1) (= 0, 2, 3) : $3a_{n-1}$ 2) (= 1): $4^{n-1} - a_{n-1}$

$\Rightarrow a_n = 3a_{n-1} + (4^{n-1} - a_{n-1}) = 2a_{n-1} + 4^{n-1}$, $n \geq 2$, and $a_1 = 3$.

① $a_n^{(h)} = c(2^n)$

② Let $a_n^{(p)} = A(4^{n-1}) \Rightarrow A(4^{n-1}) = 2A(4^{n-2}) + 4^{n-1} \Rightarrow 4A = 2A + 4 \Rightarrow A = 2.$

i.e. $a_n^{(p)} = 2(4^{n-1})$

③ $a_n = c(2^n) + 2(4^{n-1})$

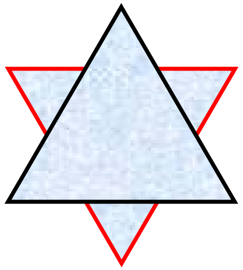
$\because a_1 = 3 = 2c + 2 \Rightarrow c = ½.$ $\therefore a_n = 2^{n-1} + 2(4^{n-1})$, $1 \leq n$.

$\Rightarrow a_{10} = 2^9 + 2(4^9) = 524800$ (= the answer using the method in 9.4)

§ 10.3 The Nonhomogeneous Recurrence Relation

Ex 10.32: snowflake curve: → 每一個邊變成4個(1/3長的)邊

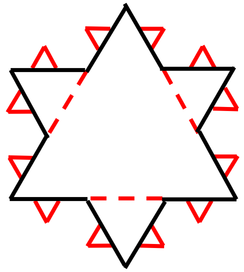
P_0



perimeter = 3

area = $\sqrt{3}/4$

P_1

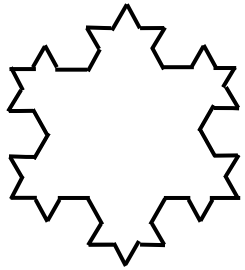


$p = (4/3) \cdot 3 = 4$

$a = \sqrt{3}/4 + 3(\sqrt{3}/4)(1/3)^2$

$= \sqrt{3}/3$

P_2



$p = 3(4/3)^2 = 16/3$

$a = \sqrt{3}/3 + 4 \cdot 3(\sqrt{3}/4)\left[(1/3)^2\right]^2$

$= 10\sqrt{3}/27$

Sol. (1/3)

For $n \geq 0$, let $a_n \equiv$ the area of the polygon P_n

$$a_{n+1} = a_n + (4^n \cdot 3)(\sqrt{3}/4)[(1/3)^{n+1}]^2$$

$$= a_n + (3\sqrt{3}/4)(4^n/(9 \cdot 9^n)) = a_n + (1/(4\sqrt{3}))(4/9)^n$$

① $a_{n+1}^{(h)} - a_n^{(h)} = 0 \Rightarrow a_n^{(h)} = c(1^n) = c$

② $\because (1/(4\sqrt{3}))(4/9)^n \neq k \cdot c, \forall k$

Let $a_n^{(p)} = A(4/9)^n \Rightarrow A(4/9)^{n+1} - A(\cancel{4/9})^n = (1/(4\sqrt{3}))(\cancel{4/9})^n$

$\Rightarrow A((4/9) - 1) = 1/(4\sqrt{3}) \Rightarrow A = -9/(20\sqrt{3})$

Ex 10.32: snowflake curve: → 每一個邊變成4個(1/3長的)邊

Sol. (2/3)

③ $\therefore a_n = c + \dfrac{-9}{20\sqrt{3}} \cdot \left(\dfrac{4}{9}\right)^n, n \geq 0$

$$a_0 = \frac{\sqrt{3}}{4} = c + \frac{-9}{20\sqrt{3}} = c - \frac{3\sqrt{3}}{20}$$

$$\Rightarrow \quad c = \frac{3\sqrt{3}}{20} + \frac{5\sqrt{3}}{20} = \frac{8\sqrt{3}}{20} = \frac{2\sqrt{3}}{5}$$

$$\Rightarrow a_n = \frac{2\sqrt{3}}{5} - \frac{9}{20\sqrt{3}} \cdot \left(\frac{4}{9}\right)^n = \frac{2\sqrt{3}}{5} - \frac{1}{5\sqrt{3}}\left(\frac{4}{9}\right)^{n-1}$$

$$= \frac{1}{5\sqrt{3}}\left[6 - \left(\frac{4}{9}\right)^{n-1}\right], n \geq 0$$

§ 10.3 The Nonhomogeneous Recurrence Relation

Ex 10.32: snowflake curve: → 每一個邊變成4個(1/3長的)邊

Sol. (3/3)

Note: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{5\sqrt{3}}\left[6-\left(\frac{4}{9}\right)^{n-1}\right] = \frac{6}{5\sqrt{3}}$

<another>

$$= \lim_{n\to\infty}\left\{\frac{\sqrt{3}}{4} + 3\cdot\frac{\sqrt{3}}{4}\sum_{i=0}^{n} 4^i\left(\frac{1}{3^{i+1}}\right)^2\right\}$$

$$= \frac{\sqrt{3}}{4} + 3\left(\frac{\sqrt{3}}{4}\right)\sum_{n=0}^{\infty} 4^n\left[\left(\frac{1}{3}\right)^{n+1}\right]^2$$

$$= \frac{\sqrt{3}}{4} + \frac{1}{4\sqrt{3}}\sum_{n=0}^{\infty}\left(\frac{4}{9}\right)^n$$

$$= \frac{\sqrt{3}}{4} + \frac{1}{4\sqrt{3}}\cdot\frac{1}{1-(4/9)} = \frac{5\sqrt{3}}{5\cdot 4} + \frac{1}{4\sqrt{3}}\cdot\frac{9\ 3\sqrt{3}}{5}$$

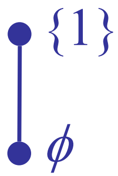
$$= \frac{8\sqrt{3}}{20} = \frac{2\sqrt{3}}{5} = \frac{6}{5\sqrt{3}}$$

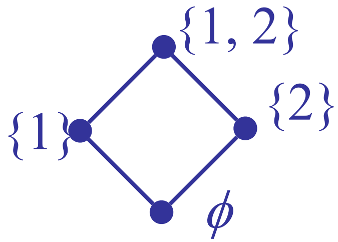
§ 10.3 The Nonhomogeneous Recurrence Relation

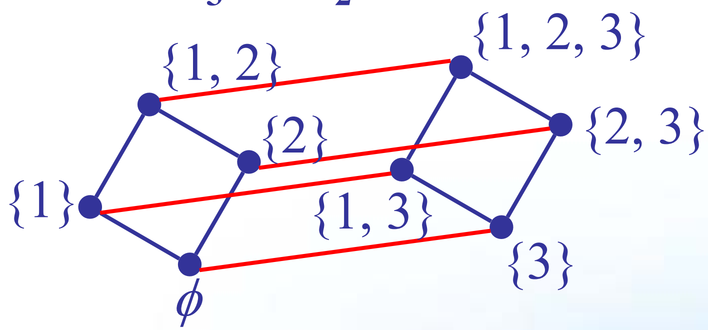
<u>Ex 10.33</u>: $\forall\ n \geq 1$, $X_n = \{1, 2, \ldots, n\}$; $\mathscr{P}(X_n) = 2^{X_n}$

Let a_n = the number of edge in the Hasse diagram for the partial order $(\mathscr{P}(X_n); \subseteq)$ = ?

Sol. (1/2) $a_1 = 1$ $\qquad$ $a_2 = 4$ $\qquad$ $a_3 = 2a_2 + 2^2$







$\because$ **the Hasse diagram for the partial orders $(\mathscr{P}(X_{n+1}, \subseteq))$**

= the Hasse diagram for the partial orders $(\mathscr{P}(X_n, \subseteq))$

+ the Hasse diagram for the partial orders $(\{T \cup \{n+1\} \mid T \in \mathscr{P}(X_n)\}, \subseteq)$

+ $\forall\ S \in \mathscr{P}(X_n)$, draw an edge from S to $S \cup \{n+1\}$

$\therefore$ $a_{n+1} = 2a_n + |\mathscr{P}(X_n)| = 2a_n + 2^n$

§ 10.3 The Nonhomogeneous Recurrence Relation

Ex 10.33: $\forall\, n \geq 1$, $X_n = \{1, 2, \ldots, n\}$; $\mathcal{P}(X_n) = 2^{X_n}$

Let a_n = the number of edge in the Hasse diagram for the partial order $(\mathcal{P}(X_n); \subseteq) = ?$

Sol. (2/2)

$\therefore\ a_{n+1} = 2a_n + |\mathcal{P}(X_n)| = 2a_n + 2^n$

$\Rightarrow \begin{cases} a_{n+1} - 2a_n = 2^n, n \geq 1 \\ a_1 = 1 \end{cases}$

① $a_{n+1}^{(h)} - 2a_n^{(h)} = 0$, $a_n^{(h)} = c(2^n)$, where $c \neq 0$

② $\because\ 2^n$ is a solution of the associated homogeneous relation.

$\therefore$ Let $a_n^{(p)} = B \cdot n \cdot 2^n \Rightarrow B(n+1) \cdot \cancel{2^{n}}{}^{+1} - 2B \cdot n \cdot \cancel{2^n} = \cancel{2^n}$

$\Rightarrow 2B(n+1) - 2n \cdot B = 1 \Rightarrow 2B = 1 \Rightarrow B = 1/2$

$\therefore\ a_n^{(p)} = n \cdot 2^{n-1}$

③ $a_n = a_n^{(h)} + a_n^{(p)} = c \cdot 2^n + n \cdot 2^{n-1} \Rightarrow a_1 = 1 = 2c + 1 \Rightarrow c = 0$

$\therefore\ a_n = n \cdot 2^{n-1}$, $\forall\, n \geq 1$

§ 10.3 The Nonhomogeneous Recurrence Relation

Ex 10.34: $\begin{cases} a_{n+2} - 4a_{n+1} + 3a_n = -200, \; n \geq 0 \\ a_0 = 3000 \\ a_1 = 3300 \end{cases}$

Sol. ① $(a_n^{(h)})^2 - 4a_n^{(h)} + 3 = 0 \Rightarrow a_n^{(h)} = c_1(3^n) + c_2(1^n) = c_1(3^n) + c_2$

② $\because f(n) = -200 = -200(1^n)$ is a solution of the associated homogeneous relation

$\therefore a_n^{(p)} = A \cdot n(1^n) = An$ for some constant A

$\Rightarrow A(n+2) - 4A(n+1) + 3An = -200$

$\therefore 2A - 4A = -200 = -2A \Rightarrow A = 100$

③ $a_n = a_n^{(h)} + a_n^{(p)} = c_1(3^n) + c_2 + 100n$

$$\begin{cases} a_0 = c_1 + c_2 = 3000 \\ a_1 = 3c_1 + c_2 + 100 = 3300 \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = 3000 \\ 3c_1 + c_2 = 3200 \end{cases} \Rightarrow \begin{cases} c_1 = 100 \\ c_2 = 2900 \end{cases}$$

$\therefore a_n = 100(3^n) + 2900 + 100n, \; n \geq 0$

Maple: 電腦計算代替人工筆算

§ 10.3 The Nonhomogeneous Recurrence Relation

Ex 10.35: **nth Fibonacci number F_n**

a_n: the number of additions performed in computing F_n, $n \geq 0$

(a) iterative algorithm

```
procedure FibNum1(n: nonnegative integer)
begin   if n = 0 then
           fib := 0
        else if n = 1 then
           fib := 1
        else  begin
                last := 1
                next_to_last := 0
                for i := 2 to n do
                   begin   temp := last
                           last := last + next_to_last
                           next_to_last := temp
                   end
                fib := last
              end
end
```

$a_0^{(i)} = a_1 = 0$

$a_n^{(i)} = n - 1$

§ 10.3 The Nonhomogeneous Recurrence Relation

Ex 10.35: nth Fibonacci number F_n

a_n: the number of additions performed in computing F_n, $n \geq 0$

(b) recursive algorithm:

```
Procedure FibNum2 (n: nonnegative integer)
begin
        if n = 0 then
          fib := 0
        else if n = 1 then
          fib := 1
        else
          fib := FibNum2 (n – 1) + FibNum2 (n – 2)
end
```

$$\Rightarrow \begin{cases} a_n = a_{n-1} + a_{n-2} + 1, \; n \geq 2 \\ a_0 = 0; \; a_1 = 0 \end{cases}$$

§ 10.3 The Nonhomogeneous Recurrence Relation

$$\begin{cases} a_n = a_{n-1} + a_{n-2} + 1,\ n \geq 2 \\ a_0 = 0;\ a_1 = 0 \end{cases}$$

Ex 10.35: ***n*th Fibonacci number F_n**

a_n: the number of additions performed in computing F_n, $n \geq 0$

① $a_n^{(h)} = a_{n-1}^{(h)} + a_{n-2}^{(h)} \Rightarrow a_n^{(h)} = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$

② $a_n^{(p)} = A \Rightarrow A = A + A + 1 \Rightarrow A = -1$

③ $a_n = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n - 1$

$$\begin{cases} a_0 = 0 = c_1 + c_2 - 1 \\ a_1 = 0 = c_1\,(\frac{1+\sqrt{5}}{2}) + c_2\,(\frac{1-\sqrt{5}}{2}) - 1 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 + c_2 = 1 \\ (c_1 + c_2) + \sqrt{5}\,(c_1 - c_2) = 2 \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = 1 \\ \sqrt{5}\,(c_1 - c_2) = 1 \end{cases}$$

$$\Rightarrow c_1 = (\frac{1+\sqrt{5}}{2\sqrt{5}}),\ c_2 = (\frac{\sqrt{5}-1}{2\sqrt{5}})$$

$$\therefore a_n = (\frac{1+\sqrt{5}}{2\sqrt{5}})(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2\sqrt{5}})(\frac{1-\sqrt{5}}{2})^n - 1 = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n+1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n+1} - 1$$

When $n \to \infty$, $(\frac{1-\sqrt{5}}{2})^{n+1} \to 0$, $a_n \to \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n+1} = \frac{1+\sqrt{5}}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n$

$\therefore$ as the value of n increases, $a_n >> a_n^{(i)}$

§ 10.3 The Nonhomogeneous Recurrence Relation

ℂ. extend:

Given a linear nonhomogeneous recurrence relation (with constant coefficient): $c_n a_n + c_{n-1} a_{n-1} + \ldots + c_{n-k} a_{n-k} = f(n)$, where $c_n, c_{n-k} \neq 0$,

① $a_n^{(h)}$: the homogeneous part of the solution a_n

② $a_n^{(p)}$: (1) If $f(n)$ is <u>not</u> a solution of the associated homogeneous relation

Table 10.2

$f(n)$	$a_n^{(p)}$
c, a constant	A, a constant
n	$A_1 n + A_0$
n^2	$A_2 n^2 + A_1 n + A_0$
$n^t, t \in \mathbb{Z}^+$	$A_t n^t + \ldots + A_1 n + A_0 = \sum_{i=0}^{t} A_i n^i$
$r^n, r \in \mathbb{R}$	$A r^n$
$\sin \theta n$	$A \sin \theta n + B \cos \theta n$
$\cos \theta n$	$A \sin \theta n + B \cos \theta n$
$n^t r^n$	$r^n (A_t n^t + A_{t-1} n^{t-1} + \ldots + A_1 n + A_0)$
$r^n \sin \theta n$	$A r^n \sin \theta n + B r^n \cos \theta n$
$r^n \cos \theta n$	$A r^n \sin \theta n + B r^n \cos \theta n$

§ 10.3 The Nonhomogeneous Recurrence Relation

ℂ. extend:

Given a linear nonhomogeneous recurrence relation (with constant coefficient): $c_n a_n + c_{n-1} a_{n-1} + \ldots + c_{n-k} a_{n-k} = f(n)$, where $c_n, c_{n-k} \neq 0$,

① $a_n^{(h)}$: the homogeneous part of the solution a_n

② $a_n^{(p)}$: (1) If $f(n)$ is not a solution of the associated homogeneous relation

(2) If $f(n)$ is a sum of constant multiples of ①'s (left) and none of these terms is a solution of the associated homogeneous relation; then $a_n^{(p)}$ = the sum of the corresponding terms in ①'s (right)

e.q.: $f(n) = n^2 + 3 \sin 2n$ and n^2, $\sin 2n$ is not a sol. ass. homo. re.

$\Rightarrow a_n^{(p)} = (A_2 n^2 + A_1 n + A_0) + (A \sin 2n + B \cos 2n)$

(3) If ∃ a summand $f_1(n)$ of $a_n^{(p)}$ contains r^n which is a solution of the associated homogeneous relation, then find smallest $s \in \mathbb{N}$ s.t. $n^s r^n$ is not a solution of the associated homogeneous relation.

$\Rightarrow n^s a_{n_1}^{(p)}$ is the corresponding part of $a_n^{(p)}$

§ 10.3 The Nonhomogeneous Recurrence Relation

Ex 10.36: $n \geq 2$, n people at a party. Each shakes hands to each other.

a_n = the total number of handshakes

Sol. (1/2) (方法一)

$$\begin{cases} a_{n+1} = a_n + n,\ n \geq 2 \\ a_2 = 1 \end{cases}$$

① $a_{n+1}^{(h)} \Rightarrow a_n^{(h)} \Rightarrow a_n^{(h)} = c(1^n) = c$

② $\because f(n) = n = n(1^n)$, and 1 is a solution of the associated homogeneous relation

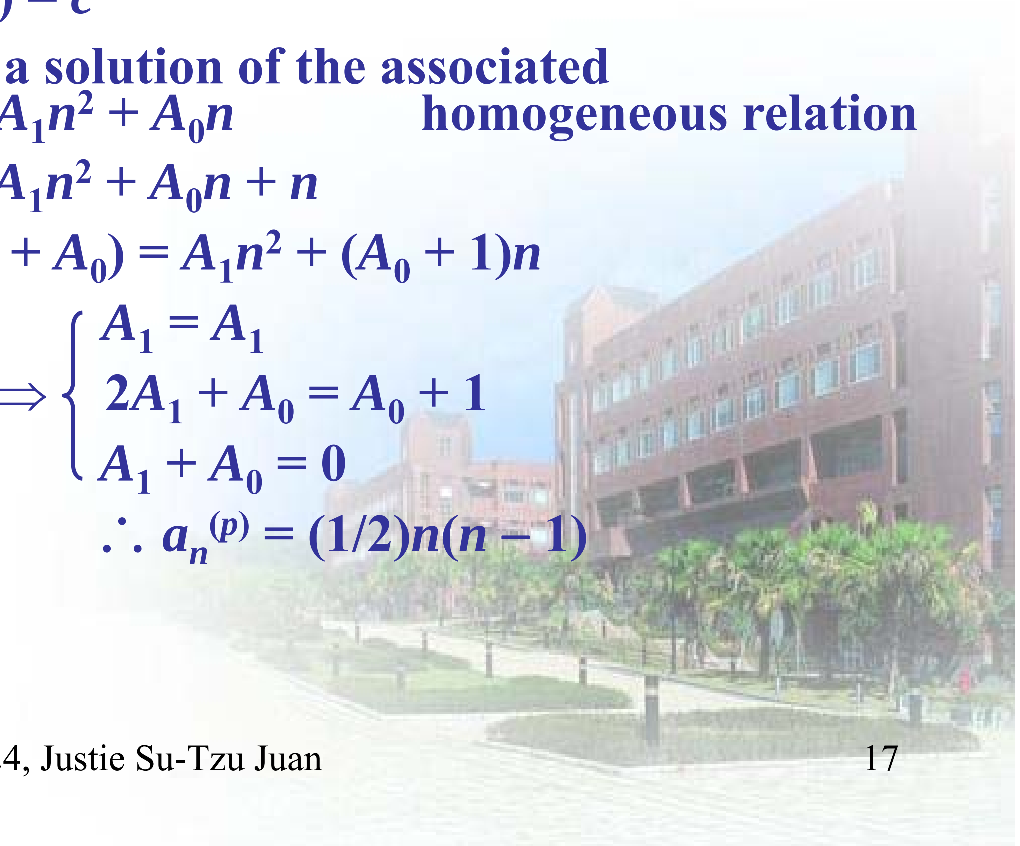
$\therefore$ Let $a_n^{(p)} = n(A_1 n + A_0) = A_1 n^2 + A_0 n$

$\Rightarrow A_1(n+1)^2 + A_0(n+1) = A_1 n^2 + A_0 n + n$

$\Rightarrow A_1 n^2 + (2A_1 + A_0)n + (A_1 + A_0) = A_1 n^2 + (A_0 + 1)n$

$$\begin{cases} A_1 n^2 = A_1 n^2 \\ (2A_1 + A_0)n = (A_0 + 1)n \\ (A_1 + A_0) = 0 \end{cases} \Rightarrow \begin{cases} A_1 = A_1 \\ 2A_1 + A_0 = A_0 + 1 \\ A_1 + A_0 = 0 \end{cases}$$

$\Rightarrow A_1 = 1/2,\ A_0 = -1/2$ $\therefore a_n^{(p)} = (1/2)n(n-1)$



§ 10.3 The Nonhomogeneous Recurrence Relation

<u>Ex 10.36</u>: $n \geq 2$, n people at a party. Each shakes hands to each other.

a_n = the total number of handshakes

Sol. (2/2) (方法一)

③ $a_n = c + (1/2)n(n-1)$

$\because a_2 = 1 = c + (1/2) \cdot 2 \cdot 1 \Rightarrow c = 0$

$\therefore a_n = (1/2)n(n-1), n \geq 0$

(方法二)

$\binom{n}{2} = n!/((n-2)!\ 2!) = (1/2)n(n-1)$ (組合)

(方法三)

graph: the number of edges in the complete graph K_n

$= \binom{n}{2} = (1/2)n(n-1)$

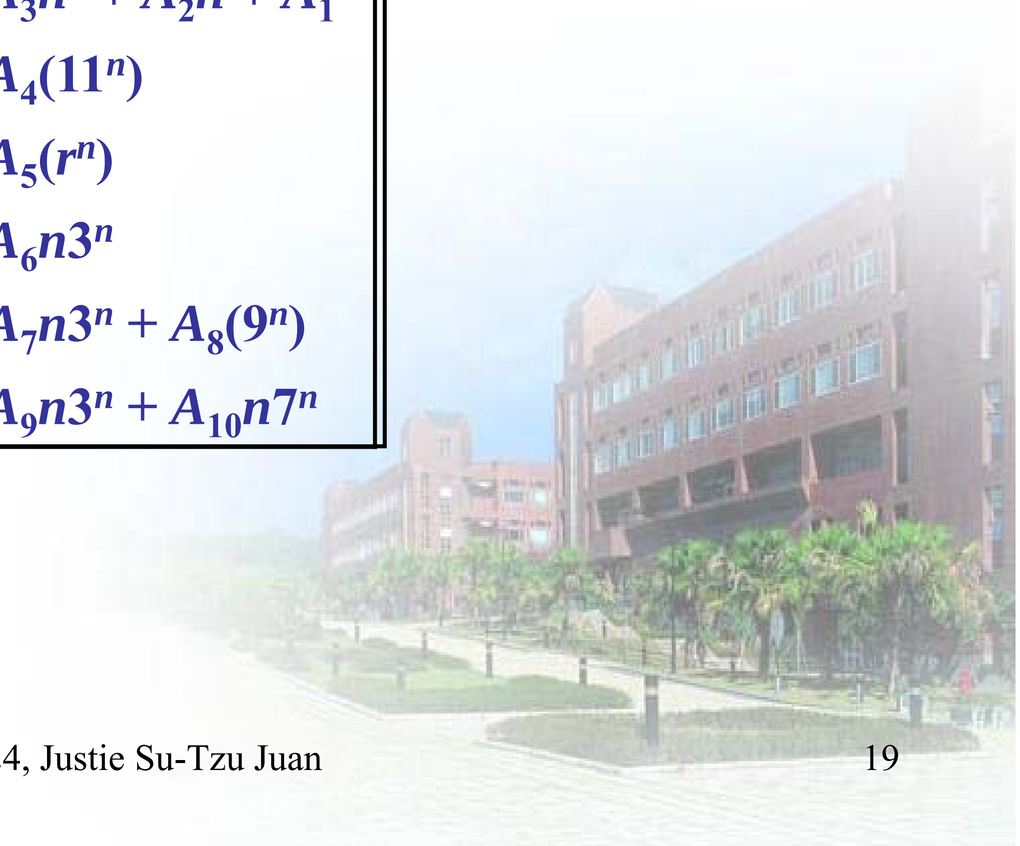
§ 10.3 The Nonhomogeneous Recurrence Relation

Ex 10.37: a) $a_{n+2} - 10a_{n+1} + 21a_n = f(n), n \geq 0$

① $a_n^{(h)} = c_1(3^n) + c_2(7^n)$, for arbitrary constants c_1, c_2

②

$f(n)$	$a_n^{(p)}$
5	A_0
$3n^2 - 2$	$A_3n^2 + A_2n + A_1$
$7(11^n)$	$A_4(11^n)$
$31(r^n), r \neq 3, 7$	$A_5(r^n)$
$6(3^n)$	A_6n3^n
$2(3^n) - 8(9^n)$	$A_7n3^n + A_8(9^n)$
$4(3^n) + 3(7^n)$	$A_9n3^n + A_{10}n7^n$



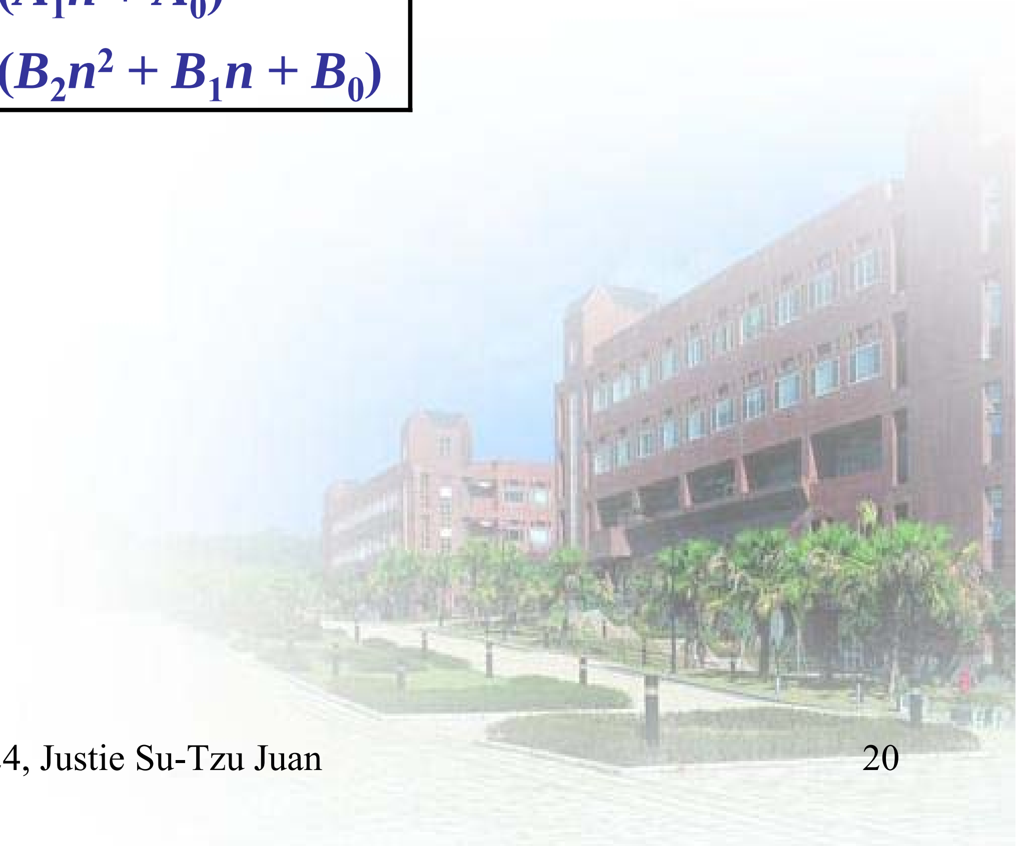
§ 10.3 The Nonhomogeneous Recurrence Relation

<u>Ex 10.37</u>: b) $a_n + 4a_{n-1} + 4a_{n-2} = f(n),\ n \geq 2$

① $a_n^{(h)} = c_1(-2)^n + c_2 n(-2)^n$, c_1, c_2 are constants

②

$f(n)$	$a_n^{(p)}$
$5(-2)^n$	$An^2(-2)^n$
$7n(-2)^n$	$n^2(-2)^n(A_1 n + A_0)$
$-11n^2(-2)^n$	$n^2(-2)^n(B_2 n^2 + B_1 n + B_0)$



§ 10.3 The Nonhomogeneous Recurrence Relation

<u>**Checklist:**</u>

1. **When $c_{n-1} = -1$ for first-order**
2. **Method of undetermined coefficient**

 □ **Table 10.2**

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

<u>Discussion (10 min)</u>:

<u>Exercise 10.3.6</u>: Solve the recurrence relation.

$$a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n),$$

where $n \geq 0$ and $a_0 = 1$, $a_1 = 4$.

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Chapter 10 Recurrence Relation

§ 10.4 The Method of Generating Functions

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

§ 10.4 The Method of Generating Functions

<u>Outline:</u>

1. **A technique will find both the homogeneous and particular solutions for a_n.**
2. **Solve a system of recurrence relations.**

§ 10.4 The Method of Generating Functions

Ex 10.38: $\begin{cases} a_n - 3a_{n-1} = n, \ n \geq 1 \\ a_0 = 1 \end{cases}$

Sol. (1/2)

$$\begin{array}{lrl} (n = 1) & a_1x^1 - 3a_0x^1 &= 1x^1 \\ (n = 2) & a_2x^2 - 3a_1x^2 &= 2x^2 \\ +) & \vdots & \vdots \\ \hline & \sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1}x^n &= \sum_{n=1}^{\infty} nx^n \quad \ldots\ldots(1) \end{array}$$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the sequence $a_0, a_1, a_2, \ldots$

then (1) $\Rightarrow (f(x) - a_0) - 3xf(x) = \sum_{n=0}^{\infty} nx^n$

$\Rightarrow (f(x) - 1) - 3xf(x) = x/(1 - x)^2$ (By Ex 9.5 (e))

$$\Rightarrow f(x) = \frac{1}{1-3x} + \frac{x}{(1-3x)(1-x)^2}$$

§ 10.4 The Method of Generating Functions

Ex 10.38: $\begin{cases} a_n - 3a_{n-1} = n, n \ge 1 \\ a_0 = 1 \end{cases}$

Sol. (2/2)

Let $\dfrac{x}{(1-3x)(1-x)^2} = \dfrac{A}{1-x} + \dfrac{B}{(1-x)^2} + \dfrac{C}{1-3x}$

$\Rightarrow x = A(1-x)(1-3x) + B(1-3x) + C(1-x)^2$

when $x = 1$: $1 = B(-2)$ $\Rightarrow B = -1/2$

when $x = (1/3)$: $(1/3) = C(2/3)^2$ $\Rightarrow C = 3/4$

when $x = 0$: $0 = A + B + C$ $\Rightarrow A = -1/4$

$\therefore f(x) = \dfrac{1}{1-3x} + \dfrac{\frac{-1}{4}}{1-x} + \dfrac{\frac{-1}{2}}{(1-x)^2} + \dfrac{\frac{3}{4}}{1-3x} = \dfrac{\frac{7}{4}}{1-3x} + \dfrac{\frac{-1}{4}}{1-x} + \dfrac{\frac{-1}{2}}{(1-x)^2}$

$\therefore a_n = (7/4)3^n - (1/4)1^n - (1/2)\binom{-2}{n}(-1)^n$

$= (7/4)3^n - (1/4) - (1/2)(n+1)$

$\therefore a_n = (7/4)3^n - (1/2)n - (3/4), n \ge 0$

(test another method that mention in Section 10.3)

§ 10.4 The Method of Generating Functions

Ex 10.39: $\begin{cases} a_{n+2} - 5a_{n+1} + 6a_n = 2,\ n \geq 0 \\ a_0 = 3;\ a_1 = 7 \end{cases}$

Sol. (1/2)

① $a_{n+2}x^{n+2} - 5a_{n+1}x^{n+2} + 6a_n x^{n+2} = 2x^{n+2}$

② $\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + 6\sum_{n=0}^{\infty} a_n x^{n+2} = 2\sum_{n=0}^{\infty} x^{n+2}$

③ $\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5x\sum_{n=0}^{\infty} a_{n+1}x^{n+1} + 6x^2\sum_{n=0}^{\infty} a_n x^n = 2x^2\sum_{n=0}^{\infty} x^n$

④ Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the solution,

then by ③: $(f(x) - a_0 - a_1x) - 5x(f(x) - a_0) + 6x^2f(x) = 2x^2/(1 - x)$

$(f(x) - 3 - 7x) - 5x(f(x) - 3) + 6x^2f(x) = 2x^2/(1 - x)$

Ex 10.39: $\begin{cases} a_{n+2} - 5a_{n+1} + 6a_n = 2, n \geq 0 \\ a_0 = 3; a_1 = 7 \end{cases}$

Sol. (2/2)

⑤ solving for $f(x)$:

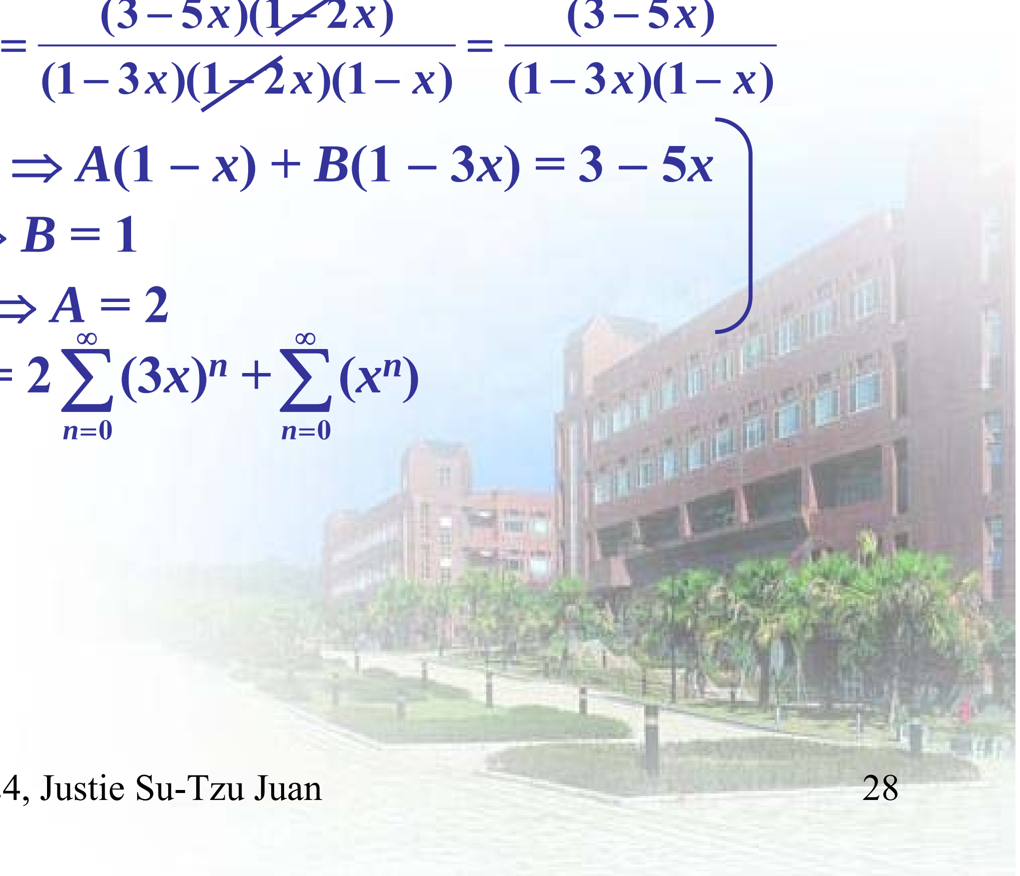
$$(1 - 5x + 6x^2)f(x) = 3 - 8x + 2x^2/(1 - x) = (3 - 11x + 10x^2)/(1 - x)$$

$$f(x) = \frac{3 - 11x + 10x^2}{(1 - 5x + 6x^2)(1 - x)} = \frac{(3 - 5x)\cancel{(1 - 2x)}}{(1 - 3x)\cancel{(1 - 2x)}(1 - x)} = \frac{(3 - 5x)}{(1 - 3x)(1 - x)}$$

$$\left(\begin{array}{l} \text{Let } f(x) = A/(1 - 3x) + B/(1 - x) \Rightarrow A(1 - x) + B(1 - 3x) = 3 - 5x \\ \quad \text{when } x = 1: B(-2) = (-2) \Rightarrow B = 1 \\ \quad \text{when } x = 1/3: A(2/3) = 4/3 \Rightarrow A = 2 \end{array} \right)$$

$$\therefore f(x) = 2/(1 - 3x) + 1/(1 - x) = 2\sum_{n=0}^{\infty}(3x)^n + \sum_{n=0}^{\infty}(x^n)$$

$$\therefore a_n = 2 \cdot 3^n + 1, n \geq 0$$



§ 10.4 The Method of Generating Functions

Ex 10.40: $n \in \mathbb{N}$, $\forall\, r \geq 0$,

let $a(n, r) = |\{[s_1, s_2, \ldots, s_r]: s_i \in \{b_1, b_2, \ldots, b_n\}, \forall\ 1 \leq i \leq r\}|$ where $\{b_1, b_2, \ldots, b_n\}$ be a set of n distinct objects. (Def:[ ]：不考慮順序但考慮重覆)

Sol. (1/2)

Consider b_n:

a) b_n is never selected $\Rightarrow a(n-1, r)$

b) b_n is selected ≥ 1 $\Rightarrow a(n, r-1)$

$\therefore\ a(n, r) = a(n-1, r) + a(n, r-1)\ \forall\ n \geq 1, r \geq 1$

Let $f_n = \sum_{r=0}^{\infty} a(n, r)x^r$ be the generation function for the sequence $a(n, 0)$, $a(n, 1), \ldots, a(n, r), \ldots$

① $a(n, r)x^r = a(n-1, r)x^r + a(n, r-1)x^r$

② $\Rightarrow \sum_{r=1}^{\infty} a(n, r)x^r = \sum_{r=1}^{\infty} a(n-1, r)x^r + \sum_{r=1}^{\infty} a(n, r-1)x^r$

§ 10.4 The Method of Generating Functions

Ex 10.40: $n \in \mathbb{N}, \forall r \geq 0,$

let $a(n, r) = |\{[s_1, s_2, \ldots, s_r]: s_i \in \{b_1, b_2, \ldots, b_n\}, \forall\ 1 \leq i \leq r\}|$ where $\{b_1, b_2, \ldots, b_n\}$ be a set of n distinct objects. (Def:[ ]：不考慮順序但考慮重覆)

Sol. (2/2)

③ $\because a(n, 0) = 1\ \forall\ n \geq 0,\ a(0, r) = 0\ \forall\ r > 0$

$\Rightarrow f_n - 1 = f_{n-1} - 1 + xf_n$

$\therefore f_n - xf_n = f_{n-1} \Rightarrow f_n = (1/(1-x))f_{n-1}$

$\Rightarrow f_n = (1/(1-x))f_{n-1} = (1/(1-x))^2 f_{n-2} = \ldots = (1/(1-x))^n f_0 = 1/(1-x)^n$

$(\because f_0 = a(0, 0) + a(0, 1)x + a(0, 2)x^2 + \ldots = 1 + 0 \cdot x + 0 \cdot x^2 + \ldots = 1)$

$\therefore f_n = (1-x)^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r}(-1)^r \cdot x^r$

$\therefore a(n, r) = \binom{-n}{r}(-1)^r = \binom{n+r-1}{r}(-1)^{2r} = \binom{n+r-1}{r}$

§ 10.4 The Method of Generating Functions

Ex 10.41: each time interval = 10^{-6} second. (microsecond)

a) high-energy neutron: two new high-energy and one low-energy neutrons.

b) low-energy neutron: one of each energy level.

Find a_n = the number of high-energy neutrons after n microsecond = ?

b_n = the number of low-energy neutrons after n microsecond = ?

with $a_0 = 1$, $b_0 = 0$

Sol. (1/3)

$$\begin{cases} a_{n+1} = 2a_n + b_n \\ b_{n+1} = a_n + b_n \end{cases} \quad \text{with} \begin{cases} a_0 = 1 \\ b_0 = 0 \end{cases}$$

① $$\begin{cases} a_{n+1}x^{n+1} = 2a_n x^{n+1} + b_n x^{n+1} \\ b_{n+1}x^{n+1} = a_n x^{n+1} + b_n x^{n+1} \end{cases}$$

② $$\begin{cases} \sum_{n=0}^{\infty} a_{n+1}x^{n+1} = 2x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} b_n x^n \\ \sum_{n=0}^{\infty} b_{n+1}x^{n+1} = x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} b_n x^n \end{cases}$$

③ let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be the generating function for the sequences $\{a_n \mid n \geq 0\}$, $\{b_n \mid n \geq 0\}$, respectively.

Sol. (2/3)

④ $\begin{cases} f(x) - a_0x^0 = 2xf(x) + xg(x) \\ g(x) - b_0x^0 = xf(x) + xg(x) \end{cases}$

$\Rightarrow \begin{cases} f(x) - 1 = 2xf(x) + xg(x) \\ g(x) = xf(x) + xg(x) \end{cases}$

$\Rightarrow \begin{cases} (1 - 2x)f(x) - xg(x) = 1 & \text{------①} \\ xf(x) + (x - 1)g(x) = 0 & \text{------②} \end{cases}$

① × $(x - 1)$ + ② × x: $[(1 - 2x)(x - 1) + x^2]f(x) = x - 1$: $f(x) = \frac{1-x}{x^2-3x+1}$

$\Rightarrow -x \cdot \frac{1-x}{x^2-3x+1} + (x-1)g(x) = 0$: $g(x) = \frac{x}{x^2-3x+1}$

$\because x^2 - 3x + 1 = (x - \frac{3+\sqrt{5}}{2})(x - \frac{3-\sqrt{5}}{2}) = (\frac{3+\sqrt{5}}{2} - x)(\frac{3-\sqrt{5}}{2} - x)$

$\therefore$ let $\gamma = \frac{3+\sqrt{5}}{2}$, $\delta = \frac{3-\sqrt{5}}{2}$, $x^2 - 3x + 1 = (\gamma - x)(\delta - x)$

<u>Note</u>: $1/\gamma = \delta$, $\delta = 1/\gamma$

§ 10.4 The Method of Generating Functions

Sol. (3/3)

$$\therefore f(x) = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1}{\gamma-x}\right) + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1}{\delta-x}\right) = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1}{\gamma}\frac{1}{1-(\frac{1}{\gamma})x}\right) + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1}{\delta}\frac{1}{1-(\frac{1}{\delta})x}\right)$$

$$g(x) = \left(\frac{-5-3\sqrt{5}}{10}\right)\left(\frac{1}{\gamma-x}\right) + \left(\frac{-5+3\sqrt{5}}{10}\right)\left(\frac{1}{\delta-x}\right)$$

$$\text{Le } f(x) = \frac{A}{\gamma-x} + \frac{B}{\delta-x} \Rightarrow A(\delta - x) + B(\gamma - x) = 1 - x$$

$$\therefore \begin{cases} -A - B = -1 \\ \delta A + \gamma B = 1 \end{cases} \Rightarrow \begin{cases} A + B = 1 & \text{-----①} \\ \frac{3-\sqrt{5}}{2}A + \frac{3+\sqrt{5}}{2}B = 1 & \text{-----②} \end{cases}$$

$$\text{①}\left(\frac{-3+\sqrt{5}}{2}\right) + \text{②}: \frac{2\sqrt{5}}{2}B = \frac{-1+\sqrt{5}}{2} \Rightarrow B = \frac{-1+\sqrt{5}}{2\sqrt{5}} = \frac{-\sqrt{5}+5}{10}$$

$$\therefore A = 1 - \frac{5-\sqrt{5}}{10} = \frac{5+\sqrt{5}}{10}$$

$$\Rightarrow a_n = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1}{r}\right)\left(\frac{1}{r}\right)^n + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1}{\delta}\right)\left(\frac{1}{\delta}\right)^n$$

$$\therefore \begin{cases} a_n = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{3-\sqrt{5}}{2}\right)^{n+1} + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{3+\sqrt{5}}{2}\right)^{n+1} \\ b_n = \left(\frac{-5-3\sqrt{5}}{10}\right)\left(\frac{3-\sqrt{5}}{2}\right)^{n+1} + \left(\frac{-5+3\sqrt{5}}{10}\right)\left(\frac{3+\sqrt{5}}{2}\right)^{n+1}, n \geq 0 \end{cases}$$

§ 10.4 The Method of Generating Functions

<u>Outline:</u>

1. **A technique:**

   Multiplying – Adding – Let $f(x)$ – Solve – Find a_n

2. **Solve a system of recurrence relations.**