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Combinatorial Mathematics

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Chapter 10 Recurrence Relations

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients (2)

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.10: $\begin{cases} F_{n+2} = F_{n+1} + F_n, \ \forall\, n \geq 0, \\ F_0 = 0,\ F_1 = 1. \end{cases}$ **(Fibonacci relation)**

Sol. Let $F_n = cr^n$, $r, c \neq 0$, $\forall\, n \geq 0$

$\Rightarrow cr^{n+2} = cr^{n+1} + cr^n$

$\Rightarrow$ the characteristic equation: $r^2 - r - 1 = 0$

$\Rightarrow$ the characteristic roots are $r = (1 \pm \sqrt{5})/2$

$\therefore$ Let the general solution: $F_n = c_1[(1+\sqrt{5})/2]^n + c_2[(1-\sqrt{5})/2]^n$

$$\begin{cases} F_0 = 0 = c_1 + c_2 \\ F_1 = 1 = c_1[(1+\sqrt{5})/2] + c_2[(1-\sqrt{5})/2] \end{cases}$$

$\Rightarrow c_1 = 1/\sqrt{5} = \sqrt{5}/5,$

$c_2 = (-1)/\sqrt{5} = (-\sqrt{5})/5$

$\therefore$ the general solution $F_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right], \forall\, n \geq 0$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.11: For $n \geq 0$, let $S = \{1, 2, \ldots, n\}$, ($n = 0$, $S = \phi$)

let a_n = # of subsets of S that contain no consecutive integers.

Find and **solve** a recurrence relation for a_n.

Sol. (1/2)

$a_0 = 1$: $\{\phi\}$

$a_1 = 2$: $\{\phi, \{1\}\}$

$a_2 = 3$: $\{\phi, \{1\}, \{2\}\}$

$$a_3 = 5: \{\phi, \{1\}, \{2\}, \{3\}, \{1, 3\}\} = \underbrace{\{\phi, \{1\}, \{2\}\}}_{(3 \notin A)} \cup \underbrace{\{\{3\}, \{1, 3\}\}}_{(3 \in A)}$$

If $A \subseteq S$ and A is to be counted in a_n:

(a) $n \in A$: $n - 1 \notin A \Rightarrow$ # of $(A - \{n\}) = a_{n-2}$.

(b) $n \notin A$: # of $A = a_{n-1}$.

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.11: For $n \geq 0$, let $S = \{1, 2, \ldots, n\}$, $(n = 0, S = \phi)$
let a_n = # of subsets of S that contain no consecutive integers.
Find and **solve** a recurrence relation for a_n.

Sol. (2/2)

Let $P = \{A \mid A \subseteq S, A$ contain no consecutive integers$\}$
then $P = B \,\dot\cup\, C$, where $B = \{A \in P \mid n \in A\}$
$C = \{A \in P \mid n \notin A\}$

$\Rightarrow |B| = a_{n-2}, |C| = a_{n-1}$

$\therefore |P| = a_n = |B| + |C| = a_{n-2} + a_{n-1}$

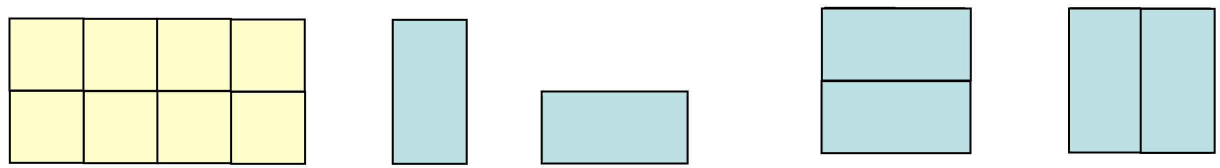
$$\therefore \begin{cases} a_n = a_{n-2} + a_{n-1}, \forall\, n \geq 2 \\ a_0 = 1;\ a_1 = 2 \end{cases}$$

$\Rightarrow a_n = F_{n+2}, n \geq 0$

$$\therefore a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right], \forall n \geq 0$$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.12:

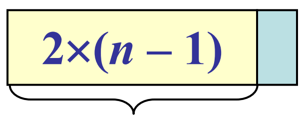


Sol. Let b_n = the number of ways we can cover a $2 \times n$ chessboard using 2×1 and 1×2 dominoes.

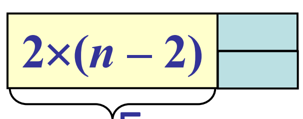
$b_1 = 1$: one 2×1,

$b_2 = 2$: two 2×1 or two 1×2.

When $n \geq 3$,

i) 

: b_{n-1}

ii) 

: b_{n-2}

$$\therefore \begin{cases} b_n = b_{n-2} + b_{n-1}, \forall\, n \geq 3 \\ b_1 = 1;\ b_2 = 2 \end{cases}$$

$$\Rightarrow b_n = F_{n+1},\ n \geq 0$$

$$\therefore b_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right], \forall n \geq 1$$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.13: (Lamé's Theorem) 略

Let $a, b \in \mathbb{Z}^+$ with $a \geq b \geq 2$. Then the number of divisions needs, on the Euclidean algorithm, to determine gcd(a, b) is at most 5 times the number of decimal digits in b.

Proof.

Use F_n and $F_n > [(1+\sqrt{5})/2]^{n-2}$

Note: The number of divisions needed, in the Euclidean algorithm, to determine gcd(a, b), for $a, b \in \mathbb{Z}^+$ with $a \geq b \geq 2$, is $\mathcal{O}(\log_{10} b)$ – that is, on the order of the number of decimal digits in b.

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.14: (Comparable relation)

a_n = # of "legal arithmetic expression without parentheses," that are made up of n symbols (+, *, /, 0, 1, 2, …, 9). a_n = ?

(+, *, / : operation symbols; 0, 1, 2, …, 9 : digits)

Sol. (1/2)

$a_1 = 10$: (0, 1, …, 9)

$a_2 = 100$: (00, 01, …, 99)

when $n \geq 3$; a_n:

1) [$n-1$ ending in d] [d]: $a_{n-1} \cdot 10$

2) [$n-2$] [o][d]: $a_{n-2} \cdot (3 \cdot 10 - 1)$ ($\because$ no [/][0])

$$\therefore \begin{cases} a_n = 10a_{n-1} + 29a_{n-2}, \text{ where } n \geq 3 \\ a_1 = 10;\ a_2 = 100 \end{cases}$$

$$\Rightarrow r^2 - 10r - 29 = 0 \Rightarrow r = 5 \pm 3\sqrt{6}$$

$$\Rightarrow a_n = \left(\frac{5}{3\sqrt{6}}\right)[(5+3\sqrt{6})^n - (5-3\sqrt{6})^n],\ n \geq 1$$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.14: (Comparable relation)

a_n = # of "legal arithmetic expression without parentheses," that are made up of n symbols (+, *, /, 0, 1, 2, …, 9). a_n = ?

(+, *, / : operation symbols; 0, 1, 2, …, 9 : digits)

Sol. (2/2)

$$\therefore \begin{cases} a_n = 10a_{n-1} + 29a_{n-2}, \text{ where } n \geq 3 \\ a_1 = 10;\ a_2 = 100 \end{cases}$$

<another>: $a_2 = 10a_1 + 29a_0 \Rightarrow 100 = 10 \cdot 10 + 29 \cdot a_0 \Rightarrow a_0 = 0$

$$\begin{cases} a_n = 10a_{n-1} + 29a_{n-2} \\ a_0 = 0;\ a_1 = 10 \end{cases}$$

$$\Rightarrow a_n = \left(\frac{5}{3\sqrt{6}}\right)[(5+3\sqrt{6})^n - (5-3\sqrt{6})^n],\ n \geq 0$$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.15: palindromes of 3, 4, 5 and 6:

(1) 3	(1') 5	(1) 4	1') 6
(2) 1+1+1	(2') 2+1+2	(2) 1+2+1	2') 2+2+2
	(1") 1+3+1	(3) 2+2	3') 3+3
	(2") 1+1+1+1+1	(4) 1+1+1+1	4') 2+1+1+2
			1") 1+4+1
			2") 1+1+2+1+1
			3") 1+2+2+1
			4") 1+1+1+1+1+1

Sol. i) Add 1 to the first and last summands.

ii) Append "1+" and "+1" to the end.

For Let $n \in \mathbb{Z}^+$, let p_n = the number of palindromes of n.

Then $\begin{cases} p_n = 2p_{n-2}, \forall\ n \geq 3 \\ p_1 = 1;\ p_2 = 2 \end{cases}$

$$\Rightarrow p_n = \begin{cases} \left(\frac{1}{2}+\frac{1}{2\sqrt{2}}\right)2^k + \left(\frac{1}{2}-\frac{1}{2\sqrt{2}}\right)2^k = 2^k = 2^{n/2}, \text{ if } n \text{ is even;} \\ 2^{(n-1)/2} \qquad\qquad, \text{ if } n \text{ is odd} \end{cases}, \text{ for } n \geq 0.$$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.16: auxiliary variables:

Find a recurrence relation for the # of binary sequences of length *n* that have no consecutive 0's.

Sol. (1/4)

(a) For $n \geq 1$, let $a_n = |P_n| = |\{x \mid x$ is a binary sequence of length n that have no consecutive 0's$\}|$

$$a_n^{(1)} = |\{x \in P_n \mid x \text{ end in } 1\}| = |P_n^{(1)}|$$

$$a_n^{(0)} = |\{x \in P_n \mid x \text{ end in } 0\}| = |P_n^{(0)}|$$

$$\Rightarrow a_n = a_n^{(1)} + a_n^{(0)} \qquad (P_n = P_n^{(1)} \dot{\cup} P_n^{(0)}) \qquad (1)$$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.16: auxiliary variables:

Find a recurrence relation for the # of binary sequences of length n that have no consecutive 0's.

Sol. (2/4)

① $a_1 = 2$: 0, 1

② $\forall\, n \geq 2$, a_n: case 1: $\underbrace{\boxed{x \in P_{n-1}^{(0)}}\ \boxed{0}}_{n-1}\ \boxed{1}$: $a_{n-1}^{(0)}$

case 2: $\underbrace{\boxed{x \in P_{n-1}^{(1)}}\ \boxed{1}}_{n-1}\ \boxed{0}$ / $\boxed{1}$: $2 \cdot a_{n-1}^{(1)}$

$\Rightarrow a_n = a_{n-1}^{(0)} + 2a_{n-1}^{(1)}$ ——— (2)

and, $\forall\, y \in P_{n-2} \Rightarrow \boxed{y}\ \boxed{1} \in P_{n-1}^{(1)}$

$\forall\ \boxed{z}\ \boxed{1} \in P_{n-1}^{(1)} \Rightarrow z \in P_{n-2}$

$\therefore a_{n-2} = a_{n-1}^{(1)}$ ———

§ 10.2 The Second-Order L… Recurrence Relatio… …cients

$a_n = a_n^{(1)} + a_n^{(0)}$ ------(1)

$a_n = a_{n-1}^{(0)} + 2a_{n-1}^{(1)}$ ------(2)

$a_{n-2} = a_{n-1}^{(1)}$ ------(3)

Ex 10.16: auxiliary variables:

Find a recurrence relation for the # of binary sequences of length *n* that have no consecutive 0's.

Sol. (3/4)

(2) $\Rightarrow a_n = \underline{a_{n-1}^{(0)} + a_{n-1}^{(1)}} + a_{n-1}^{(1)}$

$\therefore a_n = a_{n-1} + a_{n-2}$ by (1) and (3)

$\Rightarrow \begin{cases} a_n = a_{n-1} + a_{n-2}, \forall\, n \geq 3 \\ a_1 = 2;\ a_2 = 3 \ \ (11, 01, 10) \end{cases}$

⋮

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

<u>Ex 10.16</u>: auxiliary variables:

Find a recurrence relation for the # of binary sequences of length *n* that have no consecutive 0's.

Sol. (4/4)

(b) For $n \geq 1$, let a_n = the number of binary sequence of length n that have no consecutive 0's.

① $a_1 = 2; a_2 = 3$

② $\forall\, n \geq 3, a_n$: $\underbrace{\square\cdots\square}_{n-1}\,1$: a_{n-1}

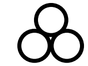
$\underbrace{\square\cdots\square}_{n-2}\,1\,0$: a_{n-2}

$$\begin{cases} a_n = a_{n-1} + a_{n-2}, n \geq 3 \\ a_1 = 2; a_2 = 3 \Rightarrow a_0 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} a_n = a_{n-1} + a_{n-2} \\ a_0 = 1; a_1 = 2 \end{cases}$$

$(a_n = F_{n+2}, n \geq 0)$ 同<u>Ex 10.11</u>

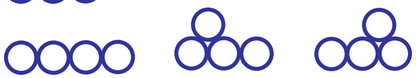
§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

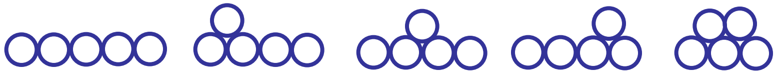
Ex 10.17: a_n = # of "arrange" (n identical pennies). s.t  and 同列相鄰

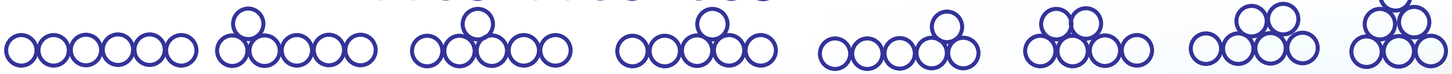
$a_1 = 1$: 

$a_2 = 1$: 

$a_3 = 2$: 

$a_4 = 3$: 

$a_5 = 5$: 

$a_6 = 8$: 

$\Rightarrow a_n = F_n$? $\Rightarrow$ ×

$a_7 = 12 \neq 13 = F_7$

$a_8 = 18 \neq 21 = F_8$

$a_9 = 26 \neq 34 = F_9$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

(Extend)

Ex 10.18: $\begin{cases} 2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n, \ n \geq 0 \\ a_0 = 0, \ a_1 = 1, \ a_2 = 2 \end{cases}$

$\Rightarrow$ Letting $a_n = cr^n$, for $c, r \neq 0$, and $n \geq 0$

$\Rightarrow$ $2r^3 - r^2 - 2r + 1 = 0 = (2r - 1)(r - 1)(r + 1)$

the characteristic roots are 1/2, 1, -1

(1/2, 1, -1 are linear independent)

$\Rightarrow$ the solution is $a_n = c_1(1)^n + c_2(-1)^n + c_3(1/2)^n$

$= c_1 + c_2(-1)^n + c_3(1/2)^n$

$$\begin{cases} a_0 = 0 = c_1 + c_2 + c_3 \\ a_1 = 1 = c_1 - c_2 + c_3/2 \\ a_2 = 2 = c_1 + c_2 + c_3/4 \end{cases} \Rightarrow \begin{cases} c_1 = 5/2 \\ c_2 = 1/6 \\ c_3 = -8/3 \end{cases}$$

$\Rightarrow$ $a_n = 5/2 + (1/6)(-1)^n - (8/3)(1/2)^n$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

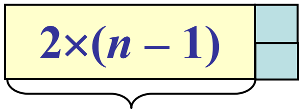
Ex 10.19:

Sol. Let a_n = the number of ways we can cover a $2 \times n$ chessboard using the two types of tiles shown above.

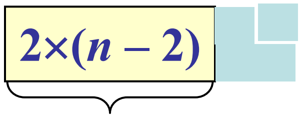
$a_1 = 1$: two 1×1, $a_2 = 5$: four 1×1 or one of each.

$a_3 = 11$: six $1 \times 1(1)$, three $1 \times 1(8)$, no $1 \times 1(2)$.

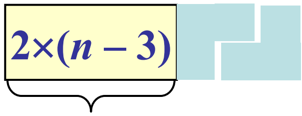
When $n \geq 3$,

i) 

: a_{n-1}

ii) 

: $4a_{n-2}$

iii) 

: $2a_{n-3}$

$$\therefore \begin{cases} a_n = a_{n-1} + 4a_{n-2} + 2a_{n-3}, \ \forall\, n \geq 4 \\ a_1 = 1;\ a_2 = 5;\ a_3 = 11 \end{cases}$$

$$\therefore a_n = (-1)^n + \frac{1}{\sqrt{3}}(1+\sqrt{3})^n - \frac{1}{\sqrt{3}}(1-\sqrt{3})^n, \quad \forall\, n \geq 1$$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Case (B): Complex Roots

Ex 10.21: $\begin{cases} a_n = 2(a_{n-1} - a_{n-2}), \forall\ n \geq 2 \\ a_0 = 1;\ a_1 = 2 \end{cases}$

Sol.

Letting $a_n = cr^n$, for $c, r \neq 0$.

$\Rightarrow r^2 - 2r + 2 = 0 \Rightarrow r = 1 \pm i$

the general solution $a_n = c_1(1 + i)^n + c_2(1 - i)^n$

(c_1, c_2 are arbitrary complex constant)

?

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

<u>Case (B)</u>: Complex Roots

<u>Recall</u>:

① **<u>DeMoivre's Theorem (</u>棣美弗定理):**

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,\ n \geq 0.$

② $z = x + iy \in \mathbb{C},\ z \neq 0$

$z = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{x^2 + y^2}$, $y/x = \tan\theta$; for $x \neq 0$

$z = yi = yi(\sin(\pi/2)) = r(\cos(\pi/2) + i\sin(\pi/2))$, for $y > 0$, for $x = 0$

$z = yi = |y|i\sin(3\pi/2) = r(\cos(3\pi/2) + i\sin(3\pi/2))$, for $y < 0$, for $x = 0$

③ $z^n = r^n(\cos n\theta + i\sin n\theta),\ \forall\ n \geq 0.$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

<u>Ex 10.20</u>: $(1+\sqrt{3}\,i)^{10} = ?$

Sol.

$$1+\sqrt{3}i = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$\therefore\ \left(1+\sqrt{3}i\right)^{10} = 2^{10}\left(\cos\frac{10\pi}{3}+i\sin\frac{10\pi}{3}\right)$$

$$= 2^{10}\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$$

$$= 2^{10}\left(\frac{-1}{2}+i\left(\frac{-\sqrt{3}}{2}\right)\right)$$

$$= -2^{9}\left(1+\sqrt{3}i\right)$$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.21: $\begin{cases} a_n = 2(a_{n-1} - a_{n-2}),\ \forall\ n \geq 2 \\ a_0 = 1;\ a_1 = 2 \end{cases}$

Sol. (1/2)

Letting $a_n = cr^n$, for $c, r \neq 0$.

$\Rightarrow r^2 - 2r + 2 = 0 \Rightarrow r = 1 \pm i$

the general solution $a_n = c_1(1 + i)^n + c_2(1 - i)^n$

(c_1, c_2 are arbitrary complex constant)

$\because 1 + i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$

$1 - i = \sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4)) = \sqrt{2}(\cos(\pi/4) - i\sin(\pi/4))$

$\Rightarrow a_n = c_1[\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))]^n + c_2[\sqrt{2}(\cos(\pi/4) - i\sin(\pi/4))]^n$

$= (\sqrt{2})^n[(c_1 + c_2)\cos(n\pi/4) + (c_1 - c_2)i\sin(n\pi/4)]$

$= (\sqrt{2})^n[k_1\cos(n\pi/4) + k_2\sin(n\pi/4)]$, where $\begin{cases} k_1 = c_1 + c_2 \\ k_2 = (c_1 - c_2)i \end{cases}$

$$a_n = (\sqrt{2}\,)^n[k_1 \cos(n\pi/4) + k_2 \sin(n\pi/4)], \text{ where } \begin{cases} k_1 = c_1 + c_2 \\ k_2 = (c_1 - c_2)i \end{cases}$$

§ 10.2 The Se… Recurrence Relation with Constant Coefficients

Ex 10.21: $\begin{cases} a_n = 2(a_{n-1} - a_{n-2}), \ \forall\ n \geq 2 \\ a_0 = 1;\ a_1 = 2 \end{cases}$

Sol. (2/2)

$$\Rightarrow \begin{cases} 1 = a_0 = k_1 \cos 0 + k_2 \sin 0 = k_1 \\ 2 = a_1 = \sqrt{2}\,(k_1 \cos(\pi/4) + k_2 \sin(\pi/4)) = (\sqrt{2}/\sqrt{2}\,)(k_1 + k_2) \end{cases} \Rightarrow \begin{cases} k_1 = 1 \\ k_2 = 1 \end{cases}$$

$$\therefore a_n = (\sqrt{2})^n(\cos(n\pi/4) + \sin(n\pi/4)),\ n \geq 0$$

(Note: $c_1 + c_2 \in \mathbb{R}$, $i(c_1 - c_2) \in \mathbb{R}$, if c_1, c_2: 共軛複數)

Ex 10. 22: 省略

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

<u>Case (C)</u>: (Repeated Real Roots)

ex: $\begin{cases} a_{n+2} = 4a_{n+1} - 4a_n, \ \forall\ n \geq 0 \\ a_0 = 1,\ a_1 = 3 \end{cases}$

Sol. (1/2)

Letting $a_n = cr^n$, where $c, r \neq 0$, $n \geq 0$.

$\Rightarrow$ characteristic equation: $r^2 - 4r + 4 = 0$

$\therefore$ characteristic roots: $r = 2$ (a root of multiplicity 2)

try $f(n)2^n$ be another independent solution:

$$f(n+2) \cdot 2^{n+2} = 4f(n+1) \cdot 2^{n+1} - 4f(n) \cdot 2^n$$

$$\Rightarrow f(n+2) \cdot \cancel{4} = \cancel{4}f(n+1) \cdot 2 - \cancel{4}f(n)$$

$$f(n+2) = 2f(n+1) - f(n)$$

$$\Rightarrow f(n) = an + b\ \forall\ a, b, \text{ and } a \neq 0$$

choose $f(n) = n$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Case (C): (Repeated Real Roots)

ex: $\begin{cases} a_{n+2} = 4a_{n+1} - 4a_n, \forall\, n \geq 0 \\ a_0 = 1, a_1 = 3 \end{cases}$

Sol. (2/2)

$\therefore n2^n$ is a second independent solution.

$(\because \forall\, n \geq 0, \nexists\, k \text{ s.t. } n2^n = k \cdot 2^n)$

$\therefore$ the general solution $= a_n = c_1(2^n) + c_2 n(2^n)$

$\begin{cases} a_0 = 1 = c_1 + 0 \\ a_1 = 3 = 2c_1 + 2c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 1/2 = 2^{-1} \end{cases}$

$\Rightarrow a_n = 2^n + (1/2)n(2^n)$

$= 2^n + n(2^{n-1}), n \geq 0$

$(= (1 + (1/2)n)\, 2^n\ )$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

In General:

If $c_n a_n + c_{n-1} a_{n-1} + \ldots + c_{n-k} a_{n-k} = 0$ with c_n ($\neq 0$), c_{n-1}, …, c_{n-k} ($\neq 0$) $\in \mathbb{R}$, and r is a characteristic root of multiplicity m, where $2 \le m \le k$, then the part of the general solution that involves the root r has the form:

$$A_0 r^n + A_1 n r^n + A_2 n^2 r^n + \ldots + A_{m-1} n^{m-1} r^n$$
$$= (A_0 + A_1 n + A_2 n^2 + \ldots + A_{m-1} n^{m-1}) \cdot r^n$$

where A_0, A_1, …, A_{m-1} are arbitrary constants.

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

<u>Ex 10.24</u>: $\begin{cases} p_n = p_{n-1} - (0.25)p_{n-2},\ n \geq 2 \\ p_0 = 0,\ p_1 = 1 \end{cases}$

$p_k < 0.01$ min. $k = ?$ (p_1 : March 1, 1999, p_n : nth week)

Sol. (1/2)

Let $p_n = cr^n$, $c, r \neq 0$

$\Rightarrow r^2 - r + (1/4) = 0 = (r - (1/2))^2$, $r = 1/2$

$\therefore$ general solution $= (c_1 + c_2 n)(1/2)^n$, $n \geq 0$

$$\begin{cases} p_0 = 0 = c_1 \\ p_1 = 1 = (c_1 + c_2)(1/2) \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 2 \end{cases} \Rightarrow p_n = n \cdot 2^{1-n},\ n \geq 0$$

The first integer n_0 s.t. $p_{n_0} < 0.01$ is

$n_0 \cdot 2^{1-n_0} < 0.01$

$200n_0 < 2^{n_0} \Rightarrow 2^{n_0} - 200n_0 > 0$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.24: $\begin{cases} p_n = p_{n-1} - (0.25)p_{n-2},\ n \geq 2 \\ p_0 = 0,\ p_1 = 1 \end{cases}$

$p_k < 0.01$ min. k = ? ($p_1 \Rightarrow$ March 1, 1999. $p_n \Rightarrow n$th week)

Sol. (2/2)

$200n_0 < 2^{n_0} \Rightarrow 2^{n_0} - 200n_0 > 0$

$n = 10 \Rightarrow 1024 - 2000 < 0$

$n = 11 \Rightarrow 2048 - 2200 < 0$

$n = 12 \Rightarrow 4096 - 2400 > 0$ √ $\therefore n_0 = 12$

$n_0 = 12$

Hence, until May 17, 1999 the probability of another new case occurring was < 0.01

3/1	3/8	3/15	3/22	3/29	4/5	4/12	4/19	4/26	5/3	5/10	**5/17**
p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	**p_{12}**

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Checklist:

1. **Definition**
2. **Formula (characteristic function): 3 cases**
   - ☐ **Fibonacci relation**
   - ☐ **Extend**
   - ☐ **DeMoivre's Theorem**
   - ☐ **Repeated real root: If $c_n a_n + c_{n-1} a_{n-1} + \ldots + c_{n-k} a_{n-k} = 0$ with c_n ($\neq 0$), $c_{n-1}, \ldots, c_{n-k}$ ($\neq 0$) $\in \mathbb{R}$, and r is a characteristic root of multiplicity m, where $2 \leq m \leq k$, then the part of the general solution that involves the root r has the form:**

$$A_0 r^n + A_1 n r^n + A_2 n^2 r^n + \ldots + A_{m-1} n^{m-1} r^n$$

$$= (A_0 + A_1 n + A_2 n^2 + \ldots + A_{m-1} n^{m-1}) \cdot r^n$$

where $A_0, A_1, \ldots, A_{m-1}$ are arbitrary constants.

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Discussion (10 min):

Exercise 10.2.1: Solve the following recurrence relations.

a) $a_n = 5a_{n-1} + 6a_{n-2}$, $n \geq 2$, $a_0 = 1$, $a_1 = 3$

b) $a_n - 6a_{n-1} + 9a_{n-2} = 0$, $n \geq 2$, $a_0 = 5$, $a_1 = 12$

c) $a_n + 2a_{n-1} + 2a_{n-2} = 0$, $n \geq 2$, $a_0 = 1$, $a_1 = 3$

Computer Science and Information Engineering
National Chi Nan University

Combinatorial Mathematics

Dr. Justie Su-Tzu Juan

Chapter 10 Recurrence Relation

§ 10.3 The Nonhomogeneous Recurrence Relation (1)

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

§ 10.3 The Nonhomogeneous Recurrence Relation

<u>Oteline</u>

1. **When $c_{n-1} = -1$ for first-order**
2. **Method of undetermined coefficient**

§ 10.3 The Nonhomogeneous Recurrence Relation

$$a_n + c_{n-1}a_{n-1} = f(n),\ n \ge 1 \qquad (1)$$

$$a_n + c_{n-1}a_{n-1} + c_{n-2}a_{n-2} = f(n),\ n \ge 1 \qquad (2)$$

where $c_{n-1} \ne 0$, $c_{n-2} \ne 0$, $f(n) \ne 0$.

A. Let $c_{n-1} = -1$ of (1): $a_n - a_{n-1} = f(n)$

$$a_1 = a_0 + f(1)$$
$$a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$$
$$a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3)$$
$$\vdots$$
$$a_n = a_0 + \sum_{i=1}^{n} f(i)$$

<u>Ex 10.25</u>: $\begin{cases} a_n - a_{n-1} = 3n^2, \text{ where } n \ge 1 \\ a_0 = 7 \end{cases}$

Sol. $f(n) = 3n^2$ $\quad \sum_{i=1}^{n} f(i) = \sum_{i=1}^{n} 3i^2 = 3\sum_{i=1}^{n} i^2 = \cancel{3}n(n+1)(2n+1)/\cancel{6}\ 2$

$\therefore\ a_n = 7 + (1/2)n(n+1)(2n+1)$

§ 10.3 The Nonhomogeneous Recurrence Relation

B. **method of undetermined coefficient**: (for certain function $f(n)$)

① Let $a_n^{(h)}$ ≡ the general solution of $f(n) = 0$ in (1) (or (2))

② Let $a_n^{(p)}$ (**particular solution**) ≡ a solution of (1) (or (2))

③ $a_n = a_n^{(h)} + a_n^{(p)}$ is the general solution of (1) (or (2))

Ex 10.26: $\begin{cases} a_n - 3a_{n-1} = 5(7^n), n \geq 1 \\ a_0 = 2 \end{cases}$

Sol. ① $a_n^{(h)} - 3a_{n-1}^{(h)} = 0 \Rightarrow a_n^{(h)} = c(3^n)$

② $\because f(n) = 5(7^n)$

Let $a_n^{(p)} = A(7^n) \Rightarrow A(7^n) - 3A(7^{n-1}) = 5(7^n), n \geq 1$

$\Rightarrow 7A - 3A = 5 \cdot 7 = 35 \Rightarrow 4A = 35 \Rightarrow A = 35/4$

$\therefore a_n^{(p)} = (35/4) \cdot 7^n = (5/4) \cdot 7^{n+1}$

③ $a_n = c(3^n) + (5/4) \cdot 7^{n+1}$

$\because a_0 = 2 = c + (35/4) \Rightarrow c = -27/4$

$\therefore a_n = (5/4) \cdot 7^{n+1} - (1/4) \cdot 3^{n+3}, n \geq 0$

§ 10.3 The Nonhomogeneous Recurrence Relation

<u>Ex 10.27</u>: $\begin{cases} a_n - 3a_{n-1} = 5(3^n), n \geq 1 \\ a_0 = 2 \end{cases}$

Sol.

① $a_n^{(h)} = c(3^n)$

② $\because f(n) = 5(3^n)$ and $a_n^{(h)}$ are not linearly independent

$\therefore$ let $a_n^{(p)} = B \cdot n(3^n)$

$\Rightarrow B \cdot n(3^n) - 3B(n-1)(3^{n-1}) = 5(3^n)$

$\Rightarrow Bn - B(n-1) = 5$

$\Rightarrow B = 5$

$\therefore a_n^{(p)} = 5n \cdot 3^n$

③ $a_n = c(3^n) + 5n \cdot 3^n = (c + 5n) \cdot 3^n$

$a_0 = 2 = c$

$\therefore a_n = (2 + 5n)3^n, n \geq 0$

§ 10.3 The Nonhomogeneous Recurrence Relation

Def:

I. $a_n + c_{n-1}a_{n-1} = kr^n,\ n \in \mathbb{Z}^+,$

$$a_n^{(p)} = \begin{cases} A \cdot r^n, A \text{ is a constant, if } r^n \text{ is \underline{not} a solution of } a_n + c_{n-1}a_{n-1} = 0; \\ B \cdot nr^n, B \text{ is a constant, otherwise.} \end{cases}$$

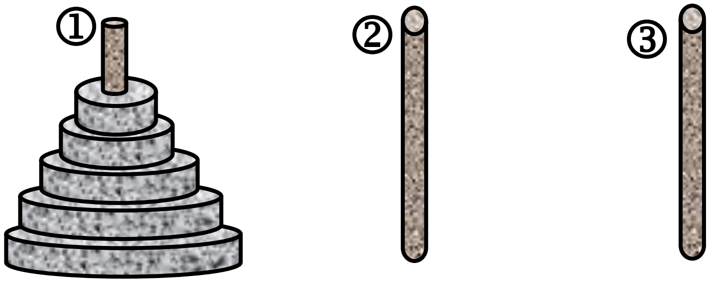
II. $a_n + c_{n-1}a_{n-1} + c_{n-2}a_{n-2} = kr^n$, k is a constant.

$$a_n^{(p)} = \begin{cases} A \cdot r^n, A \text{ is a constant, if } r^n \text{ is \underline{not} a solution of } a_n + c_{n-1}a_{n-1} + c_{n-2}a_{n-2} = 0; \\ B \cdot nr^n, B \text{ is a constant, if } a_n^{(h)} = c_1 r^n + c_2 r_1^n, \text{ where } r_1 \neq r; \\ C \cdot n^2 r^n, C \text{ is a constant, if } a_n^{(h)} = (c_1 + c_2 n) r^n. \end{cases}$$

§ 10.3 The Nonhomogeneous Recurrence Relation

Ex 10.28: **The Towers of Hanoi.**

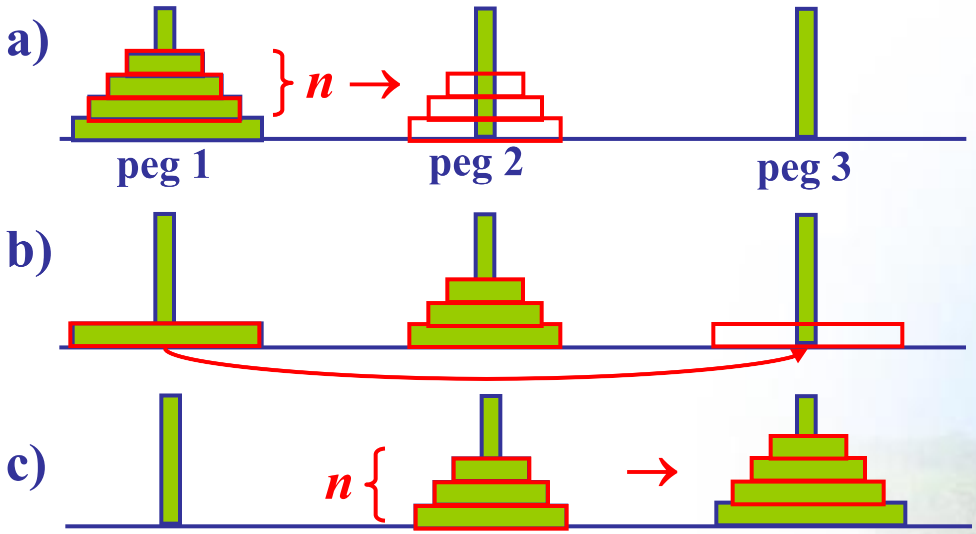
peg 1 → peg 3



Sol. (1/2)

For $n \geq 0$, let $a_n \equiv$ the minimum number of moves it takes to transfer n disks from peg 1 to peg 3 in the manner described.

then for $n + 1$ disks:



a) : need a_n steps

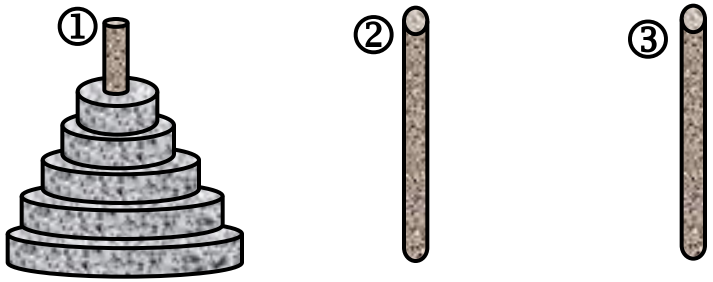
b) : need 1 step

c) : need a_n steps

§ 10.3 The Nonhomogeneous Recurrence Relation

Ex 10.28: **The Towers of Hanoi.**

peg 1 → peg 3



Sol. (2/2)

$\Rightarrow a_{n+1} = 2a_n + 1, n \geq 1; a_0 = 0$

① $a_{n+1}^{(h)} - 2a_n^{(h)} = 0 \Rightarrow a_n^{(h)} = c(2^n)$

② $\because f(n) = 1 \neq$ the solution of $a_{n+1} - 2a_n = 0$

$\therefore$ Let $a_n^{(p)} = A(1^n) = A$

$\Rightarrow A - 2A = 1 \Rightarrow A = -1$

$\Rightarrow a_n^{(p)} = -1$

③ $\therefore a_n = c(2^n) - 1$

$\because a_0 = 0 = c - 1$

$\therefore c = 1$

$\Rightarrow a_n = 2^n - 1, n \geq 0$

§ 10.3 The Nonhomogeneous Recurrence Relation

Ex 10.29: loan: S, be paid back: T period time.

the interest rate per period: r

$\Rightarrow$ payment at the end of each period: P = ?

Sol. (1/2)

Let a_n = owed after the nth payment.

$$\Rightarrow \begin{cases} a_{n+1} = a_n + ra_n - P,\ 0 \le n \le T-1 \\ a_0 = S;\ a_T = 0. \end{cases}$$

① $a_{n+1}^{(h)} - (1+r)a_n^{(h)} = 0 \Rightarrow a_n^{(h)} = c(1+r)^n$

② Let $a_n^{(p)} = A \Rightarrow A = A + r \cdot A - P \Rightarrow r \cdot A = P \Rightarrow A = P/r$

i.e. $a_n^{(p)} = P/r$

③ $a_n = c(1+r)^n + P/r$

$\because a_0 = S = c + P/r \Rightarrow c = S - P/r$

$\therefore a_n = (S - P/r)(1+r)^n + P/r,\ 0 \le n \le T$

$\because 0 = a_T = (S - P/r)(1+r)^T + P/r$

$\Rightarrow P/r = (P/r - S)(1+r)^T$

§ 10.3 The Nonhomogeneous Recurrence Relation

Ex 10.29: loan: S be paid back: T period time.

the interest rate per period: r

$\Rightarrow$ payment at the end of each period: $P = ?$

Sol. (2/2)

$\because 0 = a_T = (S - P/r)(1 + r)^T + P/r$

$\Rightarrow P/r = (P/r - S)(1 + r)^T$

$\therefore P = (P - rS)(1 + r)^T \Rightarrow P[1 - (1 + r)^T] = -Sr(1 + r)^T$

$\Rightarrow P = -Sr(1 + r)^T[1 - (1 + r)^T]^{-1}$

$= Sr[-(1 + r)^{-T}]^{-1}[1 - (1 + r)^T]^{-1}$

$= Sr[1 - (1 + r)^{-T}]^{-1}$

$$\left(\frac{-(1+r)^T}{1-(1+r)^T} = \frac{1}{1-\frac{1}{(1+r)^T}}\right)$$

$S = 1{,}000{,}000$ $r = 1.5\%/12$ $T = 20 * 12 \rightarrow P = 4825.454088819525$

$S = 1{,}000{,}000$ $r = 3\%/12$ $T = 20 * 12 \rightarrow P = 5545.97597853912$

$S = 1{,}000{,}000$ $r = 8\%/12$ $T = 20 * 12 \rightarrow P = 8364.400689934629$