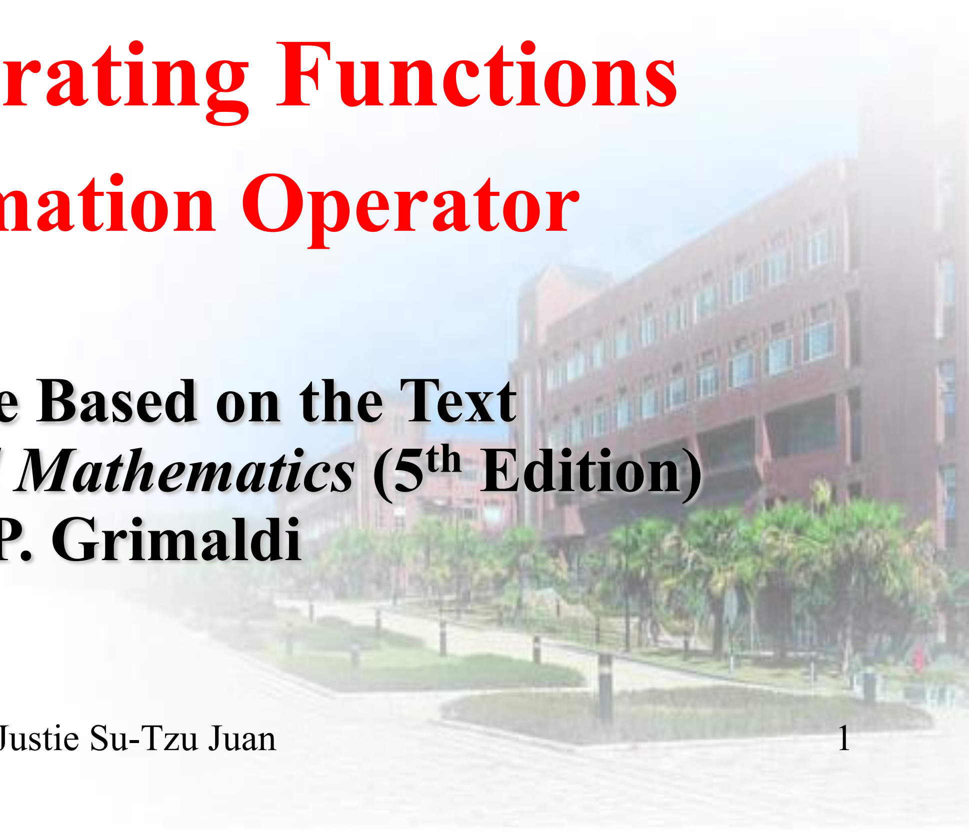
Computer Science and Information Engineering
National Chi Nan University

Combinatorial Mathematics

Dr. Justie Su-Tzu Juan

Chapter 9 Generating Functions

§ 9.5 The Summation Operator



Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

§ 9.5 The Summation Operator

Outline

1. **Technique: The convolution of the sequence $a_0, a_1, a_2,$ and the sequence 1, 1, 1, … is the sequence**

$$a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, \ldots$$

§ 9.5 The Summation Operator

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$$

$$g(x) = 1 + x + x^2 + x^3 + \ldots = 1/(1 - x)$$

$$\Rightarrow f(x) \cdot g(x) = f(x)/(1 - x) = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + \ldots$$

$\therefore f(x)/(1 - x)$ **generates the sequence of sums**

$$a_0,\ a_0 + a_1,\ a_0 + a_1 + a_2,\ a_0 + a_1 + a_2 + a_3,\ \ldots$$

= The convolution of the sequence $a_0, a_1, a_2,$ and the sequence $b_0, b_1, b_2, \ldots$ where $b_i = 1\ \forall\ n \in \mathbb{N}$.

§ 9.5 The Summation Operator

Ex 9.30: (a) ∵ $1/(1-x)$ is the G.F. for the seq. 1, 1, 1, …

∴ $(1/(1-x))/(1-x)$ is the G.F. for the seq. 1, 1+1, 1+1+1, …

i.e. $1/(1-x)^2$ is the G.F. for the seq. 1, 2, 3, …

(b) ∵ $x + x^2$ is the G.F. for the seq. 0, 1, 1, 0, 0, 0, …

∴ $(x + x^2)/(1-x)$ is the G.F. for the seq. 0, 1, 2, 2, 2, 2,…

⇒ $(x + x^2)/(1-x)^2$ is the G.F. for the seq. 0, 1, 3, 5, 7, 9,…

⇒ $(x + x^2)/(1-x)^3$ is the G.F. for the seq. 0, 1, 4, 9, 16, 25,…

$\sum_{k=1}^{n}(2k-1) = n^2$?

to verify this, look at the coefficient of x^n in $(x + x^2)/(1-x)^3$

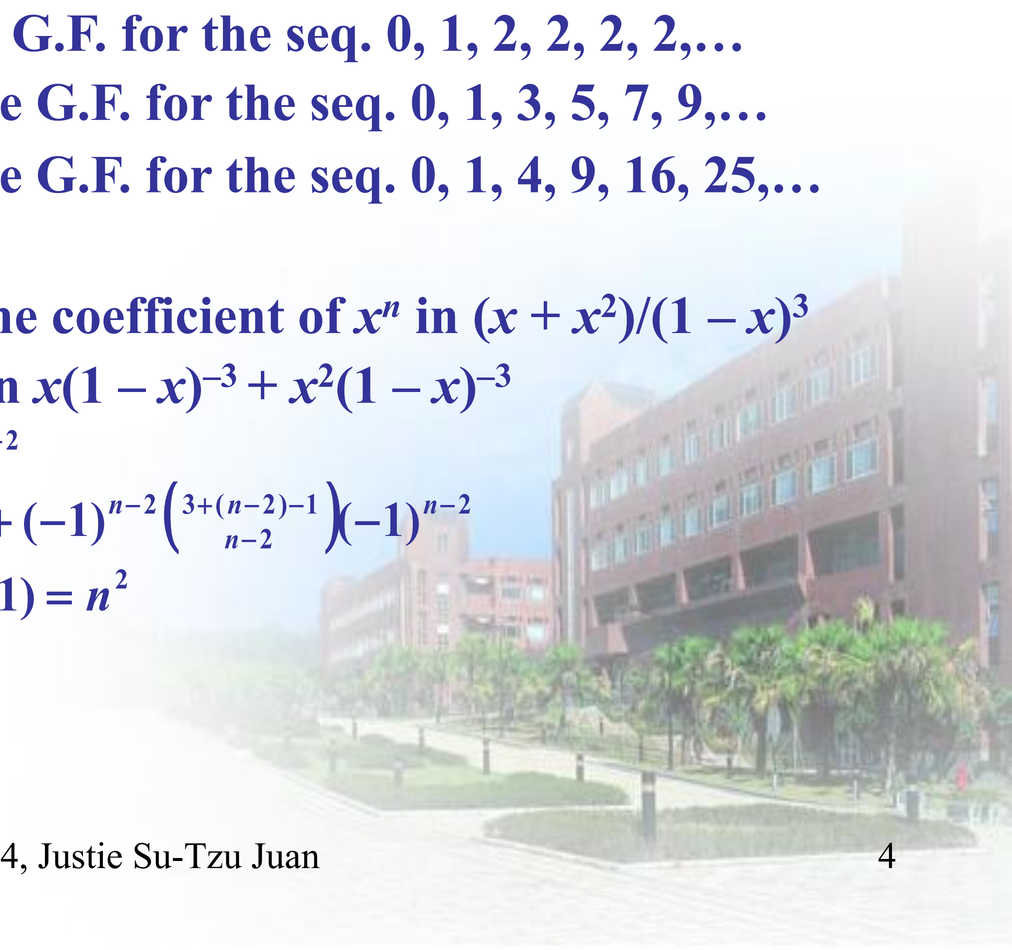
= the coefficient of x^n in $x(1-x)^{-3} + x^2(1-x)^{-3}$

$$= \binom{-3}{n-1}(-1)^{n-1} + \binom{-3}{n-2}(-1)^{n-2}$$

$$= (-1)^{n-1}\binom{3+(n-1)-1}{n-1}(-1)^{n-1} + (-1)^{n-2}\binom{3+(n-2)-1}{n-2}(-1)^{n-2}$$

$$= \frac{1}{2}(n+1)(n) + \frac{1}{2}(n)(n-1) = n^2$$

(as Example 4.7)



§ 9.5 The Summation Operator

Ex 9.31: $0^2 + 1^2 + 2^2 + 3^2 + \ldots + n^2 = ?$

Sol. (1/2)

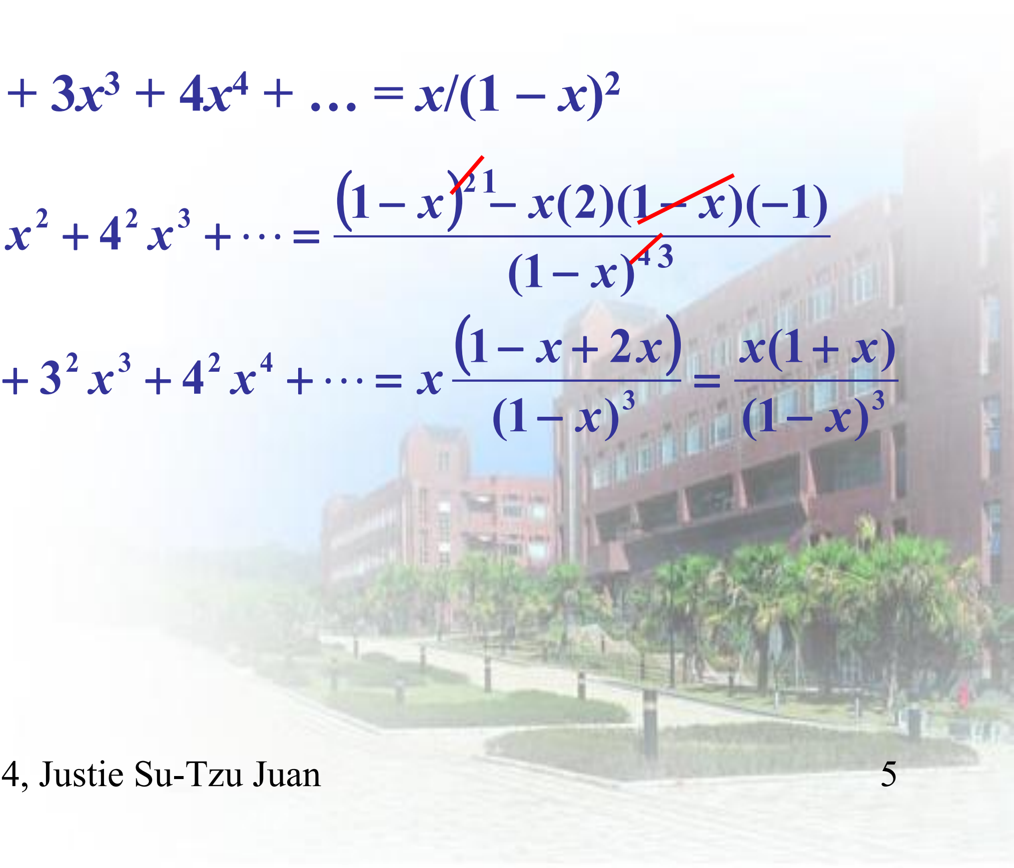
Let $g(x) = 1/(1 - x) = 1 + x + x^2 + x^3 + \ldots$

$$\frac{dg(x)}{dx} = 1 + 2x + 3x^2 + 4x^3 + \ldots = 1/(1 - x)^2$$

$$x \cdot \frac{dg(x)}{dx} = 0 + x + 2x^2 + 3x^3 + 4x^4 + \ldots = x/(1 - x)^2$$

$$\frac{d}{dx}\left[x\left(\frac{dg(x)}{dx}\right)\right] = 1 + 2^2x^1 + 3^2x^2 + 4^2x^3 + \cdots = \frac{(1-x)^{\cancel{2}1} - x(2)(\cancel{1-x})(-1)}{(1-x)^{\cancel{4}3}}$$

$$x \cdot \frac{d}{dx}\left[x\left(\frac{dg(x)}{dx}\right)\right] = 0 + x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \cdots = x\frac{(1-x+2x)}{(1-x)^3} = \frac{x(1+x)}{(1-x)^3}$$



§ 9.5 The Summ

$$\boxed{x \cdot \frac{d}{dx}\left[x\left(\frac{dg(x)}{dx}\right)\right] = 0 + x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \cdots = \frac{x(1+x)}{(1-x)^3}}$$

Ex 9.31: $0^2 + 1^2 + 2^2 + 3^2 + \ldots + n^2 = ?$

Sol. (2/2)

$\therefore \dfrac{x(1+x)}{(1-x)^3}$ is the G.F. of $0^2, 1^2, 2^2, 3^2, \ldots$

$\Rightarrow \dfrac{x(1+x)}{(1-x)^3}\dfrac{1}{(1-x)} = \dfrac{x(1+x)}{(1-x)^4}$ is the G.F. for $0^2, 0^2 + 1^2, 0^2 + 1^2 + 2^2, \ldots$

$\therefore$ the coefficient of x^n in $\dfrac{x(1+x)}{(1-x)^4}$ is $\displaystyle\sum_{i=0}^{n} i^2$

$$\frac{x(1+x)}{(1-x)^4} = (x + x^2)(1 - x)^{-4} = (x + x^2)\left[\binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + \ldots\right]$$

$\therefore$ the coefficient of $x^n = \binom{-4}{n-1}(-1)^{n-1} + \binom{-4}{n-2}(-1)^{n-2}$

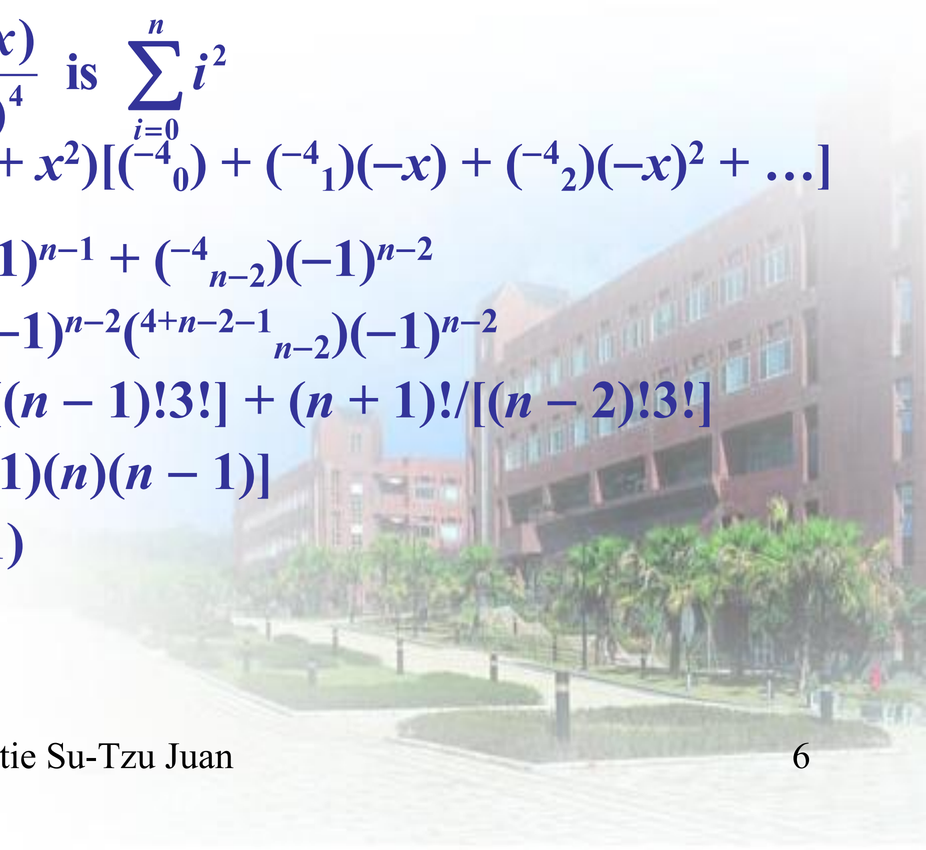
$= (-1)^{n-1}\binom{4+n-1-1}{n-1}(-1)^{n-1} + (-1)^{n-2}\binom{4+n-2-1}{n-2}(-1)^{n-2}$

$= \binom{n+2}{n-1} + \binom{n+1}{n-2} = (n + 2)!/[(n - 1)!3!] + (n + 1)!/[(n - 2)!3!]$

$= (1/6)[(n + 2)(n + 1)n + (n + 1)(n)(n - 1)]$

$= (1/6)(n)(n + 1)(n + 2 + n - 1)$

$= (1/6)n(n + 1)(2n + 1)$



§ 9.5 The Summation Operator

Checklist:

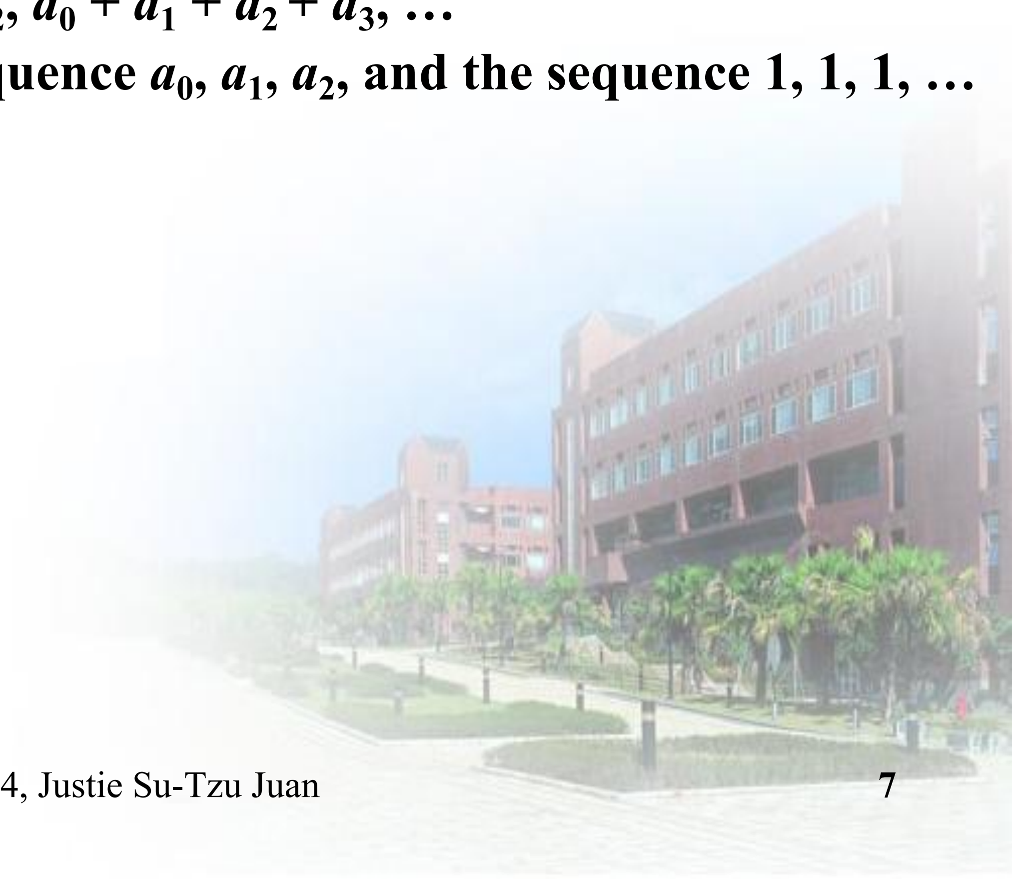
1. Technique:

☐ $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$

$f(x)/(1 - x)$ **generates the sequence of sums**

$$a_0,\ a_0 + a_1,\ a_0 + a_1 + a_2,\ a_0 + a_1 + a_2 + a_3,\ \dots$$

= The convolution of the sequence $a_0, a_1, a_2,$ **and the sequence 1, 1, 1, …**

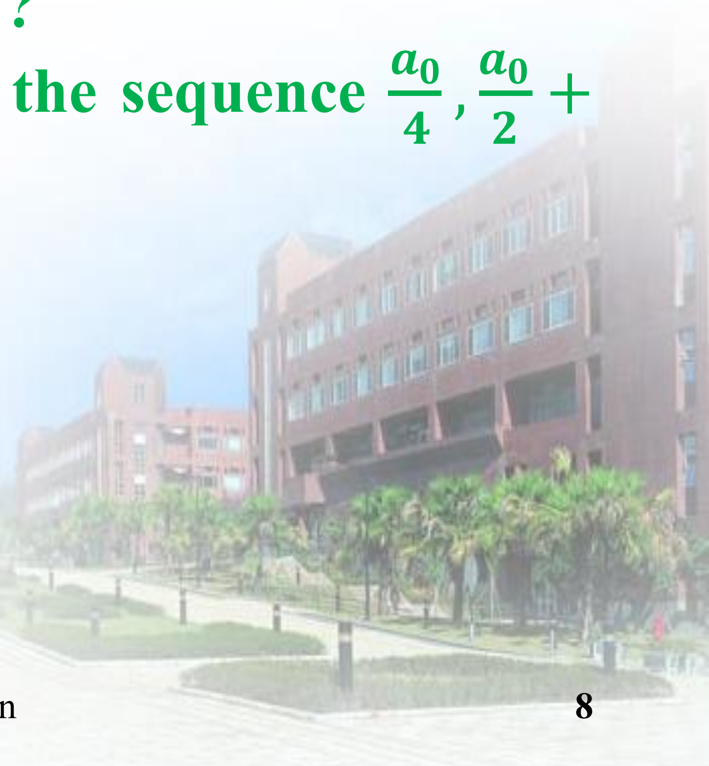


§ 9.5 The Summation Operator

Discussion (10 min):

Exercise 9.5.4: If $f(x) = \Sigma_{n=0}^{\infty} a_n x^n$,

1) **what is the generation function for the sequence a_0, $a_0 + a_1$, $a_1 + a_2$, $a_2 + a_3$, ... ?**
2) **What is the generating function for the sequence a_0, $a_0 + a_1$, $a_0 + a_1 + a_2$, $a_1 + a_2 + a_3$, $a_2 + a_3 + a_4$, ... ?**
3) **What is the generation function for the sequence $\frac{a_0}{4}, \frac{a_0}{2} + \frac{a_1}{4}, \frac{a_0}{4} + \frac{a_1}{2} + \frac{a_2}{4}, \frac{a_1}{4} + \frac{a_2}{2} + \frac{a_3}{4}, \ldots$**



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Chapter 10 Recurrence Relations

§ 10.1 The First-Order Linear Recurrence Relation

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§ 10.1 The First-Order Linear Recurrence Relation

Outline

1. **Definition**
2. **Formula**
3. **Application**
4. **Special Case**

§ 10.1 The First-Order Linear Recurrence Relation

Def: "Function $a(n)$, preferably written as a_n (for $n \geq 0$), where a_n depends on some of the prior terms $a_{n-1}, a_{n-2}, \ldots, a_1, a_0$." This study of what are called either recurrence relations or difference equations.

Def: A geometric progression is an infinite sequence of numbers $a_0 = a$, $a_1 = ar$, $a_2 = ar^2$, $a_3 = ar^3$, …; where r is called the common ratio.

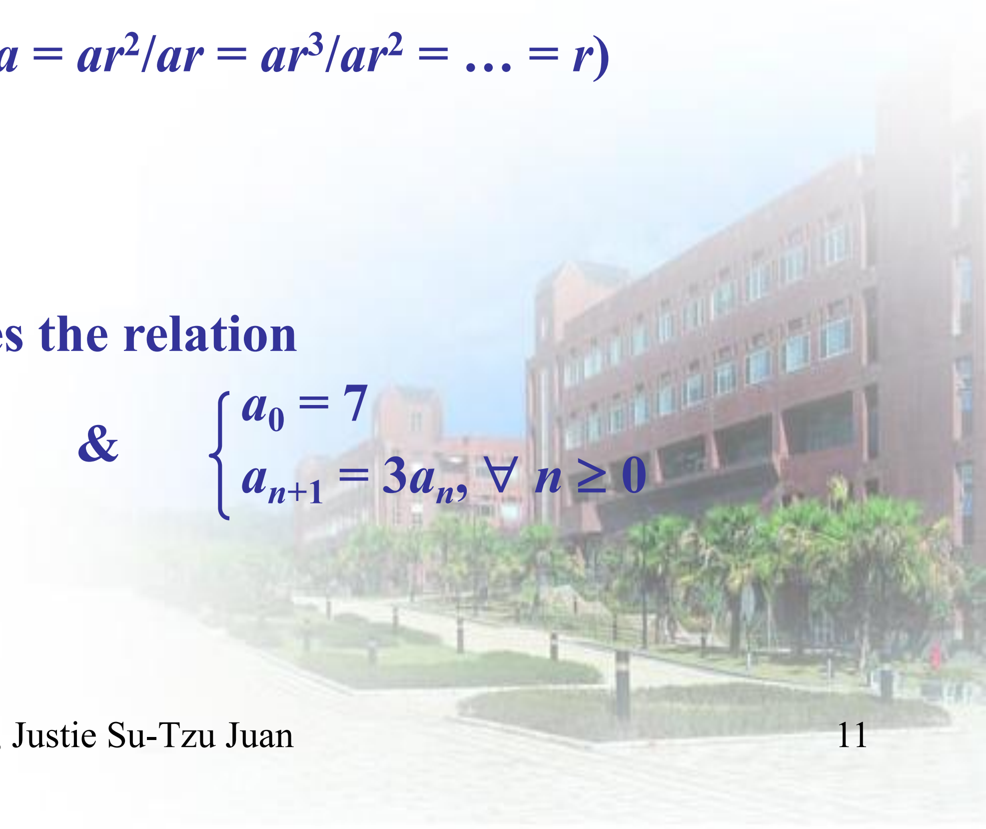
$a_1/a_0 = a_2/a_1 = a_3/a_2 = \ldots = (ar/a = ar^2/ar = ar^3/ar^2 = \ldots = r)$

ex: 5, 15, 45, 135, ….

$a_{n+1} = 3a_n, n \geq 0$

but 7, 21, 63, 189, … also satisfies the relation

$$\therefore \quad \begin{cases} a_0 = 5 \\ a_{n+1} = 3a_n, \ \forall\ n \geq 0 \end{cases} \quad \& \quad \begin{cases} a_0 = 7 \\ a_{n+1} = 3a_n, \ \forall\ n \geq 0 \end{cases}$$



§ 10.1 The First-Order Linear Recurrence Relation

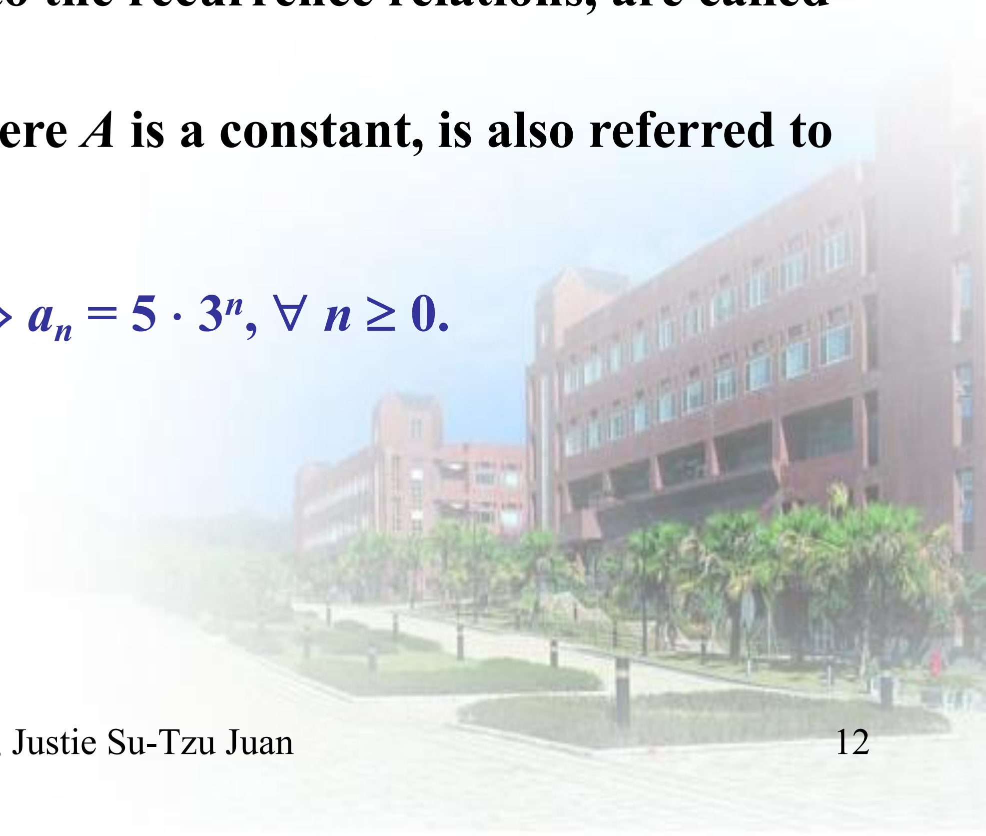
Def: ① A relation is said to be a **first order** ≡

a_{n+1} depends only on its immediate predecessor.

② In particular, a **first-order linear homogeneous** recurrence relation **with constant coefficients**:

1. The general form : $a_{n+1} = da_n$, $n \geq 0$, where d is a constant.
2. a_0 or a_1 given in addition to the recurrence relations, are called **boundary conditions**.
3. The expression $a_0 = A$, where A is a constant, is also referred to as an **initial condition**.

ex: $\begin{cases} a_0 = 5 \\ a_{n+1} = 3a_n, \; n \geq 0. \end{cases}$ $\Rightarrow a_n = 5 \cdot 3^n, \forall \; n \geq 0.$



§ 10.1 The First-Order Linear Recurrence Relation

Result: The general solution of the recurrence relation

$$\begin{cases} a_{n+1} = d \cdot a_n, \text{ where } n \geq 0,\ d \text{ is a constant and} \\ a_0 = A \end{cases}$$

is unique and is given by $a_n = A \cdot d^n$, $n \geq 0$.

Ex 10.1: $\begin{cases} a_2 = 98 \\ a_n = 7a_{n-1}, \text{ where } n \geq 1. \end{cases}$

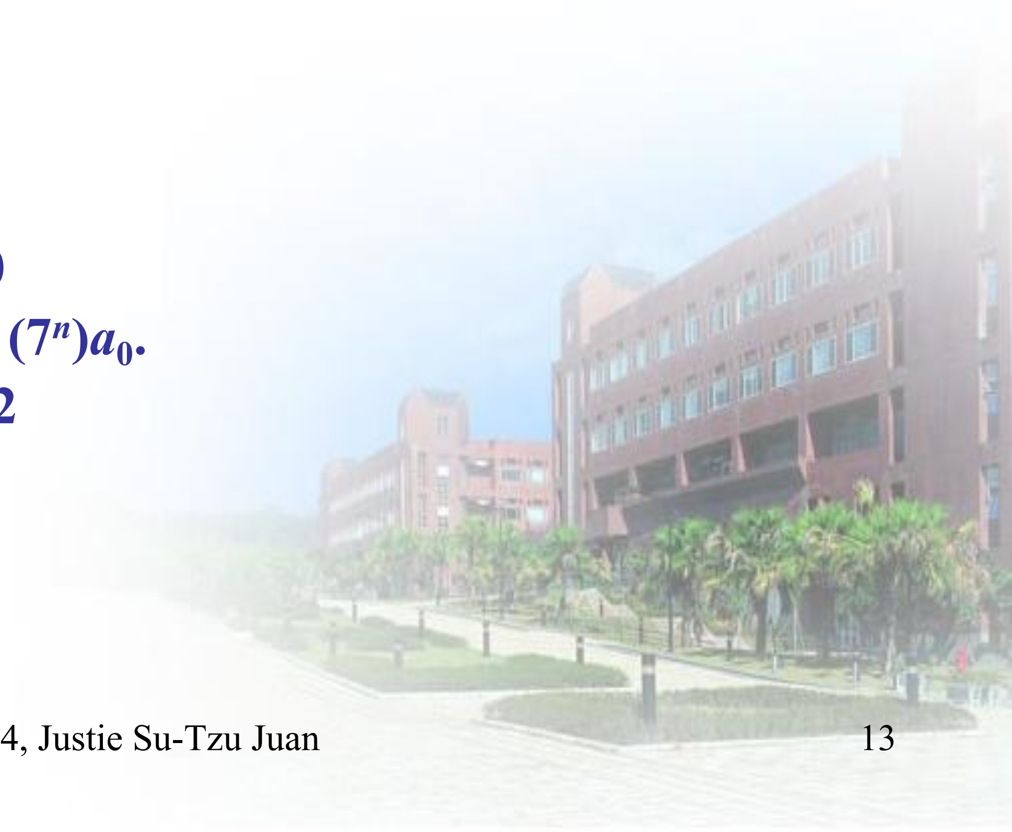
Sol.

$$\begin{cases} a_2 = 98 \\ a_{n+1} = 7a_n, \text{ where } n \geq 0 \end{cases}$$

$\therefore$ The general solution: $a_n = (7^n)a_0$.

$\because a_2 = 98 = a_0 \cdot (7^2), \Rightarrow a_0 = 2$

$\therefore a_n = 2(7^n), \forall\ n \geq 0$



§ 10.1 The First-Order Linear Recurrence Relation

Ex 10.2: Bonnie deposits \$1000; 6%/年; 每月複利; after 1 year?

Sol.

$6\%/12 = 0.5\% = 0.005$

$\forall\ 0 \le n \le 12$, let p_n denote the value of Bonnie's deposit at the end n months.

$\therefore\ p_{n+1} = p_n + 0.005p_n,\ \forall\ 0 \le n \le 11,\ p_0 = \$1000.$

↳ the interest earned on p_n during month $n + 1$.

$$\Rightarrow \begin{cases} p_{n+1} = p_n(1.005) \\ p_0 = 1000 \end{cases}$$

$\Rightarrow p_n = p_0(1.005)^n = 1000 \cdot (1.005)^n$

$\therefore$ one year $\Rightarrow 1000(1.005)^{12} = 1061.68$

§ 10.1 The First-Order Linear Recurrence Relation

Ex 10.3: The compositions of *n*:

$$\begin{cases} a_{n+1} = 2a_n, n \geq 1 \\ a_1 = 1 \end{cases}$$

let $b_n = a_{n+1}$

$$\Rightarrow \begin{cases} b_{n+1} = 2b_n, n \geq 0 \\ b_0 = 1 \end{cases}$$

so, $b_n = b_0(2^n) = 2^n$

$\Rightarrow a_n = b_{n-1} = 2^{n-1}$

(1) 3 (2) 1 + 2 (3) 2 + 1 (4) 1 + 1 + 1	(1') 4 (2') 1 + 3 (3') 2 + 2 (4') 1 + 1 + 2 (1'') 3 + 1 (2'') 1 + 2 + 1 (3'') 2 + 1 + 1 (4'') 1 + 1 + 1 + 1

§ 10.1 The First-Order Linear Recurrence Relation

Def: A recurrence relation is called linear

≡ each subscripted term appears to the first power and no product such as $a_n a_{n-1}$.

Substitution:

Ex 10.4: Find a_{12} if $a_{n+1}^2 = 5\, a_n^2$, where $a_n > 0$ for $n \geq 0$, and $a_0 = 2$.

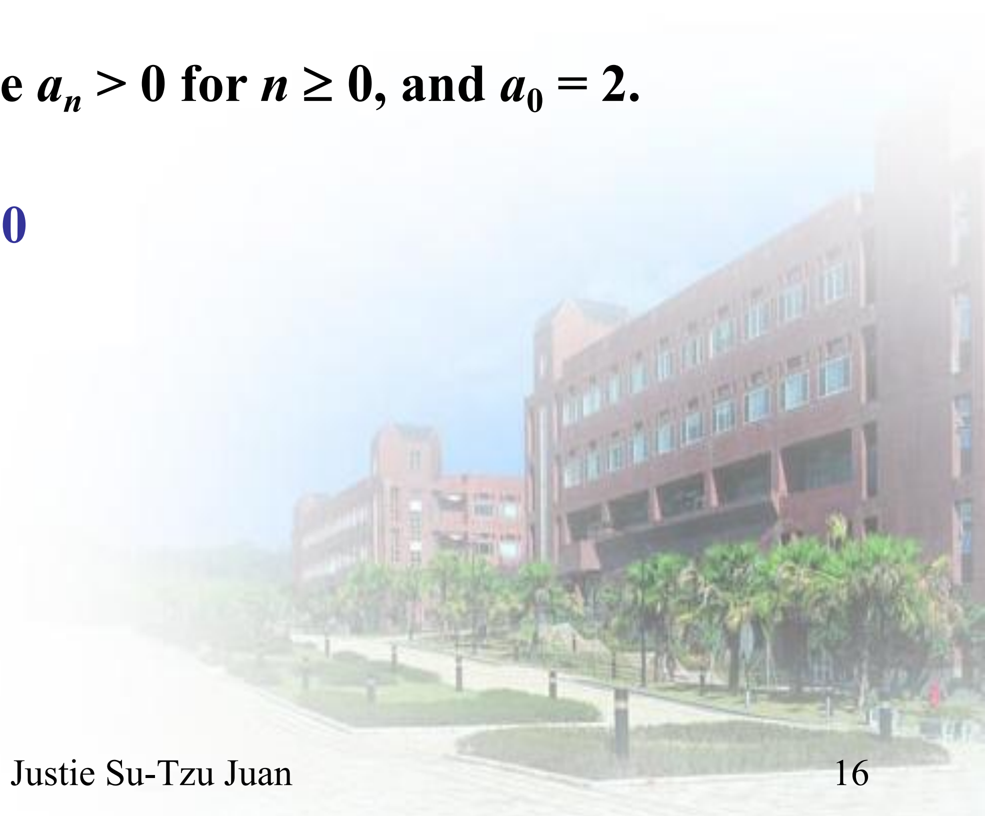
Sol.

$$\text{Let } b_n = a_n^2 \Rightarrow \begin{cases} b_{n+1} = 5b_n, \ \forall\ n \geq 0 \\ b_0 = 4 \end{cases}$$

$$\Rightarrow b_n = 4 \cdot 5^n$$

$$\Rightarrow a_n = 2(\sqrt{5})^n, \ \forall\ n \geq 0$$

$$\therefore a_{12} = 2 \cdot 5^6 = 31250$$



§ 10.1 The First-Order Linear Recurrence Relation

Def: The general first-order linear recurrence relation with constant coefficients: $a_{n+1} + c \cdot a_n = f(n)$, $n \geq 0$, where c is a constant and $f(n)$ is a function on the set $\mathbb{N}$ of nonnegative integers.

- where $f(n) = 0$, $\forall\ n \in \mathbb{N}$, the relation is called **homogeneous** (齊次);
- otherwise it is called **nonhomogeneous**.

§ 10.1 The First-Order Linear Recurrence Relation

Ex 10.5: **bubble sort**: input a positive integer n and an array $x_1, x_2, \ldots, x_n$ of real numbers that are to be sorted into ascending order. The time-complexity = ?

Sol. (1/3)

```
Procedure Bubble Sort ($n$: positive integer;
                       $x_1, x_2, x_3, \ldots, x_n$: real numbers)
  begin
    for $i := 1$ to $n - 1$ do
      for $j := n$ downto $i + 1$ do
        if $x_j < x_{j-1}$ then
          begin                    {interchange}
            temp := $x_{j-1}$
            $x_{j-1} := x_j$
            $x_j$ := temp
          end
  end
```

§ 10.1 The First-Order Linear Recurrence Relation

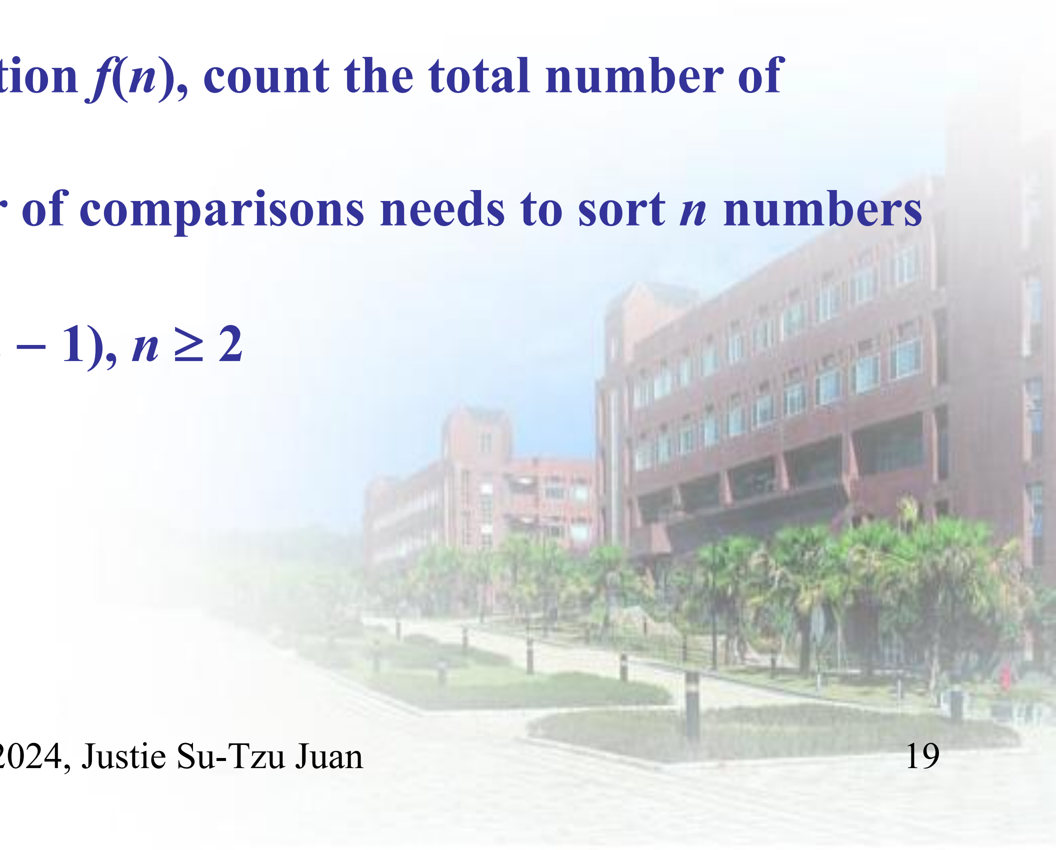
Ex 10.5: bubble sort: input a positive integer n and an array $x_1, x_2, \ldots, x_n$ of real numbers that are to be sorted into ascending order.
The time-complexity = ?

Sol. (2/3)

See p-451 example: $x_1 = 7, x_2 = 9, x_3 = 2, x_4 = 5, x_5 = 8, n = 5$.

The time-complexity function $f(n)$, count the total number of comparisons.
Let a_n denotes the number of comparisons needs to sort n numbers in this way, then:

$$\begin{cases} a_n = a_{n-1} + (n-1),\ n \geq 2 \\ a_1 = 0 \end{cases}$$



§ 10.1 The First-Order Linear Recurrence Relation

$$\begin{cases} a_n = a_{n-1} + (n-1), \ n \geq 2 \\ a_1 = 0 \end{cases}$$

Sol. (3/3)

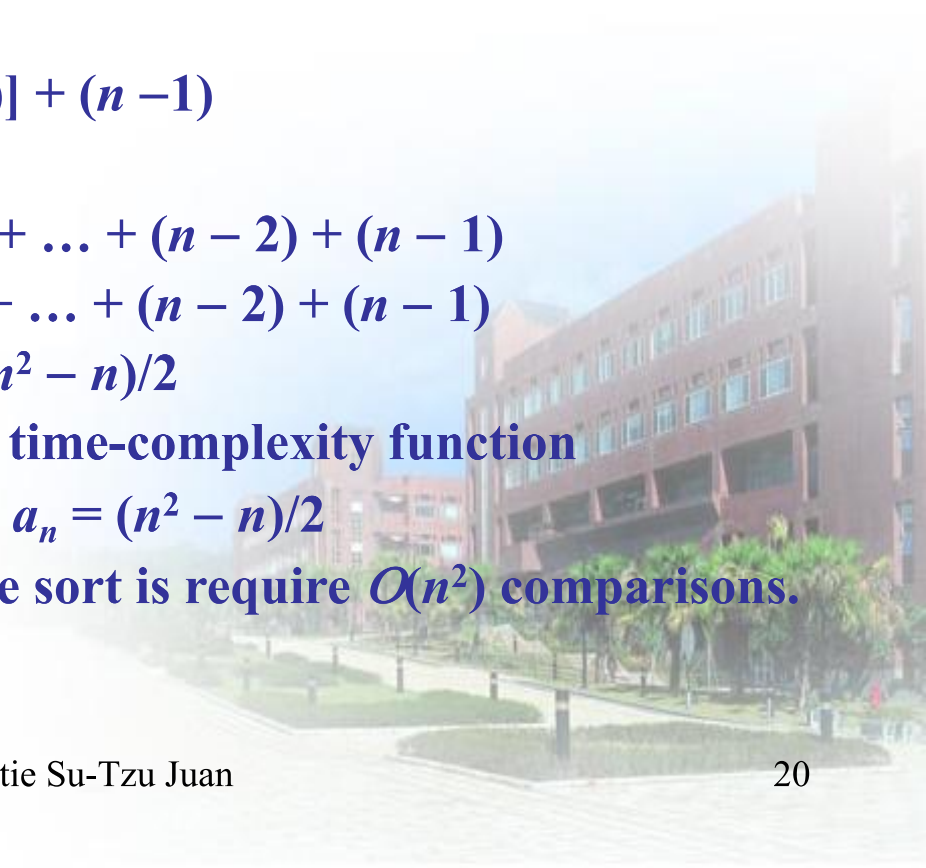
$$\begin{cases} a_1 = 0 \\ a_2 = a_1 + (2-1) = 1 \\ a_3 = a_2 + (3-1) = 1 + 2 \\ a_4 = a_3 + (4-1) = 1 + 2 + 3 \\ \qquad \vdots \\ a_n = a_{n-1} + (n-1) \end{cases}$$

$$\begin{aligned} &= [a_{n-2} + (n-2)] + (n-1) \\ &= \ldots \\ &= a_1 + 1 + 2 + 3 + \ldots + (n-2) + (n-1) \\ &= 0 + 1 + 2 + 3 + \ldots + (n-2) + (n-1) \\ &= n(n-1)/2 = (n^2 - n)/2 \end{aligned}$$

∴ the bubble sort determine the time-complexity function

$f: \mathbb{Z}^+ \to \mathbb{R}$ given by $f(n) = a_n = (n^2 - n)/2$

we write $f(n) = O(n^2)$: the bubble sort is require $O(n^2)$ comparisons.



§ 10.1 The First-Order Linear Recurrence Relation

補充: the complexity of an algorithm exactly: $f(n)$: $\mathbb{N} \to \mathbb{R}^+$, $g(n)$: $\mathbb{N} \to \mathbb{R}^+$

① $f(n) = O(g(n))$: $\exists\ c > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t. $f(n) \leq c \cdot g(n)$, $\forall\ n \geq n_0$.

② $f(n) = \Omega(g(n))$: $\exists\ c > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t. $f(n) \geq c \cdot g(n)$, $\forall\ n \geq n_0$.

③ $f(n) = \Theta(g(n))$: $f(n) = O(g(n)) = \Omega(g(n))$.

§ 10.1 The First-Order Linear Recurrence Relation

Ex 10.6: $a_0 = 0, a_1 = 2, a_2 = 6, a_3 = 12, a_4 = 20, a_5 = 30, a_6 = 42, \ldots, a_n = ?$

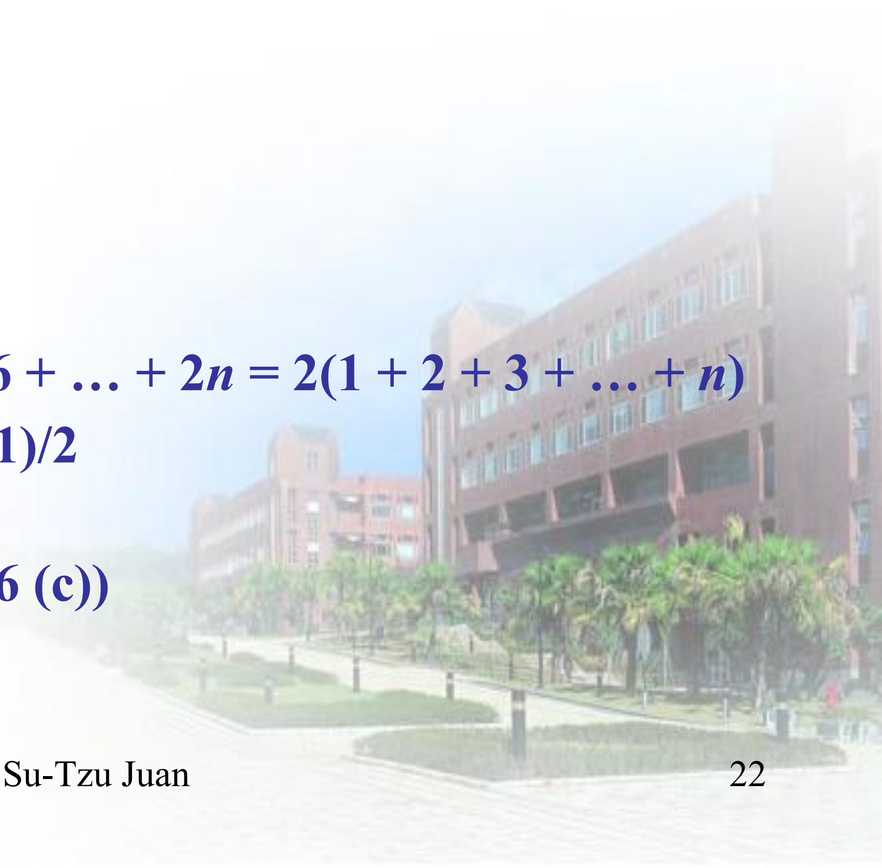
$$\begin{array}{lll} a_1 - a_0 = 2 & a_3 - a_2 = 6 & a_5 - a_4 = 10 \\ a_2 - a_1 = 4 & a_4 - a_3 = 8 & a_6 - a_5 = 12 \end{array}$$

$$\Rightarrow \begin{cases} a_n - a_{n-1} = 2n,\ n \geq 1, \\ a_0 = 0 \end{cases}$$

$$\begin{array}{rl} & \cancel{a_1} - a_0 = 2 \\ & \cancel{a_2} - \cancel{a_1} = 4 \\ & \cancel{a_3} - \cancel{a_2} = 6 \\ & \cancel{} \quad \vdots \\ +) & a_n - \cancel{a_{n-1}} = 2n \\ \hline \end{array}$$

$$\begin{aligned} a_n - a_0 &= 2 + 4 + 6 + \ldots + 2n = 2(1 + 2 + 3 + \ldots + n) \\ a_n - 0 &= 2n \cdot (n + 1)/2 \\ a_n &= n^2 + n \end{aligned}$$

$\therefore a_n = n^2 + n, \forall n \in \mathbb{N}$ (Ex9.6 (c))



§ 10.1 The First-Order Linear Recurrence Relation

Ex 10.7: a recurrence relation with a variable coefficient:

$$\begin{cases} a_n = n \cdot a_{n-1}, \text{ where } n \geq 1. \\ a_0 = 1 \end{cases}$$

Sol.

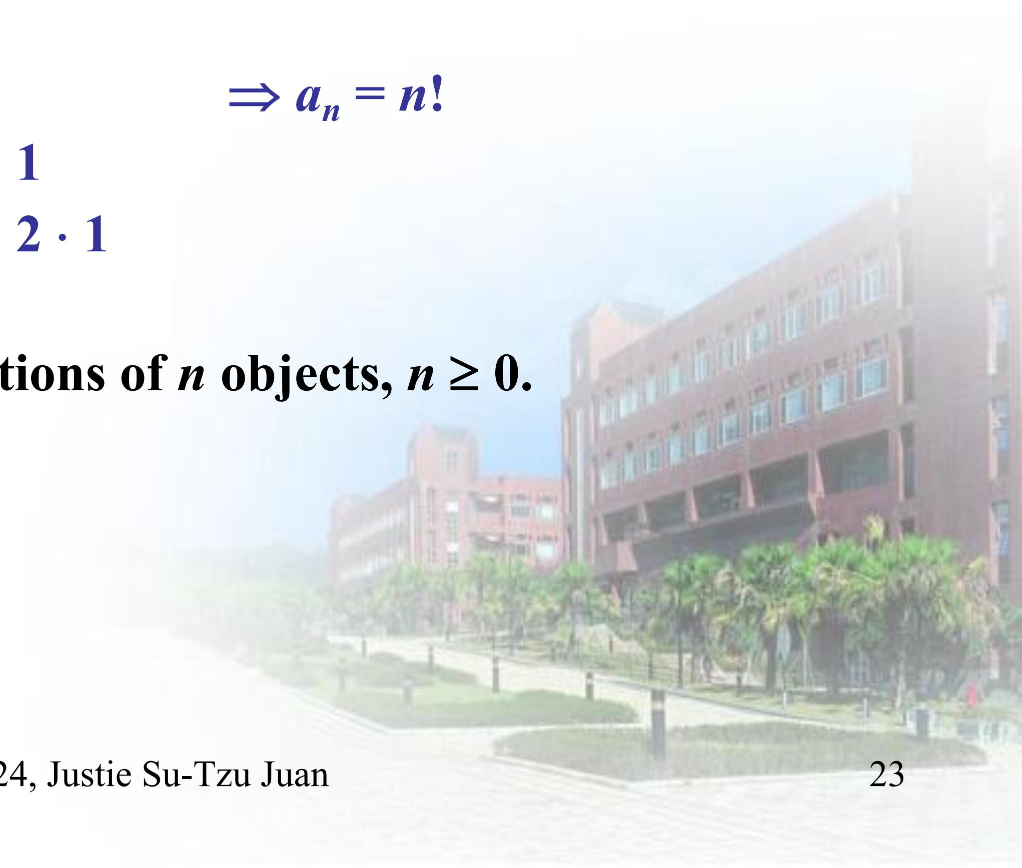
$$\begin{aligned} a_0 &= 1 \\ a_1 &= 1 \cdot a_0 = 1 \\ a_2 &= 2 \cdot a_1 = 2 \cdot 1 \qquad \Rightarrow a_n = n! \\ a_3 &= 3 \cdot a_2 = 3 \cdot 2 \cdot 1 \\ a_4 &= 4 \cdot a_3 = 4 \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

a_n (= n!) = the number of permutations of n objects, $n \geq 0$.

$a_1 = 1$: 1

$a_2 = 2 \cdot a_1$: 1 ②

② 1



§ 10.1 The First-Order Linear Recurrence Relation

$a_3 = 3 \cdot a_2$:

	1		2	3
	1	3	2	
3	1		2	
3	2		1	
	2	3	1	
	2		1	3

$a_4 = 4 \cdot a_3$:

	1		2		3	4
	1		2	4	3	
	1	4	2		3	
4	1		2		3	
4	1		3		2	
	1	4	3		2	
	1		3	4	2	
	1		3		2	4
	3		1		2	4
			:			
4	2		1		3	
	2	4	1		3	
	2		1	4	3	
	2		1		3	4

$\Rightarrow a_n = n \cdot a_{n-1} = n!$

§ 10.1 The First-Order Linear Recurrence Relation

Ex 10.8: recursive function and procedure: gcd(333, 84) = ? gcd(a, b) = ?

Sol. (1/2) $333 = 3(84) + 81,$

$84 = 1(81) + 3,$

$81 = 27(3) + 0.$

3	333	84	1
	252	81	
27	81	3	
	81		
	0		

$\Rightarrow$ By Euclidean algorithm (Section 4.4) (p-232)

gcd(333, 84) (= 3) = gcd(84, 81) = gcd(81, 3) = 3

$\Rightarrow$ gcd(333, 84) = gcd(84, 333 mod 84)

= gcd(333 mod 84, 84 mod (333 mod 84))

$\Rightarrow$ Idea:

Input $a, b \in \mathbb{Z}^+$

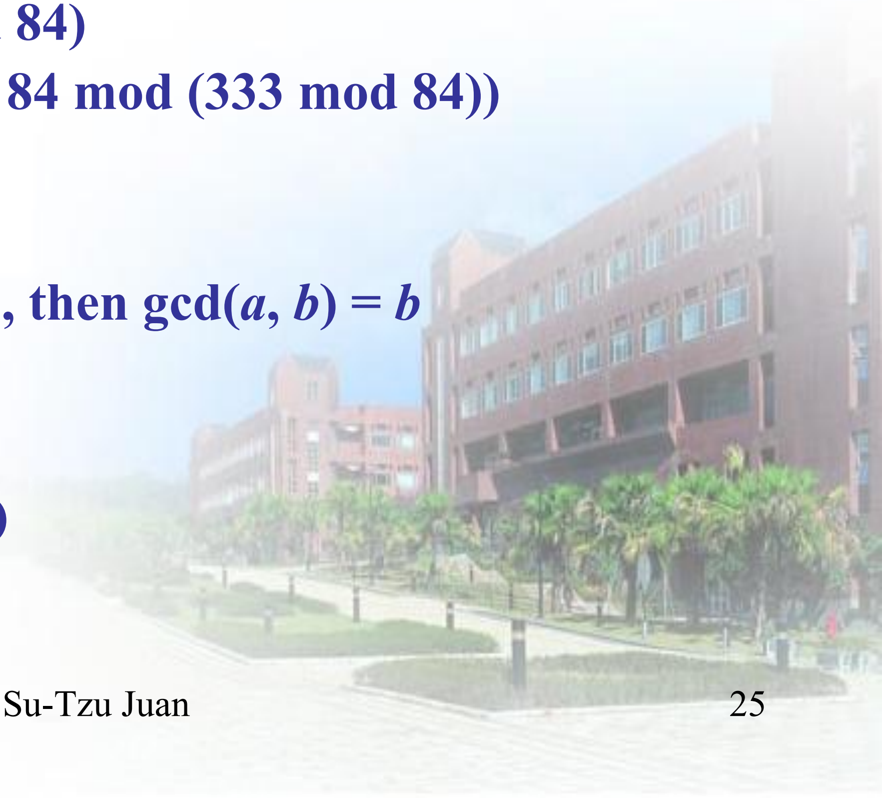
Step 1: If $b \mid a$ (or a mod $b = 0$), then gcd(a, b) = b

Step 2: If $b \nmid a$, then:

(i) set $a = b$

(ii) set $b = a$ mod b. (old a)

(iii) Return to Step 1.



§ 10.1 The First-Order

<u>Ex 10.8</u>: recursive function

Sol. (2/2)

Input $a, b \in \mathbb{Z}^+$

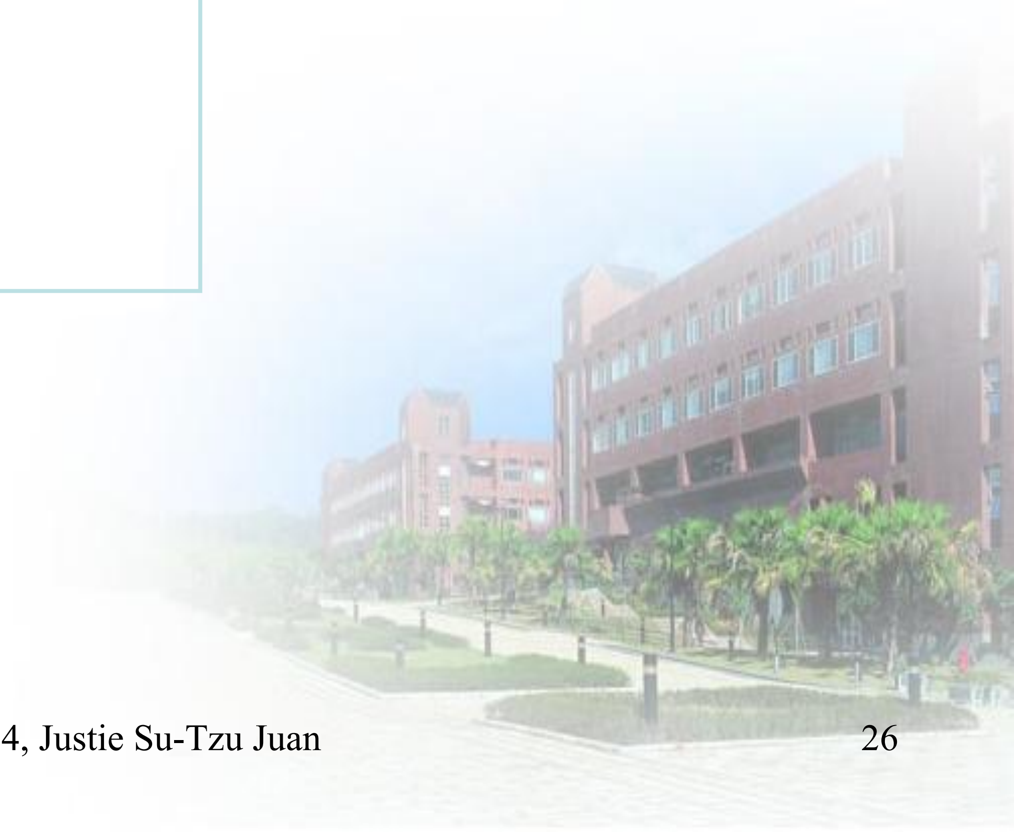
<u>Step 1</u>: If $b \mid a$ (or a mod $b = 0$), then gcd(a, b) = b

<u>Step 2</u>: If $b \nmid a$, then:

```
(i) set $a = b$
(ii) set $b = a$ mod $b$. (old $a$)
(iii) Return to Step 1.
```

```
⇒ Procedure gcd2($a$, $b$: positive integers)
begin
  if $a$ mod $b$ = 0 then
    gcd = $b$
  else
    gcd = gcd2($b$, $a$ mod $b$)
end
```

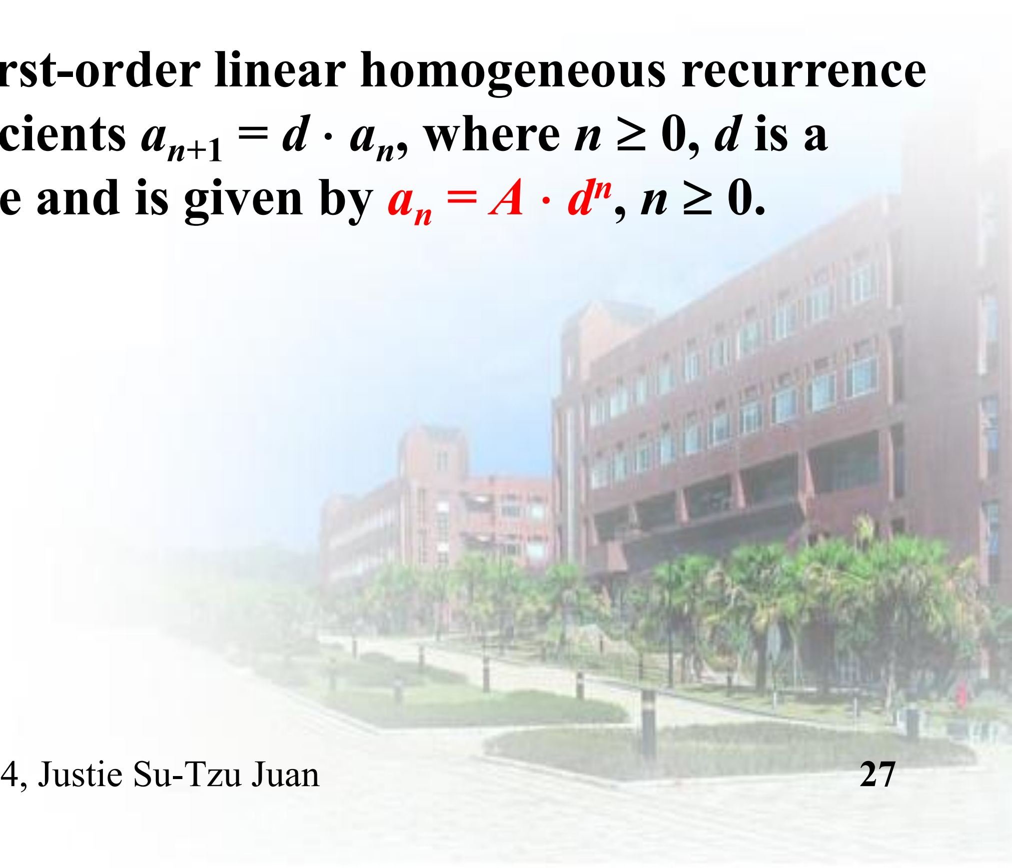
Compare with Fig 4.11 (p-234)



§ 10.1 The First-Order Linear Recurrence Relation

Outline

1. **Definition**
   - recurrence relations, geometric progression, first-order, linear, homogeneous, constant coefficients
2. **Formula:**
   - The general solution for a first-order linear homogeneous recurrence relation with constant coefficients $a_{n+1} = d \cdot a_n$, where $n \geq 0$, d is a constant and $a_0 = A$ is unique and is given by $a_n = A \cdot d^n$, $n \geq 0$.
3. **Application**
4. **Special Case**
   - Nonhomogeneous
   - Variable coefficient



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Dr. Justie Su-Tzu Juan

Chapter 10 Recurrence Relations

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients (1)

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Outline

1. **Definition**
2. **Formula (characteristic function): 3 cases**

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Def: Let $k \in \mathbb{Z}^+$, $c_n, c_{n-1}, c_{n-2}, \ldots, c_{n-k} \in \mathbb{R} - \{0\}$. a_n is a discrete function for $n \geq 0$:

1. $c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} + \ldots + c_{n-k} a_{n-k} = f(n)$, $n \geq k$, is a linear recurrence relation (with constant coefficients) of order k.
2. if $f(n) = 0$ for all $n \geq 0$; the relation is called homogeneous; otherwise, it is nonhomogeneous.

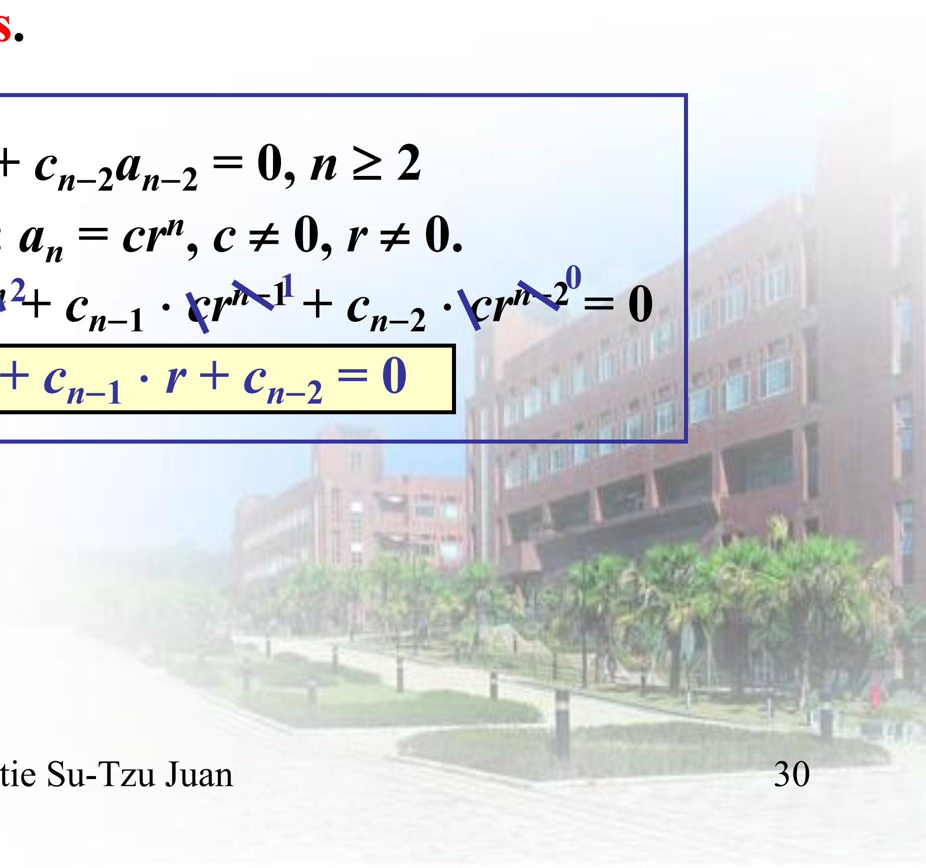
$\therefore k = 2 \Rightarrow c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0, n \geq 2$

homogeneous

seek solution: $a_n = cr^n$, $c \neq 0$, $r \neq 0$.

代入$\Rightarrow c_n \cdot \cancel{c}r^{\cancel{n}\,2} + c_{n-1} \cdot \cancel{c}r^{\cancel{n-}1} + c_{n-2} \cdot \cancel{c}r^{\cancel{n-2}\,0} = 0$

$\Rightarrow \boxed{c_n \cdot r^2 + c_{n-1} \cdot r + c_{n-2} = 0}$

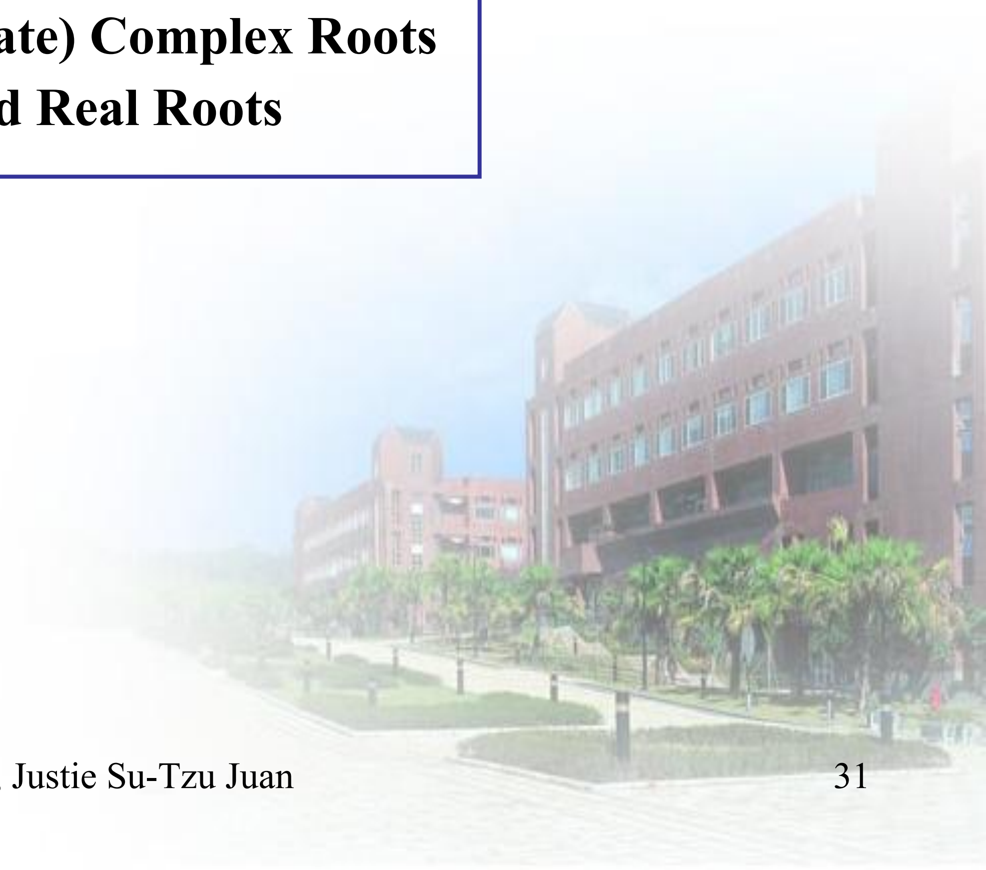


§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Def: 3. $c_n \cdot r^2 + c_{n-1} \cdot r + c_{n-2} = 0$ is called the **characteristic function**.

4. The roots r_1, r_2 of characteristic function are called **characteristic roots**.

⇒ 3 case: (A) Distinct Real Roots
(B) (Conjugate) Complex Roots
(C) Repeated Real Roots



§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Case (A): Distinct Real Roots

Ex 10.9: $\begin{cases} a_n + a_{n-1} - 6a_{n-2} = 0, \text{ where } n \geq 2, \\ a_0 = -1,\ a_1 = 8. \end{cases}$

Sol.

Let $a_n = cr^n$ with $c \neq 0$, $r \neq 0$

$\Rightarrow r^2 + r - 6 = 0$

$\Rightarrow r = 2, -3$

$\because \nexists k \in \mathbb{R}$ s.t. $2^n = k(-3)^n$ for all n. (linear independent solutions)

Let $a_n = c_1(2^n) + c_2(-3)^n$, $c_1, c_2 \in \mathbb{R}$. (general solution)

$\because a_0 = -1 = c_1 \cdot 2^0 + c_2 \cdot (-3)^0 = c_1 + c_2$

$a_1 = 8 = c_1 \cdot 2 + c_2 \cdot (-3) = 2c_1 - 3c_2$

i.e. $\begin{cases} -1 = c_1 + c_2 \\ 8 = 2c_1 - 3c_2 \end{cases} \Rightarrow \begin{cases} -2 = 2c_1 + 2c_2 \\ 8 = 2c_1 - 3c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = -2 \end{cases}$

$\Rightarrow a_n = 2^n - 2(-3)^n$, $\forall\, n \geq 0$.

(is the unique solution of the given recurrence relation)

"unique": need 2 initial conditions (values)

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.10: $\begin{cases} F_{n+2} = F_{n+1} + F_n, \forall\, n \geq 0, \\ F_0 = 0, F_1 = 1. \end{cases}$ **(Fibonacci relation)**

Sol. Let $F_n = cr^n$, $r, c \neq 0$, $\forall\, n \geq 0$

$\Rightarrow cr^{n+2} = cr^{n+1} + cr^n$

$\Rightarrow$ the characteristic equation: $r^2 - r - 1 = 0$

$\Rightarrow$ the characteristic roots are $r = (1 \pm \sqrt{5})/2$

$\therefore$ Let the general solution: $F_n = c_1[(1+\sqrt{5})/2]^n + c_2[(1-\sqrt{5})/2]^n$

$$\begin{cases} F_0 = 0 = c_1 + c_2 \\ F_1 = 1 = c_1[(1+\sqrt{5})/2] + c_2[(1-\sqrt{5})/2] \end{cases}$$

$\Rightarrow c_1 = 1/\sqrt{5} = \sqrt{5}/5$,

$c_2 = (-1)/\sqrt{5} = (-\sqrt{5})/5$

$\therefore$ the general solution $F_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right], \forall\, n \geq 0$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.11: For $n \geq 0$, let $S = \{1, 2, \ldots, n\}$, ($n = 0$, $S = \phi$)

let a_n = # of subsets of S that contain no consecutive integers.

Find and solve a recurrence relation for a_n.

Sol. (1/2)

$a_0 = 1$: $\{\phi\}$

$a_1 = 2$: $\{\phi, \{1\}\}$

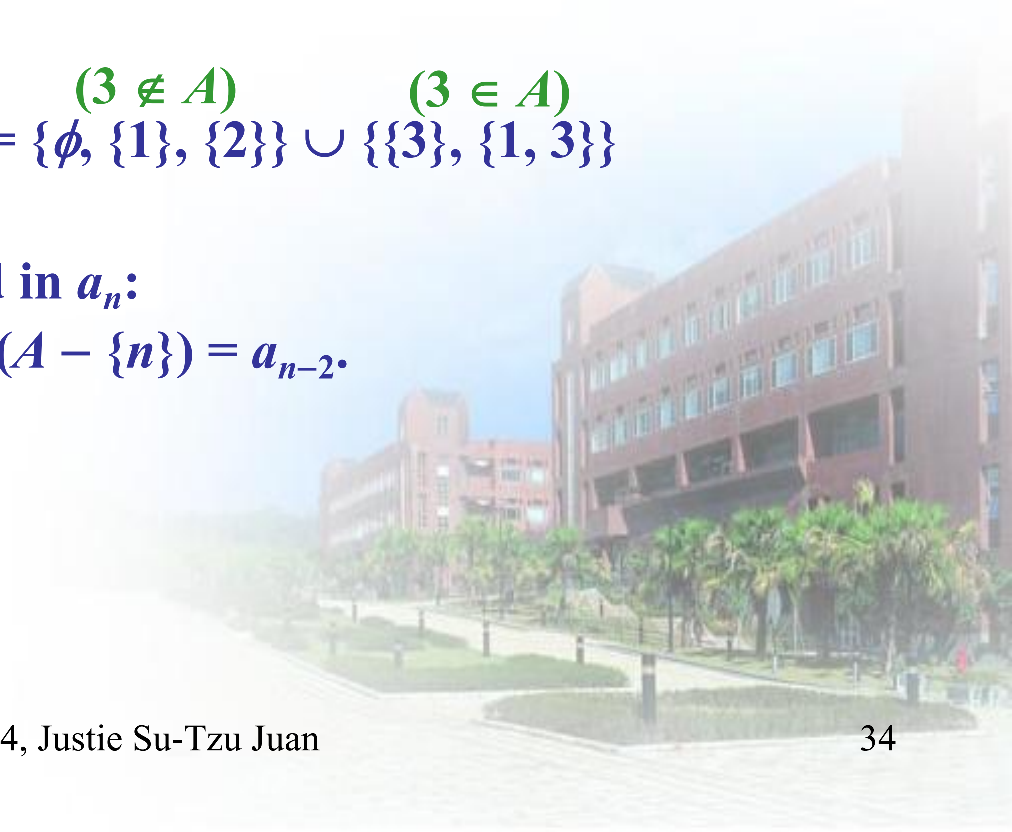
$a_2 = 3$: $\{\phi, \{1\}, \{2\}\}$

$$a_3 = 5: \{\phi, \{1\}, \{2\}, \{3\}, \{1, 3\}\} = \overset{(3 \notin A)}{\{\phi, \{1\}, \{2\}\}} \cup \overset{(3 \in A)}{\{\{3\}, \{1, 3\}\}}$$

If $A \subseteq S$ and A is to be counted in a_n:

(a) $n \in A$: $n - 1 \notin A \Rightarrow$ # of $(A - \{n\}) = a_{n-2}$.

(b) $n \notin A$: # of $A = a_{n-1}$.



§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.11: For $n \geq 0$, let $S = \{1, 2, \ldots, n\}$, ($n = 0$, $S = \phi$)

let a_n = # of subsets of S that contain no consecutive integers.

Find and **solve** a recurrence relation for a_n.

Sol. (2/2)

Let $P = \{A \mid A \subseteq S, A$ contain no consecutive integers$\}$

then $P = B \dot{\cup} C$, where $B = \{A \in P \mid n \in A\}$

$C = \{A \in P \mid n \notin A\}$

$\Rightarrow |B| = a_{n-2}, |C| = a_{n-1}$

$\therefore |P| = a_n = |B| + |C| = a_{n-2} + a_{n-1}$

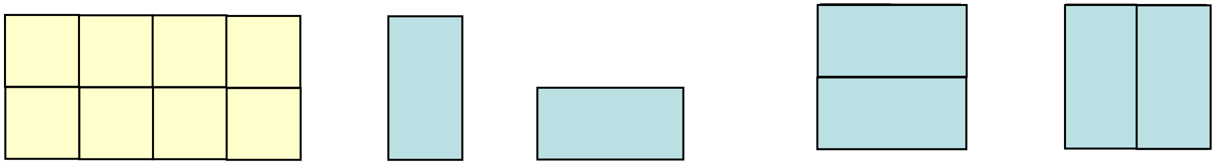
$$\therefore \begin{cases} a_n = a_{n-2} + a_{n-1}, \forall\, n \geq 2 \\ a_0 = 1;\ a_1 = 2 \end{cases}$$

$\Rightarrow a_n = F_{n+2}, n \geq 0$

$$\therefore a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right], \forall n \geq 0$$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.12:

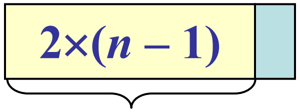


Sol. Let b_n = the number of ways we can cover a 2 × n chessboard using 2 × 1 and 1 × 2 dominoes.

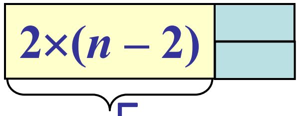
$b_1 = 1$: one 2 × 1,

$b_2 = 2$: two 2 × 1 or two 1 × 2.

When $n \geq 3$,

i) 

: b_{n-1}

ii) 

: b_{n-2}

$$\therefore \begin{cases} b_n = b_{n-2} + b_{n-1}, \ \forall\ n \geq 3 \\ b_1 = 1;\ b_2 = 2 \end{cases}$$

$$\Rightarrow b_n = F_{n+1},\ n \geq 0$$

$$\therefore b_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right], \forall n \geq 1$$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

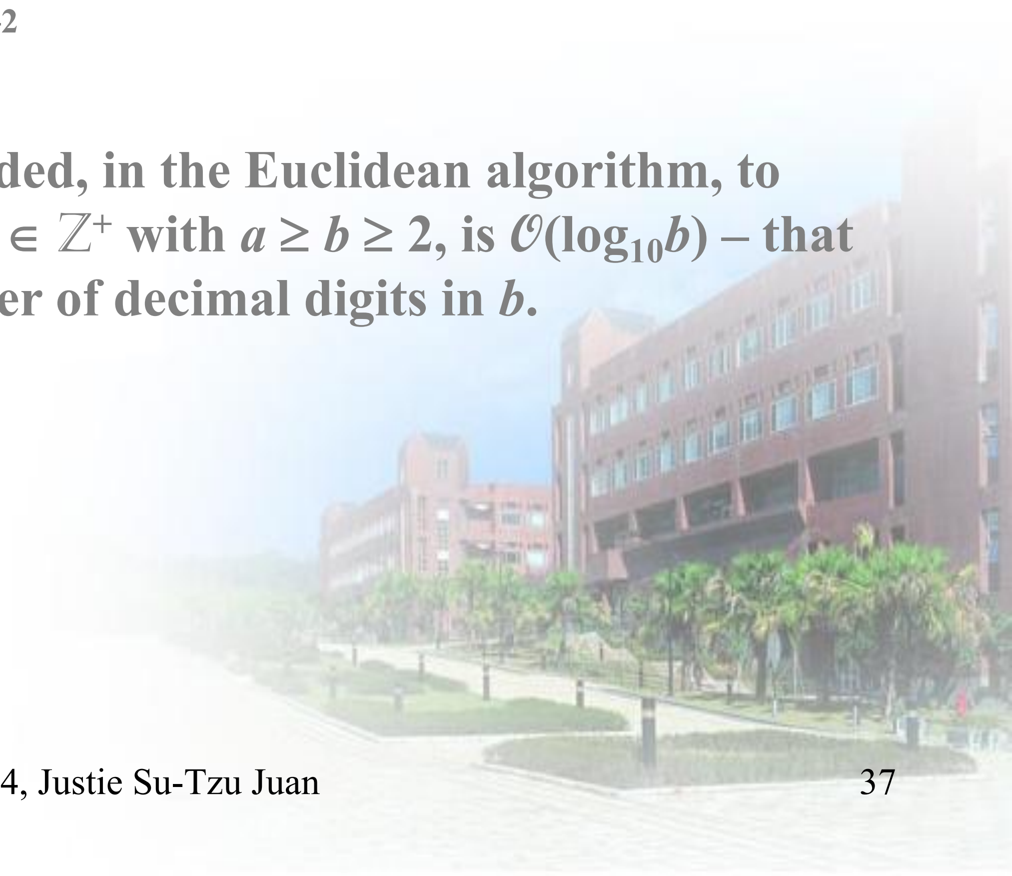
Ex 10.13: (Lamé‘s Theorem) 略

Let $a, b \in \mathbb{Z}^+$ with $a \geq b \geq 2$. Then the number of divisions needs, on the Euclidean algorithm, to determine gcd(a, b) is at most 5 times the number of decimal digits in b.

Proof.

Use F_n and $F_n > [(1+\sqrt{5})/2]^{n-2}$

Note: The number of divisions needed, in the Euclidean algorithm, to determine gcd(a, b), for $a, b \in \mathbb{Z}^+$ with $a \geq b \geq 2$, is $\mathcal{O}(\log_{10} b)$ – that is, on the order of the number of decimal digits in b.



§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.14: (Comparable relation)

a_n = # of "legal arithmetic expression without parentheses," that are made up of n symbols $(\underbrace{+, *, /}_{\text{operation symbols}}, \underbrace{0, 1, 2, \ldots, 9}_{\text{digits}})$. $a_n = ?$

Sol. (1/2)

$a_1 = 10$: (0, 1, …, 9)

$a_2 = 100$: (00, 01, …, 99)

when $n \geq 3$; a_n:

1) $\underbrace{\boxed{\qquad\qquad\qquad\qquad}\boxed{d}}_{n-1}\,\boxed{d}$: $a_{n-1} \cdot 10$

2) $\underbrace{\boxed{\qquad\qquad\qquad\qquad}}_{n-2}\boxed{o}\boxed{d}$: $a_{n-2} \cdot (3 \cdot 10 - 1)$ (∵ no $\boxed{/}\boxed{0}$)

$$\therefore \begin{cases} a_n = 10a_{n-1} + 29a_{n-2}, \text{ where } n \geq 3 \\ a_1 = 10;\ a_2 = 100 \end{cases}$$

$$\Rightarrow r^2 - 10r - 29 = 0 \Rightarrow r = 5 \pm 3\sqrt{6}$$

$$\Rightarrow a_n = \left(\frac{5}{3\sqrt{6}}\right)[(5+3\sqrt{6})^n - (5-3\sqrt{6})^n], n \geq 1$$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.14: (Comparable relation)

a_n = # of "legal arithmetic expression without parentheses," that are made up of n symbols $(\underbrace{+, *, /}_{\text{operation symbols}}, \underbrace{0, 1, 2, \ldots, 9}_{\text{digits}})$. $a_n = ?$

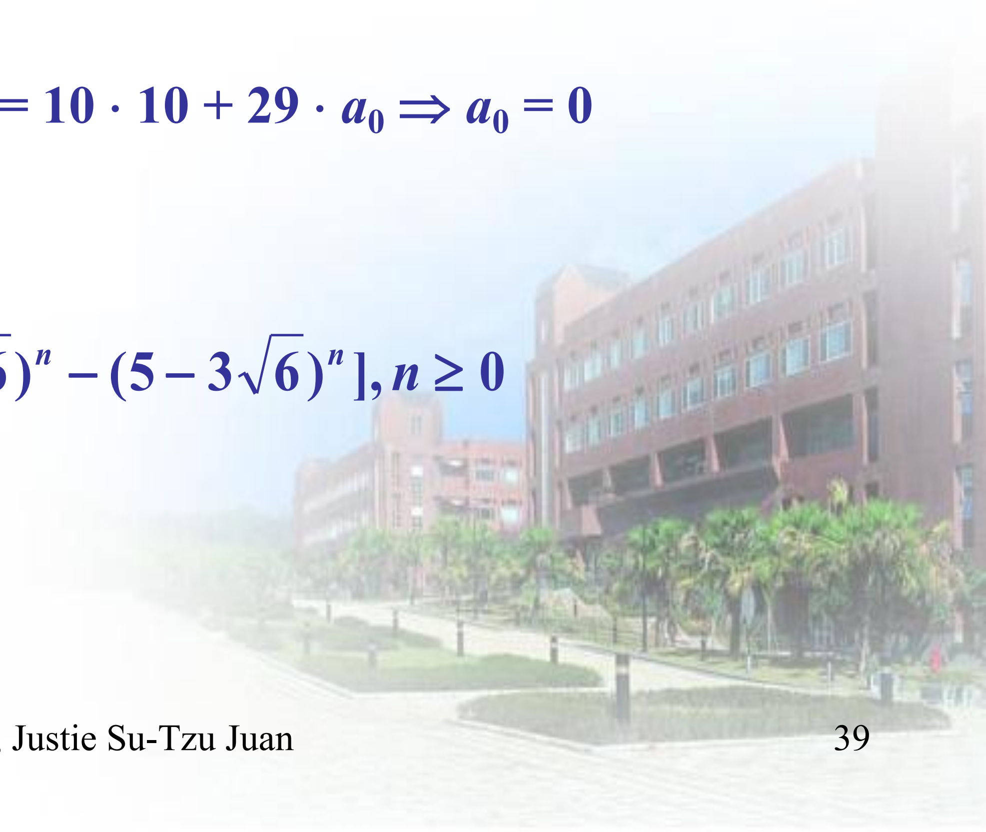
Sol. (2/2)

$$\therefore \begin{cases} a_n = 10a_{n-1} + 29a_{n-2}, \text{ where } n \geq 3 \\ a_1 = 10;\ a_2 = 100 \end{cases}$$

<another>: $a_2 = 10a_1 + 29a_0 \Rightarrow 100 = 10 \cdot 10 + 29 \cdot a_0 \Rightarrow a_0 = 0$

$$\begin{cases} a_n = 10a_{n-1} + 29a_{n-2} \\ a_0 = 0;\ a_1 = 10 \end{cases}$$

$$\Rightarrow a_n = \left(\frac{5}{3\sqrt{6}}\right)[(5+3\sqrt{6})^n - (5-3\sqrt{6})^n], n \geq 0$$



§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.15: palindromes of 3, 4, 5 and 6:

(1) 3	(1') 5	(1) 4	1') 6
(2) 1+1+1	(2') 2+1+2	(2) 1+2+1	2') 2+2+2
	(1") 1+3+1	(3) 2+2	3') 3+3
	(2") 1+1+1+1+1	(4) 1+1+1+1	4') 2+1+1+2
			1") 1+4+1
			2") 1+1+2+1+1
			3") 1+2+2+1
			4") 1+1+1+1+1+1

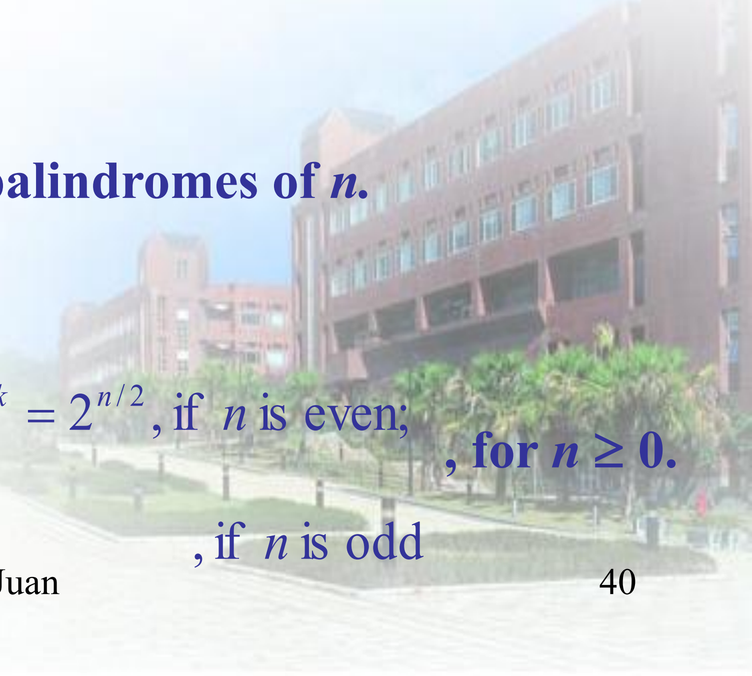
Sol. i) Add 1 to the first and last summands.

ii) Append "1+" and "+1" to the end.

For Let $n \in \mathbb{Z}^+$, let p_n = the number of palindromes of n.

Then $\begin{cases} p_n = 2p_{n-2}, \ \forall\ n \geq 3 \\ p_1 = 1;\ p_2 = 2 \end{cases}$

$$\Rightarrow p_n = \begin{cases} \left(\frac{1}{2}+\frac{1}{2\sqrt{2}}\right)2^k + \left(\frac{1}{2}-\frac{1}{2\sqrt{2}}\right)2^k = 2^k = 2^{n/2}, \text{ if } n \text{ is even;} \\ 2^{(n-1)/2} \qquad , \text{ if } n \text{ is odd} \end{cases}, \text{ for } n \geq 0.$$



§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.16: auxiliary variables:

Find a recurrence relation for the # of binary sequences of length n that have no consecutive 0's.

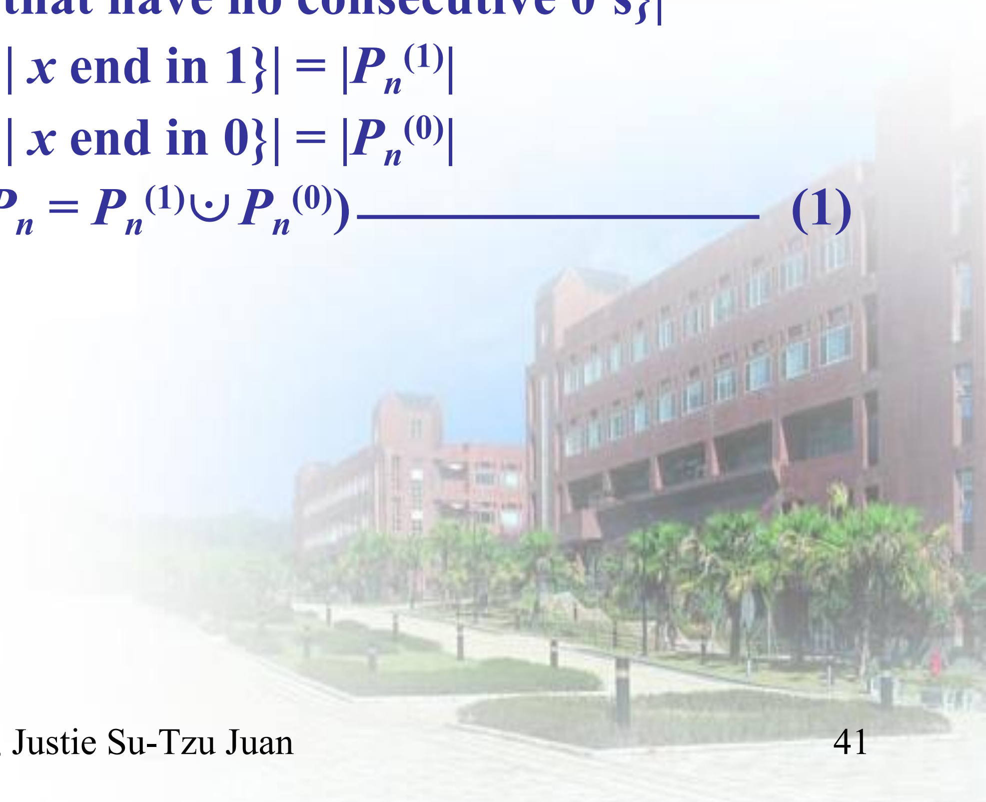
Sol. (1/4)

(a) For $n \geq 1$, let $a_n = |P_n| = |\{x \mid x \text{ is a binary sequence of length } n \text{ that have no consecutive 0's}\}|$

$$a_n^{(1)} = |\{x \in P_n \mid x \text{ end in } 1\}| = |P_n^{(1)}|$$

$$a_n^{(0)} = |\{x \in P_n \mid x \text{ end in } 0\}| = |P_n^{(0)}|$$

$$\Rightarrow a_n = a_n^{(1)} + a_n^{(0)} \qquad (P_n = P_n^{(1)} \dot\cup P_n^{(0)}) \text{———} \quad (1)$$



§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.16: auxiliary variables:

Find a recurrence relation for the # of binary sequences of length *n* that have no consecutive 0's.

Sol. (2/4)

① $a_1 = 2$: 0, 1

② $\forall\, n \geq 2,\ a_n$: case 1: $\underbrace{\boxed{x \in P_{n-1}^{(0)}}\boxed{0}}_{n-1}\boxed{1}$: $a_{n-1}^{(0)}$

case 2: $\underbrace{\boxed{x \in P_{n-1}^{(1)}}\boxed{1}}_{n-1}\begin{array}{c}\boxed{0}\\ \boxed{1}\end{array}$: $2 \cdot a_{n-1}^{(1)}$

$$\Rightarrow a_n = a_{n-1}^{(0)} + 2a_{n-1}^{(1)} \qquad \text{———} \quad (2)$$

and, $\forall\, y \in P_{n-2} \Rightarrow \boxed{\quad y \quad}\boxed{1} \in P_{n-1}^{(1)}$

$\forall\, \boxed{\quad z \quad}\boxed{1} \in P_{n-1}^{(1)} \Rightarrow z \in P_{n-2}$

$$\therefore a_{n-2} = a_{n-1}^{(1)} \qquad \text{———} \quad (3)$$

§ 10.2 The Second-Order Recurrence Relatio ients

$a_n = a_n^{(1)} + a_n^{(0)}$ ------(1)

$a_n = a_{n-1}^{(0)} + 2a_{n-1}^{(1)}$ ------(2)

$a_{n-2} = a_{n-1}^{(1)}$ ------(3)

Ex 10.16: auxiliary variables:

Find a recurrence relation for the # of binary sequences of length *n* that have no consecutive 0's.

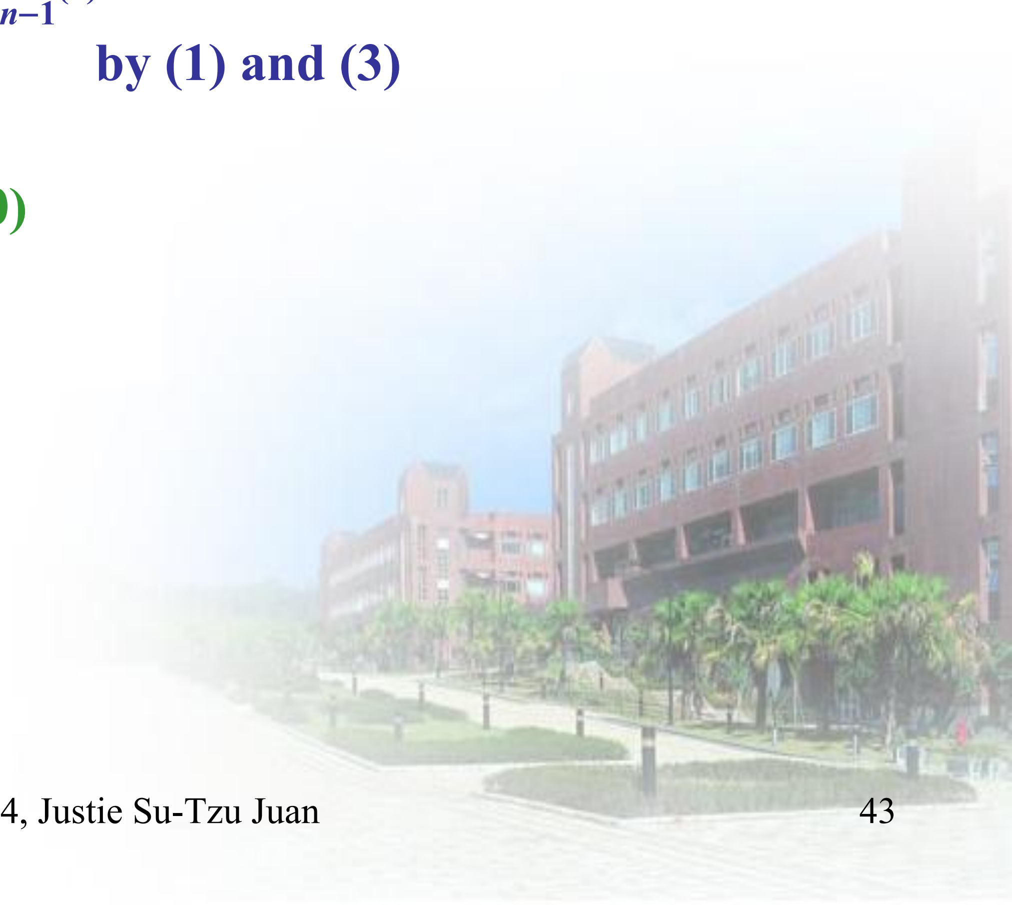
Sol. (3/4)

$$(2) \Rightarrow a_n = \underline{a_{n-1}^{(0)} + a_{n-1}^{(1)}} + a_{n-1}^{(1)}$$

$$\therefore a_n = a_{n-1} + a_{n-2}$$ **by (1) and (3)**

$$\Rightarrow \begin{cases} a_n = a_{n-1} + a_{n-2}, \forall\, n \geq 3 \\ a_1 = 2;\ a_2 = 3\ \ (11, 01, 10) \end{cases}$$

$\vdots$



§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.16: auxiliary variables:

Find a recurrence relation for the # of binary sequences of length n that have no consecutive 0's.

Sol. (4/4)

(b) For $n \geq 1$, let a_n = the number of binary sequence of length n that have no consecutive 0's.

① $a_1 = 2;\ a_2 = 3$

② $\forall\ n \geq 3,\ a_n$: $\underbrace{\boxed{}}_{n-1}\boxed{1}$: a_{n-1}

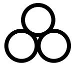
$\underbrace{\boxed{}}_{n-2}\boxed{1}\boxed{0}$: a_{n-2}

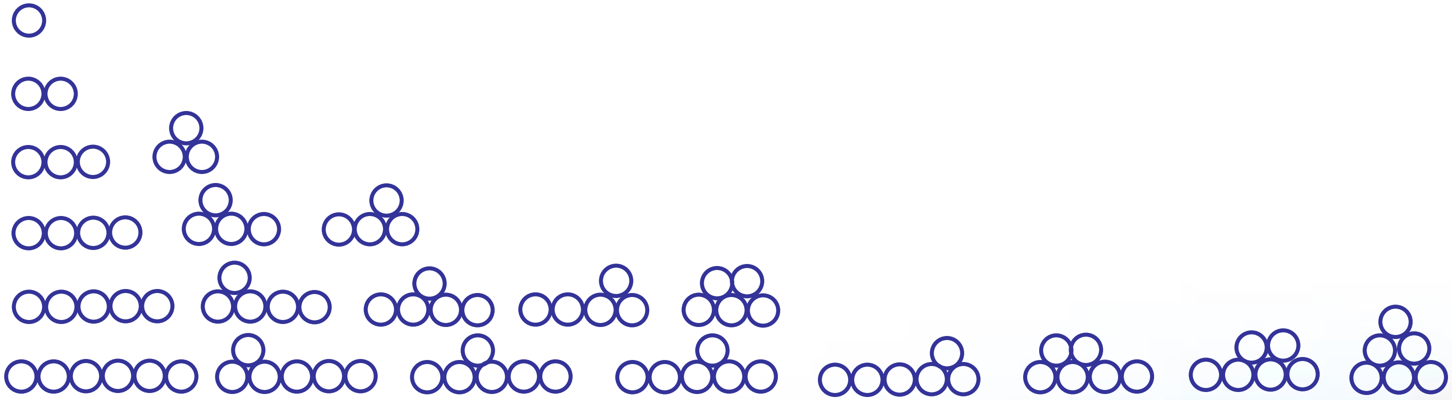
$$\begin{cases} a_n = a_{n-1} + a_{n-2},\ n \geq 3 \\ a_1 = 2;\ a_2 = 3 \Rightarrow a_0 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} a_n = a_{n-1} + a_{n-2} \\ a_0 = 1;\ a_1 = 2 \end{cases} \qquad (a_n = F_{n+2},\ n \geq 0)$$

同Ex 10.11

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

Ex 10.17: a_n = # of "arrange" (n identical pennies). s.t  and 同列相鄰



$a_1 = 1$:

$a_2 = 1$:

$a_3 = 2$:

$a_4 = 3$:

$a_5 = 5$:

$a_6 = 8$:

$\Rightarrow a_n = F_n$? $\Rightarrow$ ×

$a_7 = 12 \neq 13 = F_7$

$a_8 = 18 \neq 21 = F_8$

$a_9 = 26 \neq 34 = F_9$

§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

(Extend)

Ex 10.18: $\begin{cases} 2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n,\ n \geq 0 \\ a_0 = 0,\ a_1 = 1,\ a_2 = 2 \end{cases}$

$\Rightarrow$ Letting $a_n = cr^n$, for $c, r \neq 0$, and $n \geq 0$

$\Rightarrow$ $2r^3 - r^2 - 2r + 1 = 0 = (2r - 1)(r - 1)(r + 1)$

the characteristic roots are 1/2, 1, -1

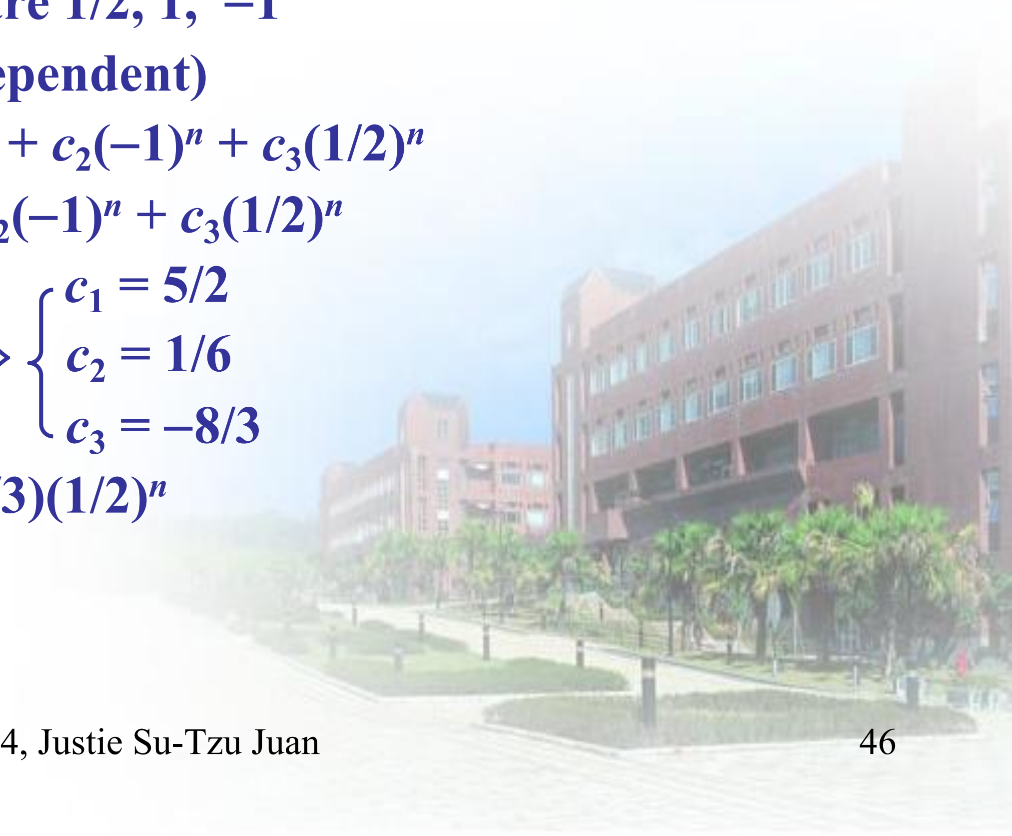
(1/2, 1, -1 are linear independent)

$\Rightarrow$ the solution is $a_n = c_1(1)^n + c_2(-1)^n + c_3(1/2)^n$

$= c_1 + c_2(-1)^n + c_3(1/2)^n$

$$\begin{cases} a_0 = 0 = c_1 + c_2 + c_3 \\ a_1 = 1 = c_1 - c_2 + c_3/2 \\ a_2 = 2 = c_1 + c_2 + c_3/4 \end{cases} \Rightarrow \begin{cases} c_1 = 5/2 \\ c_2 = 1/6 \\ c_3 = -8/3 \end{cases}$$

$\Rightarrow$ $a_n = 5/2 + (1/6)(-1)^n - (8/3)(1/2)^n$



§ 10.2 The Second-Order Linear Homogeneous Recurrence Relation with Constant Coefficients

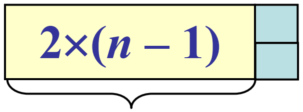
Ex 10.19:

Sol. Let a_n = the number of ways we can cover a $2 \times n$ chessboard using the two types of tiles shown above.

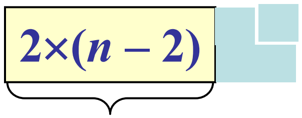
$a_1 = 1$: two 1 × 1, $a_2 = 5$: four 1 × 1 or one of each.

$a_3 = 11$: six 1 × 1(1), three 1 × 1(8), no 1 × 1(2).

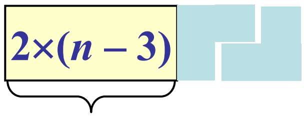
When $n \geq 3$,

i) 

: a_{n-1}

ii) 

: $4a_{n-2}$

iii) 

: $2a_{n-3}$

$$\therefore \begin{cases} a_n = a_{n-1} + 4a_{n-2} + 2a_{n-3}, \ \forall\, n \geq 4 \\ a_1 = 1;\ a_2 = 5;\ a_3 = 11 \end{cases}$$

$$\boxed{\therefore a_n = (-1)^n + \frac{1}{\sqrt{3}}(1+\sqrt{3})^n - \frac{1}{\sqrt{3}}(1-\sqrt{3})^n, \quad \forall\, n \geq 1}$$