Computer Science and Information Engineering
National Chi Nan University

Combinatorial Mathematics

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Chapter 9 Generating Functions

§ 9.3 Partitions of Integers

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

§ 9.3 Partitions of Integers

Outline

1. **Definition**
2. **Calculation and Observation**
3. **Ferrers graph**

§ 9.3 Partitions of Integers

Def: $\forall\, n \in \mathbb{Z}^+$, $p(n)$: the number of partitions of a positive integer n into positive summands.

$$\equiv |\{\Pi = \{a_1, a_2, \ldots, a_r\} \mid \sum_{i=1}^{r} a_i = n \text{ for some positive integer } r \wedge a_i \in \mathbb{Z}^+, \forall\, i \in \{1, 2, \ldots, r\}\}|.$$

e.q: $p(1) = 1$: 1

$p(2) = 2$: $2 = 1 + 1$

$p(3) = 3$: $3 = 1 + 2 = 1 + 1 + 1$

$p(4) = 5$: $4 = 1 + 3 = 2 + 2 = 1 + 1 + 2 = 1 + 1 + 1 + 1$

§ 9.3 Partitions of Integers

Q: $p(n)$ = ? If $n \in \mathbb{Z}^+$. (1/2)

the number of 1's be used as summands: $1 + x + x^2 + x^3 + \ldots$

2's be used as summands: $1 + x^2 + x^4 + x^6 + \ldots$

3's be used as summands: $1 + x^3 + x^6 + x^9 + \ldots$

⋮

$\Rightarrow \therefore p(10)$ = the coefficient of x^{10} in $f(x)$ or $g(x)$

where $f(x) = (1 + x + x^2 + x^3 + \ldots)(1 + x^2 + x^4 + x^6 + \ldots)$

power series form $(1 + x^3 + x^6 + x^9 + \ldots) \ldots (1 + x^{10} + x^{20} + x^{30} + \ldots)$

$g(x) = (1 + x + x^2 + \ldots + x^{10})(1 + x^2 + x^4 + \ldots + x^{10})$

polynomial form $(1 + x^3 + x^6 + x^9) \ldots (1 + x^{10})$

and

$$f(x) = \frac{1}{(1-x)} \frac{1}{(1-x^2)} \frac{1}{(1-x^3)} \cdots \frac{1}{(1-x^{10})} = \prod_{i=1}^{10} \frac{1}{(1-x^i)}$$

§ 9.3 Partitions of Integers

Q: $p(n)$ = ? If $n \in \mathbb{Z}^+$. (2/2)

$$\Rightarrow P(x) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)} = \sum_{i=0}^{\infty} p(i)x^i \qquad \text{(define } p(0) = 1\text{)}$$

→ it's impossible for ∞ terms.

$$\prod_{i=1}^{r} \frac{1}{(1-x^i)} = \sum_{i=0}^{r} p(i)x^i \qquad \text{for some fixed } r$$

r **很大仍然很難解, 但可解決許多類型的問題**

Ex 9.21: partition *n* minutes into 30, 60, 120 seconds.

Sol.

$a + 2b + 4c = 2n,\ a, b, c \geq 0$

$f(x) = (1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^4 + x^8 + \ldots)$

$$= \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^4}$$

the answer to the problem = the coefficient of x^{2n} in $f(x)$.

§ 9.3 Partitions of Integers

Def: $p_d(n)$ = the number of partitions of a positive integer n into distinct summands.

eq: $p(6) = 11$: 1) $1+1+1+1+1+1$ 2) $1+1+1+1+2$
3) $1+1+1+3$ 4) $1+1+4$
5) $1+1+2+2$ 6) $1+5$
7) $1+2+3$ 8) $2+2+2$
9) $2+4$ 10) $3+3$
11) 6

$\Rightarrow p_d(6) = 4$

Ex 9.22: $p_d(n) = ?$

Sol. $\forall\, k \in \mathbb{Z}^+$: either k is used or no: $(1 + x^k)$

$\Rightarrow P_d(x) = (1+x)(1+x^2)(1+x^3)\ldots = \prod_{i=1}^{\infty}(1+x^i)$

$\Rightarrow \forall\, n \in \mathbb{Z}^+$, $p_d(n)$ = the coefficient of x^n in $\prod_{i=1}^{n}(1+x^i)$ and $p_d(0) = 1$

e.q. $p_d(6)$ = the coefficient of x^6 in $(1+x)(1+x^2)\ldots(1+x^6)$
$= 4$ $(1 \cdot x^6 + x^1 \cdot x^5 + x^2 \cdot x^4 + x \cdot x^2 \cdot x^3)$

§ 9.3 Partitions of Integers

<u>Def</u>: $p_o(n)$ = the number of partitions of n into odd summands, $\forall\, n \in \mathbb{Z}^+$

$p_o(0) = 1$

eq: $p_o(6) = 4 = p_d(6)$:

1) 1 + 1 + 1 + 1 + 1 + 1
2) 1 + 1 + 1 + 1 + 2
3) 1 + 1 + 1 + 3
4) 1 + 1 + 4
5) 1 + 1 + 2 + 2
6) 1 + 5
7) 1 + 2 + 3
8) 2 + 2 + 2
9) 2 + 4
10) 3 + 3
11) 6

§ 9.3 Partitions of Integers

<u>**Ex 9.23**</u>**:** $p_o(n) = ?$

Sol.

The generating function for the sequence $p_o(0), p_o(1), p_o(2), \ldots$ is given by:

$$P_o(x) = (1 + x + x^2 + \ldots)(1 + x^3 + x^6 + \ldots)(1 + x^5 + x^{10} + \ldots)\ldots$$

$$= \frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\cdots$$

$$P_d(x) = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4)\ldots$$

$$= \frac{(1-x^2)}{(1-x)}\frac{(1-x^4)}{(1-x^2)}\frac{(1-x^6)}{(1-x^3)}\frac{(1-x^8)}{(1-x^4)}\cdots$$

$$= \frac{1}{1-x}\frac{1}{1-x^3}\cdots = P_o(x)$$

$$\Rightarrow P_d(x) = P_o(x) \Rightarrow p_d(n) = p_o(n)\ \forall\ n \geq 0$$

§ 9.3 Partitions of Integers

Ex 9.24: the number of partitions of positive integer n into odd summands, but each odd summand must occur an odd number of times = ?

e.q. 1: 1

2: ×

3: 3 = 1 + 1 + 1

4: 1 + 3

Sol.

the generating function $f(x)$:

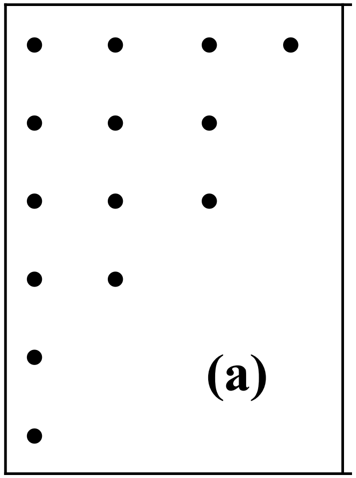
$$f(x) = (1 + x + x^3 + x^5 + \ldots)(1 + x^3 + x^9 + x^{15} + \ldots)(1 + x^5 + x^{15} + x^{25} + \ldots)\ldots = \prod_{k=0}^{\infty}\left(1 + \sum_{i=0}^{\infty} x^{(2k+1)(2i+1)}\right)$$

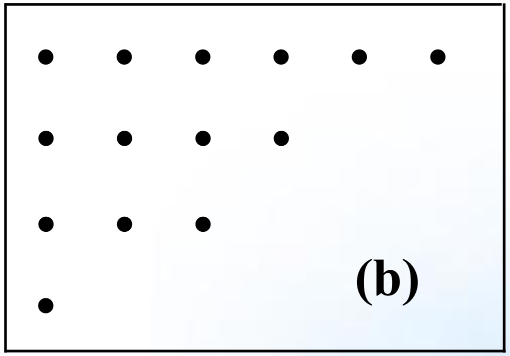
Note: $(x + x^3 + x^5 + \ldots)(x^3 + x^9 + x^{15} + \ldots)(x^5 + x^{15} + x^{25} + \ldots)\ldots$ is wrong!!

§ 9.3 Partitions of Integers

Def: **Ferrers graph**: **use rows of dots to represent a partition of an integers where per row foes not increase as we go from any row to the one below it.**

e.q: 14 = 4 + 3 + 3 + 2 + 1 + 1 = 6 + 4 + 3 + 1





§ 9.3 Partitions of Integers

Def: transposition: (b) is the transposition of (a) means (b) can be obtained from (a) by interchanging rows and columns.

Result:

The number of partitions of an integer n into m summands.

= The number of partitions of an integer n into summands where m is the largest summand.

§ 9.3 Partitions of Integers

Checklist

1. **Definition**
   - $p(n)$, $P(n)$
2. **Calculation and Observation**
   - $p_d(n)$, $p_o(n)$
3. **Ferrers graph**

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Chapter 9 Generating Functions

§ 9.4 The Exponential Generating Function

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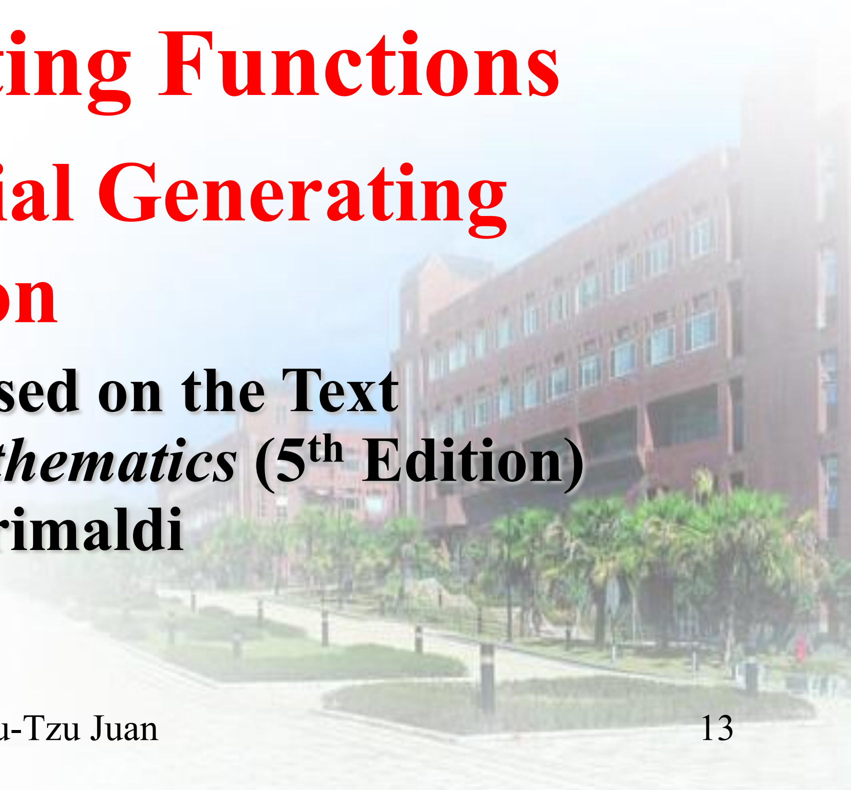


§ 9.4 The Exponential Generating Function

<u>Outline</u>

1. **Definition**
2. **Calculational Techniques**

§ 9.4 The Ex

Def 9.1: Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. The function $f(x) = a_0 + a_1x + a_2x^2 + \ldots = \sum_{i=0}^{\infty} a_i x^i$ is called the **generating function** for the given sequence.

ordinary generating function → selection problem

↕

? ← arrangement problem

$$(1+x)^n = C(n,0) + C(n,1)x + C(n,2)x^2 + C(n,3)x^3 + \ldots + C(n,n)x^n$$
$$= P(n,0) + P(n,1)\frac{x}{1!} + P(n,2)\frac{x^2}{2!} + P(n,3)\frac{x^3}{3!} + \cdots + P(n,n)\frac{x^n}{n!}$$

Def 9.2: For a sequence $a_0, a_1, a_2, \ldots$ of real numbers,

$$f(x) = a_0 + a_1x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$$

is called the **exponential generating function** for the given sequence. (Compare with Def 9.1)

§ 9.4 The Exponential Generating Function

<u>Ex 9.25</u>: Maclaurin series for e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

⇒ e^x is the exponential generating function for the sequence 1, 1, 1, ….

(e^x is the generating function for the sequence 1, 1, 1/2!, 1/3!, 1/4!,…)

§ 9.4 The Exponential Generating Function

Ex 9.26: In how many ways can four of the letters in ENGINE be arranged?

Sol. (1/2) E × 2, N × 2, G × 1, I × 1:

E	E	N	N	4!/(2!2!)	E	G	N	N	4!/2!
E	E	G	N	4!/2!	E	I	N	N	4!/2!
E	E	I	N	4!/2!	G	I	N	N	4!/2!
E	E	G	I	4!/2!	E	I	G	N	4!

E $\rightarrow 1 + x + (x^2/2!)$

N $\rightarrow 1 + x + (x^2/2!)$

G $\rightarrow 1 + x$

I $\rightarrow 1 + x$

$\Rightarrow$ the exponential generating function is $f(x) = [1 + x + (x^2/2!)]^2(1 + x)^2$

claim: the required answer = the coefficient of $x^4/4!$ in $f(x)$

§ 9.4 The Exponential Generating Function

Sol. (2/2)

claim: the required answer = the coefficient of $x^4/4!$ in $f(x)$

Proof. $f(x) = [1 + x + (x^2/2!)][1 + x + (x^2/2!)](1 + x)(1 + x)$

$\rightarrow$ there are 8 way to get $x^4/4!$

1) $(x^2/2!)(x^2/2!)(1)(1)$ $\Leftrightarrow$ E E N N

$= x^4/2!2! = (4!/(2!2!))(x^4/4!)$ $\Rightarrow 4!/(2!2!)$

2) $(x^2/2!)(1)(x)(x)$ $\Leftrightarrow$ E E G I

$= x^4/2! = (4!/2!)(x^4/4!)$ $\Rightarrow 4!/2!$

:

the term involving $x^4/4!$ is

$$\left(\frac{x^4}{2!2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+\frac{x^4}{2!}+x^4\right)$$

$$=\left[\left(\frac{4!}{2!2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+\left(\frac{4!}{2!}\right)+4!\right]\left(\frac{x^4}{4!}\right)$$

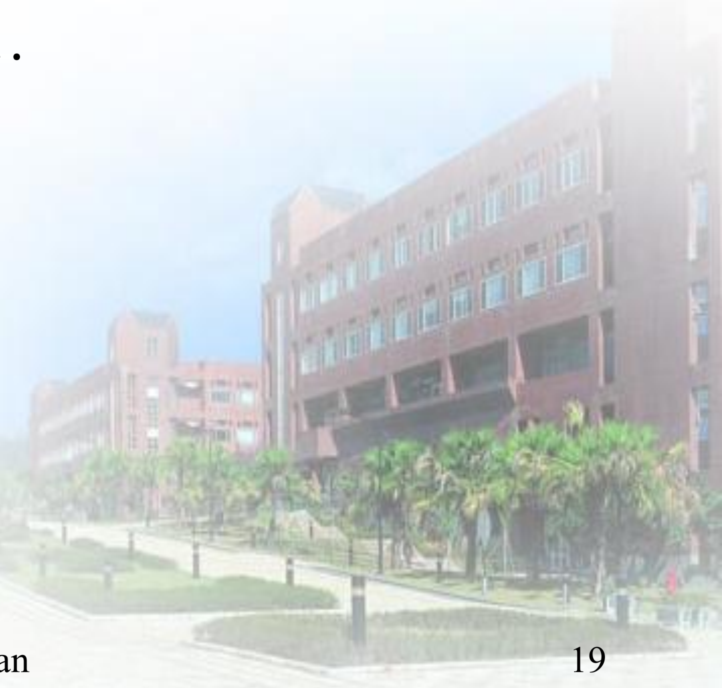
by the table, claim is true. $(= 102(x^4/4!))$

§ 9.4 The Exponential Generating Function

Ex 9.27:

$$\begin{cases} e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots \\ e^{-x} = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \cdots \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{e^x + e^{-x}}{2} = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots \\ \dfrac{e^x - e^{-x}}{2} = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots \end{cases}$$



§ 9.4 The Exponential Generating Function

Ex 9.28: 48 flags: red × 12, white × 12, blue ×12, black × 12.
12 flags on a vertical pole ⇒ a signal.
a) blue: even; black: odd?

Sol.

The exponential generating function:

$$f(x)=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^2\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)$$

$$=(e^x)^2\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right)=\left(\frac{1}{4}\right)(e^{2x})(e^{2x}-e^{-2x})$$

$$=\frac{1}{4}\left(e^{4x}-1\right)=\frac{1}{4}\left(\sum_{i=0}^{\infty}\frac{(4x)^i}{i!}-1\right)=\left(\frac{1}{4}\right)\sum_{i=1}^{\infty}\frac{(4x)^i}{i!}$$

∴the coefficient of $x^{12}/12!$ in $f(x) = (1/4)(4^{12}) = 4^{11}$

§ 9.4 The Exponential Generating Function

<u>Ex 9.28</u>: 48 flags: red × 12, white × 12, blue ×12, black × 12.

12 flags on a vertical pole ⇒ a signal.

b) white ≥ 3 or = 0 ?

Sol. (1/2)

The exponential generating function:

$$g(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^3 \left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$$

$$= (e^x)^3 \left(e^x - x - \frac{x^2}{2!}\right)$$

$$= e^{4x} - xe^{3x} - \left(\frac{1}{2}\right) x^2 e^{3x}$$

$$= \sum_{i=0}^{\infty} \frac{(4x)^i}{i!} - x \sum_{i=0}^{\infty} \frac{(3x)^i}{i!} - \left(\frac{x^2}{2}\right)\left(\sum_{i=0}^{\infty} \frac{(3x)^i}{i!}\right)$$

§ 9.4 The Exponential Generating Function

<u>Ex 9.28</u>: 48 flags: red × 12, white × 12, blue ×12, black × 12.

12 flags on a vertical pole ⇒ a signal.

b) white ≥ 3 or = 0 ?

$$g(x) = \sum_{i=0}^{\infty} \frac{(4x)^i}{i!} - x\sum_{i=0}^{\infty} \frac{(3x)^i}{i!} - \left(\frac{x^2}{2}\right)\left(\sum_{i=0}^{\infty} \frac{(3x)^i}{i!}\right)$$

Sol. (2/2)

∴the turm $x^{12}/12!$ in $g(x)$

$$= 4^{12}\left(\frac{x^{12}}{12!}\right) - x\left(\frac{(3x)^{11}}{11!}\right) - \left(\frac{x^2}{2}\right)\left(\frac{(3x)^{10}}{10!}\right)$$

$$= 4^{12}\left(\frac{x^{12}}{12!}\right) - 3^{11}\cdot(12)\left(\frac{x^{12}}{12!}\right) - \left(\frac{1}{2}\right)(3^{10})\cdot 12\cdot 11\cdot\left(\frac{x^{12}}{12!}\right)$$

$$= [4^{12} - 12\cdot 3^{11} - \frac{1}{2}\cdot 12\cdot 11\cdot 3^{10}]\left(\frac{x^{12}}{12!}\right) = [4^{12} - 12\cdot 3^{11} - 6\cdot 11\cdot 3^{10}]\left(\frac{x^{12}}{12!}\right)$$

$= 10{,}754{,}218(x^{12}/12!)$

∴ the coefficient of $x^{12}/12!$ = 10,754,218.

§ 9.4 The Exponential Generating Function

Ex 9.29: 11 new employees: assigned to one of four subdivisions. such that each get at least one.

Sol.

The exponential generating function:

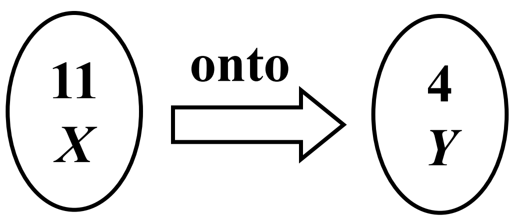
$$f(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right)^4 = (e^x - 1)^4 = e^{4x} - 4e^{3x} + 6e^{2x} - 4e^x + 1$$

The answer is the coefficient of $x^{11}/11!$ in $f(x)$:

$= 4^{11} - 4(3^{11}) + 6(2^{11}) - 4(1^{11})$

$= \sum_{i=0}^{4} (-1)^i \binom{4}{i} (4 - i)^{11}$

= The number of onto function $g: X \to Y$ where $|X| = 11$, $|Y| = 4$.



$\Rightarrow 4^{11} - \binom{4}{1}3^{11} + \binom{4}{2}2^{11} - \binom{4}{4}1^{11}$

§ 9.4 The Exponential Generating Function

Checklist:

1. **Definition**

- □ Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. For a sequence $a_0, a_1, a_2, \ldots$ of real numbers,

$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$$

is called the **exponential generating function** for the given sequence.

2. **Calculational Techniques**

- □ Maclaurin series for e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

- □ **Human beings** are different from each other, need to be "arranged".

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Chapter 9 Generating Functions

§ 9.5 The Summation Operator

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§ 9.5 The Summation Operator

Outline

1. **Technique: The convolution of the sequence a_0, a_1, a_2, and the sequence 1, 1, 1, … is the sequence**

$$a_0,\ a_0 + a_1,\ a_0 + a_1 + a_2,\ a_0 + a_1 + a_2 + a_3,\ \ldots$$

§ 9.5 The Summation Operator

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$$

$$g(x) = 1 + x + x^2 + x^3 + \ldots = 1/(1 - x)$$

$$\Rightarrow f(x) \cdot g(x) = f(x)/(1 - x) = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + \ldots$$

$\therefore$ $f(x)/(1 - x)$ generates the sequence of sums

$$a_0,\ a_0 + a_1,\ a_0 + a_1 + a_2,\ a_0 + a_1 + a_2 + a_3,\ \ldots$$

= The convolution of the sequence $a_0, a_1, a_2,$ and the sequence $b_0, b_1, b_2, \ldots$ where $b_i = 1\ \forall\ n \in \mathbb{N}$.

§ 9.5 The Summation Operator

<u>Ex 9.30</u>: (a) $\because$ $1/(1-x)$ is the G.F. for the seq. 1, 1, 1, …

$\therefore$ $(1/(1-x))/(1-x)$ is the G.F. for the seq. 1, 1+1, 1+1+1, …

i.e. $1/(1-x)^2$ is the G.F. for the seq. 1, 2, 3, …

(b) $\because$ $x+x^2$ is the G.F. for the seq. 0, 1, 1, 0, 0, 0, …

$\therefore$ $(x+x^2)/(1-x)$ is the G.F. for the seq. 0, 1, 2, 2, 2, 2,…

$\Rightarrow$ $(x+x^2)/(1-x)^2$ is the G.F. for the seq. 0, 1, 3, 5, 7, 9,…

$\Rightarrow$ $(x+x^2)/(1-x)^3$ is the G.F. for the seq. 0, 1, 4, 9, 16, 25,…

$\sum_{k=1}^{n}(2k-1)=n^2$?

to verify this, look at the coefficient of x^n in $(x+x^2)/(1-x)^3$

= the coefficient of x^n in $x(1-x)^{-3}+x^2(1-x)^{-3}$

$$=\binom{-3}{n-1}(-1)^{n-1}+\binom{-3}{n-2}(-1)^{n-2}$$

$$=(-1)^{n-1}\binom{3+(n-1)-1}{n-1}(-1)^{n-1}+(-1)^{n-2}\binom{3+(n-2)-1}{n-2}(-1)^{n-2}$$

$$=\frac{1}{2}(n+1)(n)+\frac{1}{2}(n)(n-1)=n^2$$

(as Example 4.7)

§ 9.5 The Summation Operator

<u>Ex 9.31</u>: $0^2 + 1^2 + 2^2 + 3^2 + \ldots + n^2 = ?$

Sol. (1/2)

Let $g(x) = 1/(1-x) = 1 + x + x^2 + x^3 + \ldots$

$$\frac{dg(x)}{dx} = 1 + 2x + 3x^2 + 4x^3 + \ldots = 1/(1-x)^2$$

$$x \cdot \frac{dg(x)}{dx} = 0 + x + 2x^2 + 3x^3 + 4x^4 + \ldots = x/(1-x)^2$$

$$\frac{d}{dx}\left[x\left(\frac{dg(x)}{dx}\right)\right] = 1 + 2^2x^1 + 3^2x^2 + 4^2x^3 + \cdots = \frac{(1-x)^{\cancel{2}\,1} - x(2)\cancel{(1-x)}(-1)}{(1-x)^{\cancel{4}\,3}}$$

$$x \cdot \frac{d}{dx}\left[x\left(\frac{dg(x)}{dx}\right)\right] = 0 + x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \cdots = x\frac{(1-x+2x)}{(1-x)^3} = \frac{x(1+x)}{(1-x)^3}$$

§ 9.5 The Summ

$$x \cdot \frac{d}{dx}\left[x\left(\frac{dg(x)}{dx}\right)\right] = 0 + x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \cdots = \frac{x(1+x)}{(1-x)^3}$$

Ex 9.31: $0^2 + 1^2 + 2^2 + 3^2 + \ldots + n^2 = ?$

Sol. (2/2)

$\therefore \dfrac{x(1+x)}{(1-x)^3}$ is the G.F. of $0^2, 1^2, 2^2, 3^2, \ldots$

$\Rightarrow \dfrac{x(1+x)}{(1-x)^3}\dfrac{1}{(1-x)} = \dfrac{x(1+x)}{(1-x)^4}$ is the G.F. for $0^2, 0^2 + 1^2, 0^2 + 1^2 + 2^2, \ldots$

$\therefore$ the coefficient of x^n in $\dfrac{x(1+x)}{(1-x)^4}$ is $\sum_{i=0}^{n} i^2$

$$\frac{x(1+x)}{(1-x)^4} = (x + x^2)(1 - x)^{-4} = (x + x^2)\left[\binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + \ldots\right]$$

$\therefore$ the coefficient of $x^n = \binom{-4}{n-1}(-1)^{n-1} + \binom{-4}{n-2}(-1)^{n-2}$

$= (-1)^{n-1}\binom{4+n-1-1}{n-1}(-1)^{n-1} + (-1)^{n-2}\binom{4+n-2-1}{n-2}(-1)^{n-2}$

$= \binom{n+2}{n-1} + \binom{n+1}{n-2} = (n + 2)!/[(n - 1)!3!] + (n + 1)!/[(n - 2)!3!]$

$= (1/6)[(n + 2)(n + 1)n + (n + 1)(n)(n - 1)]$

$= (1/6)(n)(n + 1)(n + 2 + n - 1)$

$= (1/6)n(n + 1)(2n + 1)$

§ 9.5 The Summation Operator

Checklist:

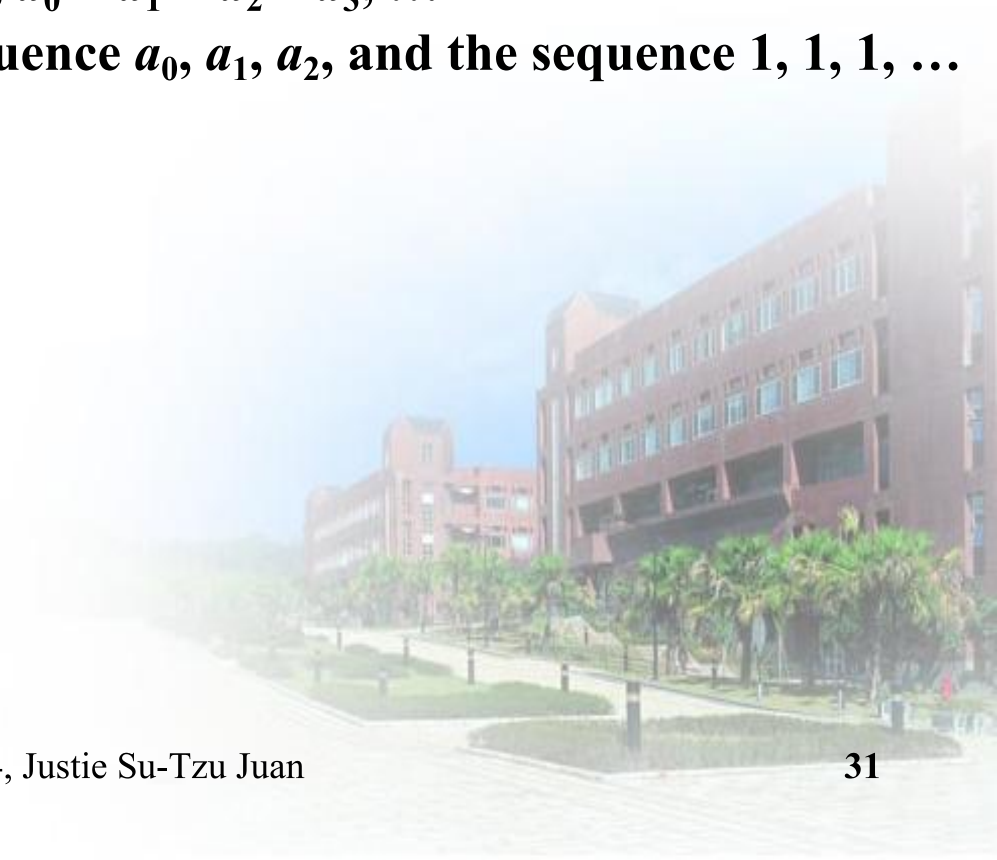
1. Technique:

- $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$

  $f(x)/(1 - x)$ generates the sequence of sums

  $$a_0,\ a_0 + a_1,\ a_0 + a_1 + a_2,\ a_0 + a_1 + a_2 + a_3,\ \ldots$$

  = The convolution of the sequence $a_0, a_1, a_2,$ and the sequence 1, 1, 1, …



§ 9.5 The Summation Operator

Discussion (10 min):

Exercise 9.5.4: If $f(x) = \Sigma_{n=0}^{\infty} a_n x^n$,

1) what is the generation function for the sequence a_0, $a_0 + a_1$, $a_1 + a_2$, $a_2 + a_3$, ... ?

2) What is the generating function for the sequence a_0, $a_0 + a_1$, $a_0 + a_1 + a_2$, $a_1 + a_2 + a_3$, $a_2 + a_3 + a_4$, ... ?

3) What is the generation function for the sequence $\frac{a_0}{4}, \frac{a_0}{2} + \frac{a_1}{4}, \frac{a_0}{4} + \frac{a_1}{2} + \frac{a_2}{4}, \frac{a_1}{4} + \frac{a_2}{2} + \frac{a_3}{4}, \ldots$