Computer Science and Information Engineering
National Chi Nan University

Combinatorial Mathematics

Dr. Justie Su-Tzu Juan

Chapter 9 Generating Functions

§ 9.2 Definition and Examples: Calculational Techniques

Slides for a Course Based on the Text
***Discrete & Combinatorial Mathematics* (5th Edition)**
by Ralph P. Grimaldi

§ 9.2 Definition and Examples: Calculational Techniques

Outline

1. **Definition**
2. **Calculational Techniques**
3. **Examples**

§ 9.2 Definition and Examples: Calculational Techniques

<u>Def 9.1</u>: Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers.

The function $f(x) = a_0 + a_1x + a_2x^2 + \ldots = \sum_{i=0}^{\infty} a_i x^i$

is called the generating function for the given sequence.

<u>Ex 9.4</u>: $\forall\, n \in \mathbb{Z}^+$, $(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$

$\Rightarrow (1 + x)^n$ is the generating function for the sequence

$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}, 0, 0, 0, \ldots$.

<u>Ex 9.5</u>: a) $\forall\, n \in \mathbb{Z}^+$, $(1 - x^{n+1}) = (1 - x)(1 + x + x^2 + \ldots + x^n)$

$\Rightarrow (1 - x^{n+1})/(1 - x)$ is the generating function for the sequence $\underbrace{1, 1, 1, \ldots, 1}_{n+1\text{ 個}}, 0, 0, 0, \ldots$.

§ 9.2 Definition and Examples: Calculational Techniques

Ex 9.5:

b) $1 = (1 - x)(1 + x + x^2 + \ldots)$

$\Rightarrow$ $1/(1 - x)$ is the generating function for the sequence 1, 1, 1, ….

Note: $|x| < 1$: $1/(1 - x) = \sum_{i=0}^{\infty} x^i$: this range of values that the geometric series $1 + x + x^2 + \ldots$ converges.

在此章不考慮 convergence的問題, 重點是在G.F.的係數

c) $$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{i=0}^{\infty} x^i \Rightarrow \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{i=0}^{\infty}(i+1)x^i$$

$\Rightarrow$ $1/(1 - x)^2$ is the generating function of the seq. 1, 2, 3, ….

$\Rightarrow$ $x/(1 - x)^2$ is the generating function of the seq. 0, 1, 2, 3, ….

§ 9.2 Definition and Examples: Calculational Techniques

<u>Ex 9.5:</u>

d) $\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \frac{x+1}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + \ldots. = \sum_{i=1}^{\infty} i^2 x^{i-1} = \sum_{i=0}^{\infty} (i+1)^2 x^i$

$\Rightarrow$ **$(x + 1)/(1 - x)^3$ is the generating function of the sequence $1^2, 2^2, 3^2, \ldots.$**

e) $f_0(x) = 1/(1 - x) = 1 + x + x^2 + x^3 + \ldots$

$f_1(x) = x(d/dx)f_0(x) = x/(1 - x)^2 = 0 + x + 2x^2 + 3x^3 + \ldots$

$f_2(x) = x(d/dx)f_1(x) = (x^2 + x)/(1 - x)^3 = 0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \ldots$

$f_3(x) = x(d/dx)f_2(x) = (x^3 + 4x^2 + x)/(1 - x)^4$

$= 0^3 + 1^3x + 2^3x^2 + 3^3x^3 + \ldots$

$f_4(x) = x(d/dx)f_3(x) = (x^4 + 11x^3 + 11x^2 + x)/(1 - x)^5$

$= 0^4 + 1^4x + 2^4x^2 + 3^4x^3 + \ldots$

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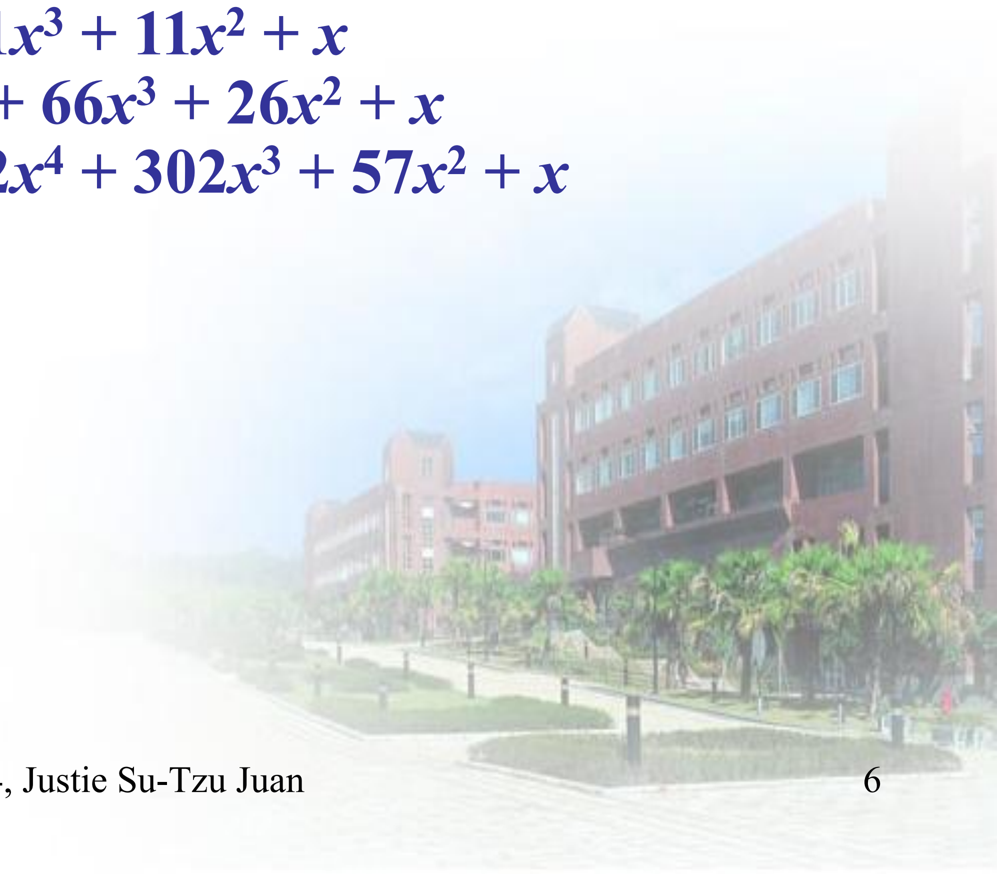
Ex 9.5:

e)

Eulerian numbers

(in Ex 4.18)

$$x$$
$$x^2 + x$$
$$x^3 + 4x^2 + x$$
$$x^4 + 11x^3 + 11x^2 + x$$
$$x^5 + 26x^4 + 66x^3 + 26x^2 + x$$
$$x^6 + 57x^5 + 302x^4 + 302x^3 + 57x^2 + x$$



§ 9.2 Definition and Examples: Calculational Techniques

Def (Ex 4.18): Eulerian number $a_{m,k}$

$$\begin{cases} a_{m,k} = (m-k)a_{m-1,k-1} + (k+1)a_{m-1,k}, & 0 \le k \le m-1 \\ a_{0,0} = 1,\ a_{m,k} = 0,\ k \ge m,\ a_{m,k} = 0,\ k < 0. \end{cases}$$

ex: Row Sum

$m = 1$: $1 \cdot 0 + 1 \cdot 1 = 1$ — $1!$

$m = 2$: $2 \cdot 0 + 1 \cdot 1 = 1$, $1 \cdot 1 + 2 \cdot 0 = 1$ — $2!$

$m = 3$: $3 \cdot 0 + 1 \cdot 1 = 1$, $2 \cdot 1 + 2 \cdot 1 = 4$, $1 \cdot 1 + 3 \cdot 0 = 1$ — $3!$

$m = 4$: $4 \cdot 0 + 1 \cdot 1 = 1$, $3 \cdot 1 + 2 \cdot 4 = 11$, $2 \cdot 4 + 3 \cdot 1 = 11$, $1 \cdot 1 + 4 \cdot 0 = 1$ — $4!$

$5 \cdot 0 + 1 \cdot 1 = 1$, $4 \cdot 1 + 2 \cdot 11 = 26$, $3 \cdot 11 + 3 \cdot 11 = 66$, $2 \cdot 11 + 4 \cdot 1 = 26$, $1 \cdot 1 + 5 \cdot 0 = 1$

$m = 5$ — $5!$

§ 9.2 Definition and Examples: Calculational Techniques

<u>Ex 9.6</u>: (a) $f(x) = 1/(1-x)$ is the G.F. for the sequence 1, 1, 1, …

$\Rightarrow g(x) = f(x) - x^2 = (1/(1-x)) - x^2$ is the G.F. for the sequence
1, 1, 0, 1, 1, 1, …

$\Rightarrow h(x) = f(x) + 2x^3 = (1/(1-x)) + 2x^3$ is the G.F. for the sequence
1, 1, 1, 3, 1, 1, …

(b) Find a G.F. for the sequence 0, 2, 6, 12, 20, 30, 42, …?

$$\left.\begin{aligned} a_0 &= 0 = 0^2 + 0 \\ a_1 &= 2 = 1^2 + 1 \\ a_2 &= 6 = 2^2 + 2 \\ a_3 &= 12 = 3^2 + 3 \\ a_4 &= 20 = 4^2 + 4 \end{aligned}\right\} \Rightarrow$$

$a_n = n^2 + n$ for each $n \geq 0$

using <u>Ex 9.5</u> (c), (d)

$$\therefore \frac{x(x+1)}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{2x}{(1-x)^3}$$

is the G.F. for the given sequence.

(若看不出$a_n = n^2 + n$, 在 Chap 10 有說明)

§ 9.2 Definition and Examples: Calculational Techniques

已知: $\forall\, n \in \mathbb{Z}^+$, $(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$.

(a) $n < 0$?

(b) n is not necessarily an integer?

Note: $\forall\, n, r \in \mathbb{Z}^+$, and $n \geq r > 0$

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

§ 9.2 Definition and Examples: Calculational Techniques

Def: (b) If $n \in \mathbb{R}$, use $\dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$ as the definition of $\binom{n}{r}$

(a) If $n \in \mathbb{Z}^+$, define:

$$\binom{-n}{r} = \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{r!}$$

$$= (-1)^r \frac{(n)(n+1)(n+2)(n+3)\cdots(n+r-1)}{r!}$$

$$= (-1)^r \cdot (n+r-1)!/[r!(n-1)!] = (-1)^r\binom{n+r-1}{r}$$

(c) $\forall\, n \in \mathbb{R}$, define $\binom{n}{0} = 1$

Def: (a) $\forall\, n \in \mathbb{R}, r \in \mathbb{Z}^+$: $\binom{n}{r} = \dfrac{n(n-1)\cdots(n-r+1)}{r!}$

(b) $\forall\, n \in \mathbb{R}, r = 0$: $\binom{n}{0} = 1$

§ 9.2 Definition and Examples: Calculational Techniques

Ex 9.7: Generalized the binomial theorem:

$\forall\, n \in \mathbb{Z}^+$, By Maclaurin series expansion for $(1 + x)^{-n} =$
$1 + (-n)x + (-n)(-n-1)x^2/2! + (-n)(-n-1)(-n-2)x^3/3! + \ldots$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!} x^r = \sum_{r=0}^{\infty} (-1)^r \binom{n+r-1}{r} x^r$$

$$\Rightarrow (1 + x)^{-n} = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \ldots = \sum_{r=0}^{\infty} \binom{-n}{r} x^r$$

$\therefore (1 + x)^{-n}$ is the G.F. for the sequence $\binom{-n}{0}, \binom{-n}{1}, \ldots$.

Ex 9.8: Find the coefficient of x^5 in $(1 - 2x)^{-7}$.

Sol.

Let $y = -2x$: $(1 + y)^{-7} = \sum_{r=0}^{\infty} \binom{-7}{r} y^r = \sum_{r=0}^{\infty} \binom{-7}{r} (-2x)^r$

when $r = 5 \Rightarrow \binom{-7}{r}(-2)^r = \binom{-7}{5}(-2)^5 = (-1)^5 \binom{7+5-1}{5}(-32)$

$= -\binom{11}{5}(-32) = 32\binom{11}{5} = 14{,}784.$

§ 9.2 Definition and Examples: Calculational Techniques

Ex 9.9: when $n \in \mathbb{R}$:

the Maclaurin series expansion for $(1 + x)^n$ is

$1 + nx + n(n - 1)x^2/2! + (n)(n - 1)(n - 2)x^3/3! + \ldots$

$$= 1 + \sum_{r=1}^{\infty} \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} x^r$$

$$\Rightarrow (1 + 3x)^{-1/3} = 1 + \sum_{r=1}^{\infty} \frac{(-1/3)(-4/3)(-7/3)\cdots((-3r+2)/3)}{r!} (3x)^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1)^r 1 \cdot 4 \cdot 7 \cdots (3r-2)}{r!} x^r$$

$\therefore (1 + 3x)^{-1/3}$ is the G.F. for the sequence

$1, -1, (1 \cdot 4)/2!, -(1 \cdot 4 \cdot 7)/3!, \ldots, (-1)^r 1 \cdot 4 \cdot \ldots \cdot (3r - 2)/r!, \ldots.$

§ 9.2 Definition and Examples: Calculational Techniques

<u>Ex 9.10</u>: Determine the coefficient of x^{15} in $f(x) = (x^2 + x^3 + x^4 + \ldots)^4$.

Sol.

$\because (x^2 + x^3 + x^4 + \ldots) = x^2(1 + x + x^2 + \ldots) = x^2/(1 - x)$

$\therefore$ the coefficient of x^{15} in $f(x)$

= the coefficient of x^{15} in $x^8/(1 - x)^4$

= the coefficient of x^7 in $1/(1 - x)^4 = (1 - x)^{-4}$

$= \binom{-4}{7}(-1)^7 = (-1)^7\binom{4+7-1}{7}(-1)^7 = (-1)^{14}\binom{10}{7}$

= 120

In general: the coefficient of x^n in $f(x)$

$$= \begin{cases} 0, \text{ if } 0 \le n \le 7; \\ \text{the coefficient of } x^{n-8} \text{ in } (1 - x)^{-4} \end{cases}$$

$= \binom{-4}{n-8}(-1)^{n-8} = \binom{n-5}{n-8}$, if $n > 7$.

§ 9.2 Definition and Examples: Calculational Techniques

Table 9.2: **(i) For all $m, n \in \mathbb{Z}^+$, $a \in \mathbb{R}$**

1) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$
2) $(1+ax)^n = \binom{n}{0} + \binom{n}{1}ax + \binom{n}{2}a^2x^2 + \ldots + \binom{n}{n}a^nx^n$
3) $(1+x^m)^n = \binom{n}{0} + \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \ldots + \binom{n}{n}x^{nm}$
4) $(1-x^{n+1})/(1-x) = 1 + x + x^2 + \ldots + x^n$
5) $1/(1-x) = 1 + x + x^2 + \ldots = \sum_{i=0}^{\infty} x^i$
6) $1/(1+x)^n = (1+x)^{-n} = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \ldots$
$= \sum_{i=0}^{\infty} \binom{-n}{i}x^i = \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i}x^i$
7) $1/(1-x)^n = (1-x)^{-n} = \binom{-n}{0} + \binom{-n}{1}(-x) + \binom{-n}{2}(-x)^2 + \ldots$
$= \sum_{i=0}^{\infty} \binom{-n}{i}(-x)^i = \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i}(-x)^i$
$= \sum_{i=0}^{\infty} \binom{n+i-1}{i}x^i$

(ii) If $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{i=0}^{\infty} b_i x^i$ and $h(x) = f(x)g(x)$,
then $h(x) = \sum_{i=0}^{\infty} c_i x^i$, where $\forall\, k \geq 0$,
$c_k = a_0b_k + a_1b_{k-1} + \ldots + a_{k-1}b_1 + a_kb_0 = \sum_{i=0}^{k} a_ib_{k-i}$

§ 9.2 Definition and Examples: Calculational Techniques

Ex 9.11: In how many ways can we select, with repetitions allowed, r objects from n distinct objects?

Sol.

$0 \sim r$	$0 \sim r$	$\cdots$	$0 \sim r$
object 1	object 2		object n

$\Rightarrow$ the G.F. is $f(x) = (1 + x + x^2 + \ldots)^n$

$\therefore$ the answer = the coefficient of x^r in $f(x)$ = ?

$$\because (1 + x + x^2 + \ldots)^n = \left(\frac{1}{(1-x)}\right)^n = \frac{1}{(1-x)^n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$$

$$\therefore ? = \binom{n+r-1}{r}$$

(Found in Chapter 1)

Ex 9.12: Use G. F. to solve the problem of counting the **compositions** of $n \in \mathbb{Z}^+$.

Solve.

# of 1-summand = the coef. of x^n in $(x + x^2 + \ldots + x^n + \ldots) = 1$

# of 2-summands = the coef. of x^n in $(x + x^2 + \ldots + x^n + \ldots)^2$

$$(x + x^2 + \ldots + x^n + \ldots)^2 = [x/(1-x)]^2 = x^2/(1-x)^2$$

ex: $n = 4$: the coefficient of x^4 in $(x + x^2 + \ldots + x^n)^2 = 3$

$x^1 \cdot x^3, x^2 \cdot x^2, x^3 \cdot x^1 \rightarrow 1 + 3, 2 + 2, 3 + 1$

# of 3-summands = the coefficient of x^n in $[x/(1-x)]^3$

⋮

of i-summands = the coefficient of x^n in $[x/(1-x)]^i$

$\Rightarrow$ Answer = the coefficient of x^n in $f(x) = \sum_{i=1}^{\infty}[x/(1-x)]^i$

$$f(x) = \sum_{i=1}^{\infty}[x/(1-x)]^i = x/(1-x) \sum_{i=0}^{\infty}[x/(1-x)]^i$$

$$= x/(1-x)\,[1/(1 - x/(1-x))] = x/(1-2x)$$

$$= 2^0x + 2^1x^2 + \ldots + 2^{n-1}x^n + \ldots$$

$\Rightarrow$ Answer = 2^{n-1} (Examples 1.37, 3.11, 4.12)

Ex 9.13: **palindromes of 6 and 7:**

	6			7	
1)	6	(1)	1)	7	(1)
2)	1+4+1	(1)	2)	1+5+1	(1)
3)	2+2+2	(2)	3)	2+3+2	(2)
4)	1+1+2+1+1		4)	1+1+3+1+1	
5)	3+3	(4)	5)	3+1+3	(4)
6)	1+2+2+1		6)	1+2+1+2+1	
7)	2+1+1+2		7)	2+1+1+1+2	
8)	1+1+1+1+1+1		8)	1+1+1+1+1+1+1	

$8 = 1 + (1 + 2 + 4) = 1 + (1 + 2^1 + 2^2) = 1 + (2^3 - 1) = 2^3$

Center : 7 1 palindrome

Center : 5 1 (= 2^{1-1})palindrome $(7 - 5) / 2 = 1$

Center : 3 2 (= 2^{2-1})palindrome $(7 - 3) / 2 = 2$

Center : 1 4 (= 2^{3-1})palindrome $(7 - 1) / 2 = 3$

In general: n **has** $1 + (1 + 2^1 + \ldots + 2^{k-1}) = 2^k = 2^{\lfloor n/2 \rfloor}$ **palindromes.**

(where $k = \lfloor n/2 \rfloor$)

萊福槍砲彈

Ex 9.14: 24 rifle shells → 4 police officers such that each officer gets ≥ 3, ≤ 8.

Sol.

each: $x^3 + x^4 + \ldots + x^8$

$\Rightarrow$ the resulting G.F. $= f(x) = (x^3 + x^4 + \ldots + x^8)^4$

$= x^{12}(1 + x + \ldots + x^5)^4 = x^{12}((1 - x^6)/(1 - x))^4$

$\therefore$ the coefficient of x^{24} in $f(x)$

$=$ the coefficient of x^{12} in $((1 - x^6)/(1 - x))^4$

$=$ the coefficient of x^{12} in $[1 - \binom{4}{1}x^6 + \binom{4}{2}x^{12} - \binom{4}{3}x^{18} + \binom{4}{4}x^{24}]$

$\cdot [\binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + \ldots]$

$= [\binom{-4}{12}(-1)^{12} - \binom{4}{1}\binom{-4}{6}(-1)^6 + \binom{4}{2}\binom{-4}{0}]$

$= [\binom{15}{12} - \binom{4}{1}\binom{9}{6} + \binom{4}{2}\binom{3}{0}] = 125.$

Ex 9.15: Verify $\forall\, n \in \mathbb{Z}^+$, $\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$

Sol.

$(1 + x)^{2n} = [(1 + x)^n]^2 \;\forall\, n \in \mathbb{Z}^+$:

$\therefore$ the coefficient of x^n in $(1 + x)^{2n} = \binom{2n}{n}$

and the coefficient of x^n in $[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n]^2$

$$= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \ldots + \binom{n}{n}\binom{n}{0}$$

$$= \binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \ldots + \binom{n}{n}\binom{n}{n}$$

$$(\because \binom{n}{i} = \binom{n}{n-i} \;\forall\, n \geq r \geq 0)$$

$$= \sum_{i=0}^{n} \binom{n}{i}^2$$

§ 9.2 Definition and Examples: Calculational Techniques

Ex 9.16: Determine the coefficient of x^8 in $\dfrac{1}{(x-3)(x-2)^2}$

Sol 1.

Note: $1/(x - a) = (-1/a)(1/(1 - (x/a)))$
$= (-1/a)[1 + x/a + (x/a)^2 + \ldots] \ \forall\ a \neq 0$

$$\therefore \frac{1}{(x-3)(x-2)^2} = \left(\frac{-1}{3}\right)\left[1+\left(\frac{x}{3}\right)+\left(\frac{x}{3}\right)^2+\ldots\right]\left(\frac{1}{4}\right)\left[\binom{-2}{0}+\binom{-2}{1}\left(\frac{-x}{2}\right)+\ldots\right]$$

Sol 2. (1/3) Let $\dfrac{1}{(x-3)(x-2)^2} = \dfrac{A}{x-3} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$

$\Rightarrow 1 = A(x-2)^2 + B(x-2)(x-3) + C(x-3)$

$\Rightarrow 0 \cdot x^2 + 0 \cdot x + 1 = (A + B)x^2 + (-4A - 5B + C)x$
$+ (4A + 6B - 3C)$

§ 9.2 Definition and Examples: Calculational Techniques

Ex 9.16: Determine the coefficient of x^8 in $\dfrac{1}{(x-3)(x-2)^2}$

Sol. (2/3)

$$\Rightarrow 0 \cdot x^2 + 0 \cdot x + 1 = (A+B)x^2 + (-4A-5B+C)x + (4A+6B-3C)$$

$$\therefore \begin{cases} A+B=0 \\ -4A-5B+C=0 \\ 4A+6B-3C=1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=-1 \end{cases}$$

$$\therefore \frac{1}{(x-3)(x-2)^2} = \frac{1}{x-3} - \frac{1}{x-2} - \frac{1}{(x-2)^2}$$

$$= \left(\frac{-1}{3}\right)\frac{1}{\left(1-\frac{x}{3}\right)} - \left(\frac{-1}{2}\right)\left(\frac{1}{1-\frac{x}{2}}\right) + \left(\frac{-1}{4}\right)\frac{1}{\left(1-\frac{x}{2}\right)^2}$$

$$\frac{1}{(x-3)(x-2)^2}=\frac{A}{x-3}+\frac{B}{x-2}+\frac{C}{(x-2)^2}\quad \begin{cases}A=1\\ B=-1\\ C=-1\end{cases}$$

Ex 9.16: Determine the coefficient of x^8 in $\dfrac{1}{(x-3)(x-2)^2}$

Sol. (3/3)

$$\therefore\quad \frac{1}{(x-3)(x-2)^2}=\left(\frac{-1}{3}\right)\frac{1}{\left(1-\frac{x}{3}\right)}-\left(\frac{-1}{2}\right)\left(\frac{1}{1-\frac{x}{2}}\right)+\left(\frac{-1}{4}\right)\frac{1}{\left(1-\frac{x}{2}\right)^2}$$

$$=\left(\frac{-1}{3}\right)\sum_{i=0}^{\infty}\left(\frac{x}{3}\right)^i+\left(\frac{1}{2}\right)\sum_{i=0}^{\infty}\left(\frac{x}{2}\right)^i$$

$$+\left(\frac{-1}{4}\right)\left[\binom{-2}{0}+\binom{-2}{1}\left(\frac{-x}{2}\right)+\binom{-2}{2}\left(\frac{-x}{2}\right)^2+\ldots\right]$$

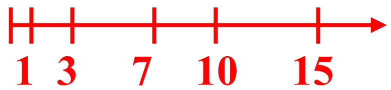
∴ the coefficient of x^8 is : $\left(\frac{-1}{3}\right)\left(\frac{1}{3}\right)^8+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^8+\left(\frac{-1}{4}\right)\binom{-2}{8}\left(\frac{-1}{2}\right)^8$

$$=-\left[\left(\frac{1}{3}\right)^9+7\left(\frac{1}{2}\right)^{10}\right]$$

§ 9.2 Definition and Examples: Calculational Techniques

Ex 9.17: Use G.F. to determine how many four-element subsets of $S = \{1, 2, 3, \ldots, 15\}$ contain no consecutive integers.

Sol. (1/2)



a) ① $\{1, 3, 7, 10\}$

$\Rightarrow 1 \le 1 < 3 < 7 < 10 \le 15 \Leftrightarrow 0, 2, 4, 3, 5 \Rightarrow 0 + 2 + 4 + 3 + 5 = 14$

② $\{2, 5, 11, 15\}$

$\Rightarrow 1 \le 2 < 5 < 11 < 15 \le 15 \Leftrightarrow 1, 3, 6, 4, 0 \Rightarrow 1 + 3 + 6 + 4 + 0 = 14$

Let $c_1 + c_2 + c_3 + c_4 + c_5 = 14$ when $0 \le c_1, c_5$, and $c_2, c_3, c_4 \ge 2$

$\Rightarrow$ Answer = the coef. of x^{14} in

$$f(x) = (1 + x + x^2 + x^3 + \ldots)^2(x^2 + x^3 + x^4 + \ldots)^3$$

= the coef. of x^{14} in $x^6(1 - x)^{-5}$

= the coef. of x^8 in $(1 - x)^{-5}$

$\Rightarrow$ Answer $= \binom{-5}{8}(-1)^8 = \binom{12}{8} = 495$

§ 9.2 Definition and Examples: Calculational Techniques

Ex 9.17: Use G.F. to determine how many four-element subsets of S = {1, 2, 3, …, 15} contain no consecutive integers.

Sol. (2/2)

b) Another way:

① {1, 3, 7, 10}

$\Rightarrow 0 < 1 < 3 < 7 < 10 < 16 \Leftrightarrow 0, 1, 3, 2, 5 \Rightarrow 0 + 1 + 3 + 2 + 5 = 11$

② {2, 5, 11, 15}

$\Rightarrow 0 < 2 < 5 < 11 < 15 < 16 \Leftrightarrow 1, 2, 5, 3, 0 \Rightarrow 1 + 2 + 5 + 3 + 0 = 11$

Let $b_1 + b_2 + b_3 + b_4 + b_5 = 11$ when $0 \le b_1, b_5$, and $b_2, b_3, b_4 \ge 1$

$\Rightarrow$ Answer = the coefficient of x^{11} in $g(x)$

where $g(x) = (1 + x + x^2 + \ldots)(x + x^2 + x^3 + \ldots)^3(1 + x + x^2 + \ldots)$

$= x^3(1 - x)^{-5}$

$\Rightarrow$ Answer = $\binom{-5}{8}(-1)^8 = 495$

§ 9.2 Definition and Examples: Calculational Techniques

Ex 9.19: $\forall\, k \in \mathbb{N}$,

$f(x) = x/(1-x)^2$ is the G.F. for $a_0, a_1, a_2, \ldots, a_k = k$ and

$g(x) = x(x+1)/(1-x)^3$ is the G.F. for $b_0, b_1, b_2, \ldots, b_k = k^2$,

$h(x) = f(x) \cdot g(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \ldots$

$\therefore h(x) = f(x) \cdot g(x)$ is the G.F. for $c_0, c_1, c_2, \ldots$.

$$c_k = a_0b_k + a_1b_{k-1} + a_2b_{k-2} + \ldots + a_{k-2}b_2 + a_{k-1}b_1 + a_kb_0$$

for example: $c_0 = 0 \cdot 0^2 = 0$

$c_1 = 0 \cdot 1^2 + 1 \cdot 0^2 = 0$

$c_2 = 0 \cdot 2^2 + 1 \cdot 1^2 + 2 \cdot 0^2 = 1$

$c_3 = 0 \cdot 3^2 + 1 \cdot 2^2 + 2 \cdot 1^2 + 3 \cdot 0^2 = 6$

$\vdots$

$c_k = \sum_{i=0}^{k} i(k-i)^2$

§ 9.2 Definition and Examples: Calculational Techniques

Def: $c_0, c_1, c_2, \ldots, c_n = \sum_{i=0}^{n} a_i b_{n-i}, \ldots$ from two generating functions $f(x)$ (for $a_0, a_1, \ldots$), $g(x)$ (for $b_0, b_1, \ldots$) is called **convolution** 捲積 of the sequence $a_0, a_1, a_2, \ldots$, and $b_0, b_1, b_2, \ldots$.

Ex 9.20: For $f(x) = 1/(1-x) = 1 + x + x^2 + x^3 + \ldots$

$g(x) = 1/(1+x) = 1 - x + x^2 - x^3 + \ldots$

$$f(x)g(x) = \frac{1}{(1-x)(1+x)} = \frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots$$

∴ 因此, the sequence 1, 0, 1, 0, 1, 0, … is the convolution of the sequence1, 1, 1, … and 1, −1, 1, −1,….

§ 9.2 Definition and Examples: Calculational Techniques

Checklist:

1. **Definition**
   - Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. The function $f(x) = a_0 + a_1x + a_2x^2 + \ldots$ is called the **generating function** for the given sequence

2. **Calculational Techniques**
   - Table 9.2
   - Decomposition and simplification of fractions
   - Convolution

3. **Examples**