Before §8.4 & §8.5

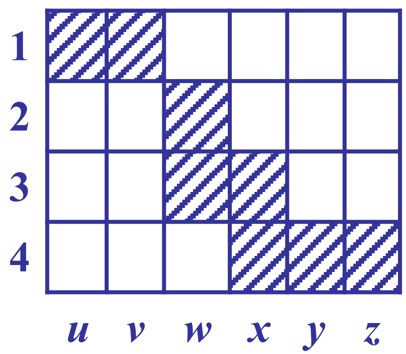
<u>Ex 8.17</u>: $A = \{1, 2, 3, 4\}$, $B = \{u, v, w, x, y, z\}$.
How many 1-1 function $f: A \rightarrow B$ satisfy none of the following conditions: $c_1: f(1) = u$ or v

$c_2: f(2) = w$

$c_3: f(3) = w$ or x

$c_4: f(4) = x, y$ or z

Sol.



$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = S_0 - S_1 + S_2 - S_3 + S_4$$
$$= (6!/2!) - (5!/2! + \ldots)$$
$$+ (4!/2! + \ldots)$$
$$- (3!/2! + \ldots)$$
$$+ (2!/2! + \ldots)$$
$$= ?$$

Computer Science and Information Engineering
National Chi Nan University

Combinatorial Mathematics

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Chapter 8 The Principle of Inclusion and Exclusion

§ 8.4 Rook Polynomials

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

§ 8.4 Rook Polynomials

<u>Outline</u>

1. **Definitions**
2. **Calculate**

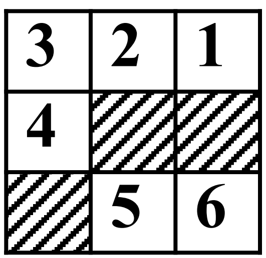
§ 8.4 Rook Polynomials

Def: 1. **rook** (or **castle**): moved horizontally or vertically over any space.

2. r_k (or $r_k(C)$): number of ways in which k rooks can be placed on C s.t. no two of them can take each other.

Note: 1. r_1 = the number of squares on the board.

ex:



$r(C, x) = 1 + 6x + 8x^2$

$r_1(C) = 6$ $(r_0(C) = 1)$

$r_2(C) = |\{\{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 5\}, \{4, 6\}\}| = 8$

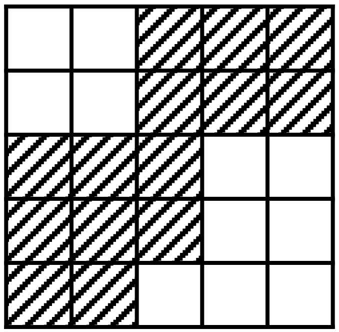
Def: 3. the **rook polynomial** $r(C, x)$

$\equiv \forall\, k \geq 0$, the coefficient of $x^k = r_k(C)$

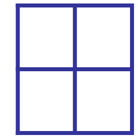
$\Rightarrow$ i.e. $r(C, x) = \sum_{k=0}^{\infty} r_k(C) \cdot x^k$

§ 8.4 Rook Polynomials

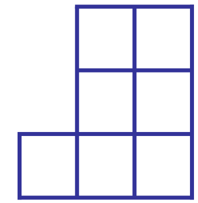
ex:



C



C_1



C_2

$r(C_1, x) = 1 + 4x + 2x^2$

$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$

$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$

$= r(C_1, x) \cdot r(C_2, x)$

Sol.

r_3 for C: (a) 3 from C_2 : $2 \cdot 1 = 2$

(b) 2 from C_2, 1 from C_1: $10 \cdot 4 = 40$

(c) 1 from C_2, 2 from C_1: $7 \cdot 2 = 14$

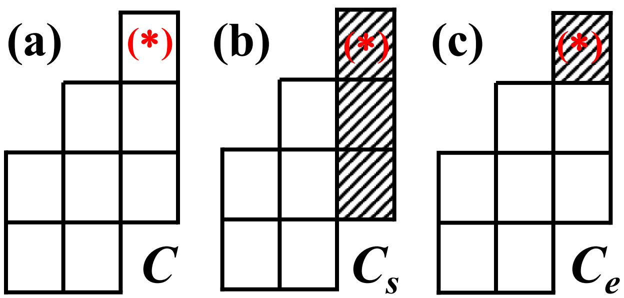
} 56

$56 = 2 \cdot 1 + 10 \cdot 4 + 7 \cdot 2$ = the coefficient of x^3 in $r(C_1, x) \cdot r(C_2, x)$

§ 8.4 Rook Polynomials

Note: If C is a chessboard made up of pairwise disj. subboards $C_1, C_2, \ldots, C_n$, then $r(C, x) = r(C_1, x) \cdot r(C_2, x) \cdot \ldots \cdot r(C_n, x)$.

ex: (a) (b) (c)



$$r_k(C) = r_{k-1}(C_s) + r_k(C_e)$$

(a) Place one on (*) : $r_{k-1}(C_s)$

(b) Do not place on (*) : $r_k(C_e)$

$$\Rightarrow r_k(C)x^k = r_{k-1}(C_s)x^k + r_k(C_e)x^k$$

$$\therefore \sum_{k=1}^{\infty} r_k(C)x^k = x \sum_{k=1}^{\infty} r_{k-1}(C_s)x^{k-1} + \sum_{k=1}^{\infty} r_k(C_e)x^k$$

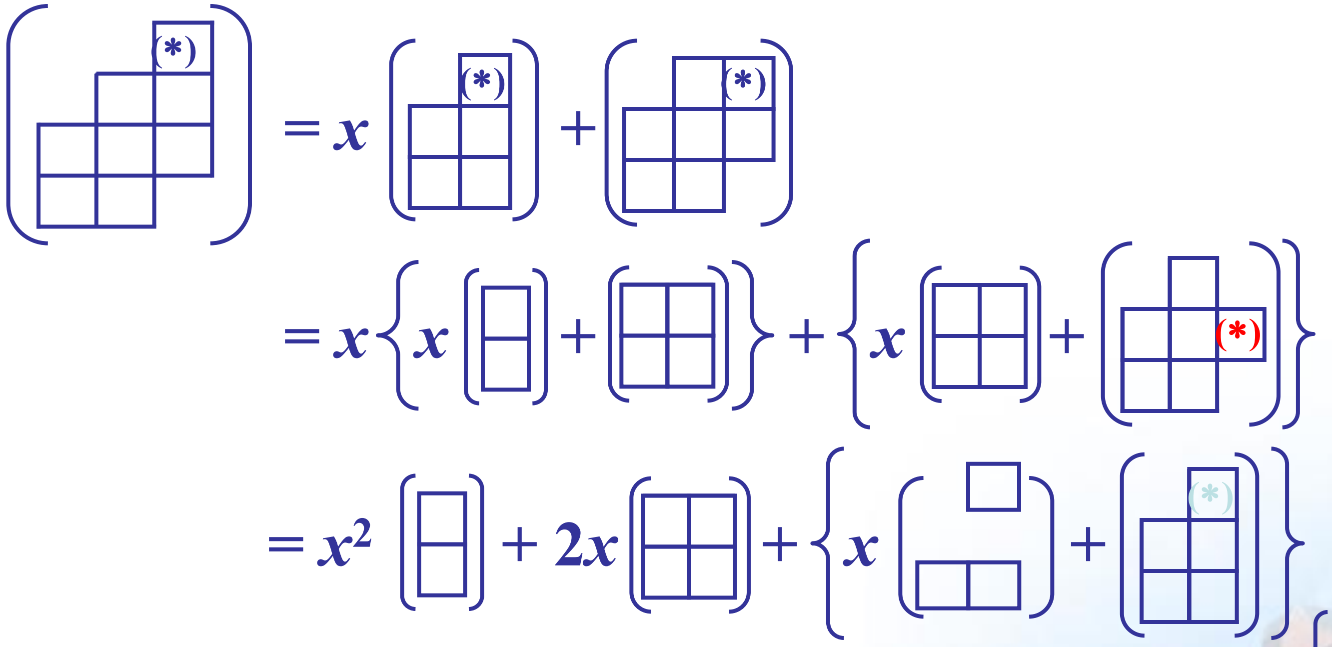
$$\Rightarrow 1 + \sum_{k=1}^{\infty} r_k(C)x^k = x \cdot r(C_s, x) + \sum_{k=1}^{\infty} r_k(C_e)x^k + 1$$

$$\Rightarrow r(C, x) = x \cdot r(C_s, x) + r(C_e, x)$$

§ 8.4 Rook Polynomials

$$r(C, x) = x \cdot r(C_s, x) + r(C_e, x)$$

ex:



$= x^2(1 + 2x) + 2x(1 + 4x + 2x^2) + x(1 + 3x + x^2) +$ 

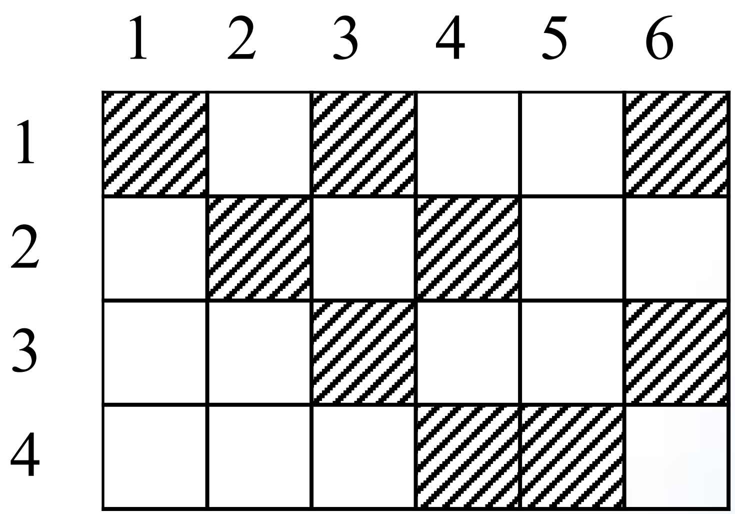
$= 3x + 12x^2 + 7x^3 + x(1 + 2x) + (1 + 4x + 2x^2)$

$= 1 + 8x + 16x^2 + 7x^3$

§ 8.4 Rook Polynomials

Discussion (10 min):

Ex: Find the rook polynomial for the following chessboard.



§ 8.4 Rook Polynomials

Checklist:

1. **Definitions**
   - **rook** (or **castle**)
   - r_k (or $r_k(C)$)
   - **rook polynomial** $r(C, x)$

2. **Calculate**
   - **Factoring (pairwise disjoint):** $r(C, x) = r(C_1, x) \cdot \ldots \cdot r(C_n, x)$.
   - $r_k(C) = r_{k-1}(C_s) + r_k(C_e)$

Computer Science and Information Engineering
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Combinatorial Mathematics

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Chapter 8 The Principle of Inclusion and Exclusion

§ 8.5 Arrangement with Forbidden Positions

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

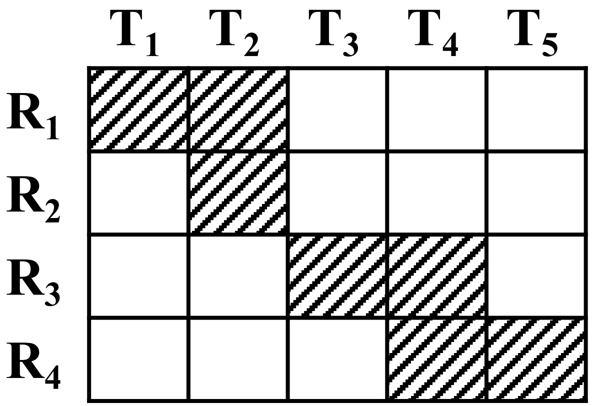
§ 8.5 Arrangement with Forbidden Positions

Outline

1. **Understanding**
2. **Calculate**

§ 8.5 Arrangement with Forbidden Positions

Ex 8.15:



(a) R_1 will not sit at T_1 or T_2.
(b) R_2 will not sit at T_2.
(c) R_3 will not sit at T_3 or T_4.
(d) R_4 will not sit at T_4 or T_5.

Sol. (1/2)

Let condition c_i: R_i in forbidden position.

$S = 5!$

$S_1 = N(c_1) + N(c_2) + N(c_3) + N(c_4)$

$\quad = (4! + 4!) + 4! + (4! + 4!) + (4! + 4!) = 7 \cdot (4!)$

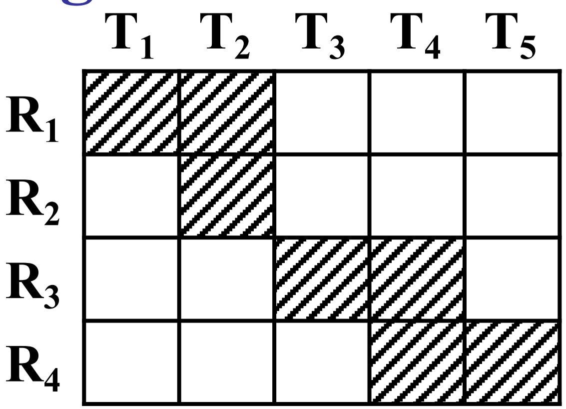
S_2: $N(c_1c_2) = 3!$ $\qquad N(c_1c_3) = 3! + 3! + 3! + 3!$

$\vdots \qquad\qquad \vdots$

$S_2 = 16 \cdot (3!)$

§ 8.5 Arrangement with Forbidden Positions

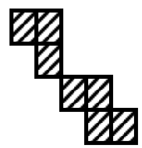
Ex 8.15:



(a) R_1 will not sit at T_1 or T_2.
(b) R_2 will not sit at T_2.
(c) R_3 will not sit at T_3 or T_4.
(d) R_4 will not sit at T_4 or T_5.

Sol. (2/2)

$$\left.\begin{array}{l}\Rightarrow 7 = r_1(C) \\ \quad 16 = r_2(C)\end{array}\right\} \quad C:$$



$$\Rightarrow \forall\ 0 \le i \le 4,\ S_i = r_i(C) \cdot (5-i)!$$

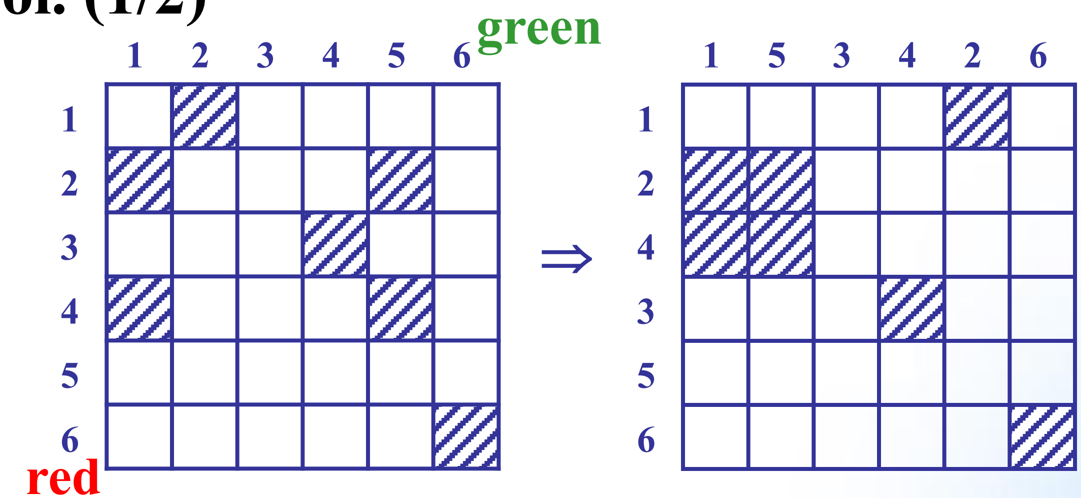
$$\because r(C, x) = (1 + 3x + x^2)(1 + 4x + 3x^2)$$
$$= 1 + 7x + 16x^2 + 13x^3 + 3x^4$$

$$\therefore N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = S_0 - S_1 + S_2 - S_3 + S_4$$
$$= 5! - 7(4!) + 16(3!) - 13(2!) + 3(1!)$$
$$= \sum_{i=0}^{4}(-1)^i r_i (5-i)! = 25$$

§ 8.5 Arrangement with Forbidden Positions

red green
(a, b)

Ex 8.16: two dice: one red; other green.
row six times.
If (1, 2), (2,1), (2, 5), (3, 4), (4, 1), (4, 5), (6, 6) did not occur, what's the probability that we obtain all six values on both dices?
Sol. (1/2)



$s(1) = \{2\}$
$s(2) = \{1, 5\}$
$s(3) = \{4\}$
$s(4) = \{1, 5\}$
$s(5) = \{\}$
$s(6) = \{6\}$

$$r(C, x) = (1 + 4x + 2x^2)(1 + x)^3 = 1 + 7x + 17x^2 + 19x^3 + 10x^4 + 2x^5$$

§ 8.5 Arrangement with Forbidden Positions

red green

(a, b)

Ex 8.16: two dice: one red; other green.
row six times.
If (1, 2), (2,1), (2, 5), (3, 4), (4, 1), (4, 5), (6, 6) did not occur, what's the probability that we obtain all six values on both dices?
Sol. (2/2)

$\forall\ 1 \le i \le 6$, let condition c_i: $(i, x), x \in s(i)$ occur.

$$\therefore 6!N(\bar{c}_1\bar{c}_2\ldots\bar{c}_6) = (6!)\sum_{i=0}^{6}(-1)^i S_i = (6!)\sum_{i=0}^{6}(-1)^i r_i \cdot (6-i)!$$

$$= 6![6! - 7(5!) + 17(4!) - 19(3!) + 10(2!) - 2(1!) + 0(0!)]$$

$$= 6!(192) = 138240$$

$\because |S| = 29^6$

$\Rightarrow 138240/(29)^6 \fallingdotseq 0.00023.$

§ 8.5 Arrangement with Forbidden Positions

Ex 8.17: $A = \{1, 2, 3, 4\}$, $B = \{u, v, w, x, y, z\}$.
How many 1-1 function $f: A \to B$ satisfy none of the following conditions:

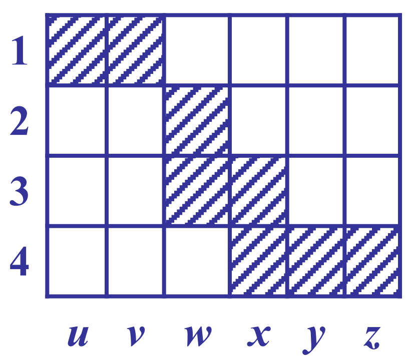
$c_1: f(1) = u$ or v

$c_2: f(2) = w$

$c_3: f(3) = w$ or x

$c_4: f(4) = x, y$ or z

Sol.



$$r(C, x) = (1 + 2x)(1 + 6x + 9x^2 + 2x^3)$$
$$= 1 + 8x + 21x^2 + 20x^3 + 4x^4$$
$$\therefore N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = S_0 - S_1 + S_2 - S_3 + S_4$$
$$= (6!/2!) - 8(5!/2!) + 21(4!/2!) - 20(3!/2!) + 4(2!/2!)$$
$$= \sum_{i=0}^{4}(-1)^i r_i (6 - i)!/2! = 76$$

§ 8.5 Arrangement with Forbidden Positions

ex: $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

How many 1-1 function $h: A \to A$ **where** $h(i) \neq i, \forall i \in A$.

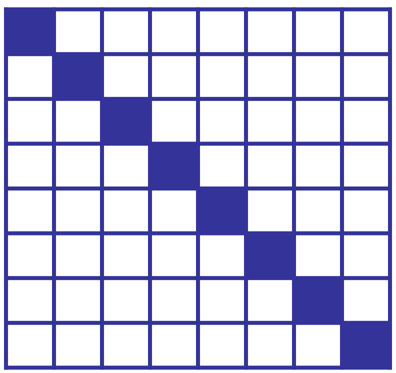
Sol.

$$r(C, x) = (1 + x)^8 = \sum_{k=0}^{8} \binom{8}{k} x^k$$

$$\therefore N(\bar{c}_1\bar{c}_2\ldots\bar{c}_8) = \binom{8}{0}8! - \binom{8}{1}7! + \binom{8}{2}6! - \ldots + \binom{8}{8}0!$$

$$= 8![1 - 1 + 1/(2!) - 1/(3!) + \ldots + 1/(8!)]$$

$$= d_8$$



§ 8.5 Rook Polynomials

Discussion (10 min):

Exercise 8.5.11: A computer dating service wants to match each of four women with one of six men. According to the information these applicants provided when they joined the service, we can draw the following conclusions.

- **Woman 1 would not be compatible with man 1, 3, or 6.**
- **Woman 2 would not be compatible with man 2 or 4.**
- **Woman 3 would not be compatible with man 3 or 6.**
- **Woman 4 would not be compatible with man 4 or 5.**

In how many ways can the service successfully match each of the five women with a compatible partner?

§ 8.5 Arrangement with Forbidden Positions

<u>Checklist:</u>

1. Understanding

- ☐ **Define "conditions" correctly**

2. Calculate

- ☐ **The Principle of Inclusion and Exclusion**
- ☐ **Rook polynomial**

§ 8.5 Rook Polynomials

Let’s Kahoot!

https://play.kahoot.it/v2/?quizId=d9638913-594f-4831-9fba-b88cde64235a&hostId=e3b5c5c7-c22d-4353-a580-53c46d132332

(modified from holder1149**)**

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Chapter 9 Generating Functions

§ 9.1 Introductory Examples

Slides for a Course Based on the Text
***Discrete & Combinatorial Mathematics* (5th Edition)**
by Ralph P. Grimaldi

§ 9.1 Introductory Examples

Outline

1. **Observation**
2. **Definition**

§ 9.1 Introductory Examples

$c_1 + c_2 + c_3 + c_4 = 25$, with $0 \le c_i \quad \forall\ 1 \le i \le 4$ → Chap 1

$c_1 + c_2 + c_3 + c_4 = 25$, with $0 \le c_i < 10\ \forall\ 1 \le i \le 4$ → Chap 8

$c_1 + c_2 + c_3 + c_4 = 25$, with $0 \le c_i < 10\ \forall\ 1 \le i \le 4$

and c_2: even, c_3: a multiple of 3

→ Generating Function

§ 9.1 Introductory Examples

Ex 9.1: 12 oranges: Grace at least 4, Mary at least 2, Frank at least 2, no more then 5.

Sol.

Grace	4	4	4	4	5	5	5	5	6	6	6	7	7	8
Mary	3	4	5	6	2	3	4	5	2	3	4	2	3	2
Frank	5	4	3	2	5	4	3	2	4	3	2	3	2	2

$\Rightarrow c_1 + c_2 + c_3 = 12$ where $4 \le c_1$, $2 \le c_2$, $2 \le c_3 \le 5$.

$$f(x) = (x^4 + x^5 + x^6 + x^7 + x^8)\,(x^2 + x^3 + x^4 + x^5 + x^6)\,(x^2 + x^3 + x^4 + x^5)$$

$$4 + 3 + 5 = 12 \;\rightarrow\; x^4 \cdot x^3 \cdot x^5$$

$$4 + 4 + 4 = 12 \;\rightarrow\; x^4 \cdot x^4 \cdot x^4$$

$$\vdots \qquad\qquad \vdots$$

the number of distributions = the coefficient of x^{12} in $f(x)$

$f(x)$ is called a **generating function** for the distribution

§ 9.1 Introductory Examples

why...

Note: 1. $(x^4 + \ldots + x^8)(x^2 + \ldots + x^6)(x^2 + \ldots + x^5)$

2. Table faster than G.F.?

Ex 9.2: red, green, white, black jelly beans

How many way can Douglas select 24 of these so that

$\begin{cases} \text{white: even} \\ \text{black: at least 6} \end{cases}$

Sol.

red, green: $(1 + x + x^2 + \ldots + x^{24})$

white: $(1 + x^2 + x^4 + x^6 + \ldots + x^{24})$

black: $(x^6 + x^7 + x^8 + \ldots + x^{24})$

$f(x) = (1 + x + \ldots + x^{24})^2(1 + x^2 + x^4 + \ldots + x^{24})(x^6 + x^7 + \ldots + x^{24})$

Answer = the coefficient of x^{24} in $f(x)$.

§ 9.1 Introductory Examples

Ex 9.3: $c_1 + c_2 + c_3 + c_4 = 25$, if $0 \le c_i$ for all $1 \le i \le 4$. How many solutions?

Sol.

(1) Let $f(x) = (1 + x + x^2 + \ldots + x^{25})^4$ → **polynomial in x**

the coefficient of x^{25} in $f(x)$

(2) for easier to compute: → **power series in x**

Let $g(x) = (1 + x + x^2 + \ldots + x^{25} + x^{26} + \ldots)^4$ **冪級數**

Note: $\forall\ k \ge 26$, x^k are never need.

§ 9.1 Introductory Examples

Checklist:

1. **Observation**
   - ☐ The relationship between the way to find the answer of the problem and the way to calculate some coefficient of a function.
2. **Definition**
   - ☐ Find a generating function for the answer of the problem.

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Chapter 9 Generating Functions

§ 9.2 Definition and Examples: Calculational Techniques (1)

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***Discrete & Combinatorial Mathematics* (5th Edition)**
by Ralph P. Grimaldi

§ 9.2 Definition and Examples: Calculational Techniques

Def 9.1: Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers.
The function $f(x) = a_0 + a_1x + a_2x^2 + \ldots = \sum_{i=0}^{\infty} a_i x^i$
is called the **generating function** for the given sequence.

Ex 9.4: $\forall\, n \in \mathbb{Z}^+, (1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$
$\Rightarrow (1 + x)^n$ is the generating function for the sequence
$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}, 0, 0, 0, \ldots.$

Ex 9.5: a) $\forall\, n \in \mathbb{Z}^+, (1 - x^{n+1}) = (1 - x)(1 + x + x^2 + \ldots + x^n)$
$\Rightarrow (1 - x^{n+1})/(1 - x)$ is the generating function for the
sequence $\underbrace{1, 1, 1, \ldots, 1}_{n+1\text{ 個}}, 0, 0, 0, \ldots.$