Computer Science and Information Engineering
National Chi Nan University

Combinatorial Mathematics

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Chapter 8 The Principle of Inclusion and Exclusion

§ 8.1 The Principle of Inclusion and Exclusion

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5^th^ Edition)**
by Ralph P. Grimaldi

§ 8.1 The Principle of Inclusion and Exclusion

<u>Outline</u>

1. **Symbols**
2. **The Principle of Inclusion and Exclusion**
3. **Application**

§ 8.1 The Principle of Inclusion and Exclusion

Recall: https://www.youtube.com/watch?v=vVZwe3TCJT8

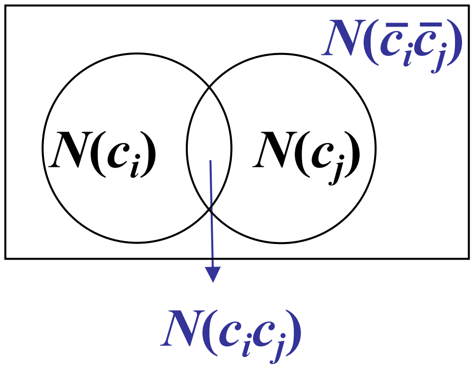
Def: Let S be a set; $|S| = N$.

1) $N_i = \{x \in S \mid x$ satisfy condition $c_i\}$, where $i = 1, 2, \ldots, t$.
2) $N(c_i) = |N_i|$, where $i = 1, 2, \ldots, t$.
3) $N(c_ic_j)$ $(= |\{x \mid x \in N_i \cap N_j\}|) = |N_i \cap N_j|$, $\forall\ i \neq j$, and $i, j \in \{1, 2, \ldots, t\}$.
4) $N(c_ic_jc_k) = |N_i \cap N_j \cap N_k|$, $\forall\ i \neq j \neq k$, and $i, j, k \in \{1, 2, \ldots, t\}$.
5) $N(\bar{c}_i) = N - N(c_i) = |\{x \in S \mid x \notin N_i\}|$, $\forall\ 1 \leq i \leq t$.
6) $N(\bar{c}_i\bar{c}_j) = |\{x \in S \mid x \notin N_i \cup N_j\}|$, $\forall\ i \neq j$, $\forall\ 1 \leq i, j \leq t$.

§ 8.1 The Principle of Inclusion and Exclusion

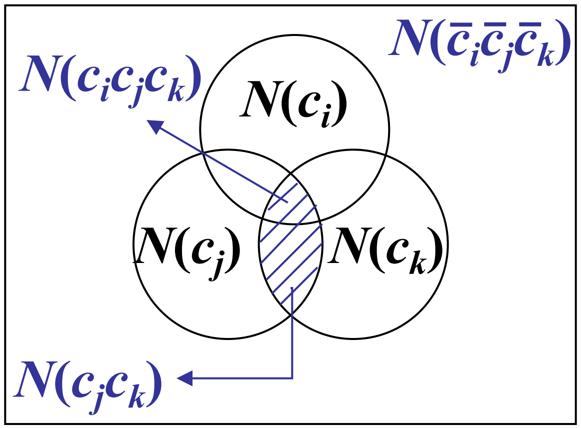
Note: $N(\bar{c}_i\bar{c}_j) \neq N(\overline{c_i c_j})$

Ex 8.1:



$$N(\bar{c}_i\bar{c}_j) = N - [N(c_i) + N(c_j)] + N(c_i c_j)$$

Ex 8.2:



$$N(\bar{c}_i\bar{c}_j\bar{c}_k) = N - [N(c_i) + N(c_j) + N(c_k)] + [N(c_i c_j) + N(c_j c_k) + N(c_i c_k)] - N(c_i c_j c_k)$$

§ 8.1 The Principle of Inclusion and Exclusion

Thm 8.1: **The Principle of Inclusion and Exclusion** (排容、容斥、包除原理)

$$\overline{N} = N(\bar{c}_1\bar{c}_2\ldots\bar{c}_t)$$

$$= N - [N(c_1) + N(c_2) + \ldots + N(c_t)]$$

$$+ [N(c_1c_2) + N(c_1c_3) + \ldots + N(c_1c_t) + N(c_2c_3) + \ldots + N(c_{t-1}c_t)]$$

$$- [N(c_1c_2c_3) + N(c_1c_2c_4) + \ldots + N(c_1c_2c_t) + N(c_1c_3c_4) + \ldots$$

$$+ N(c_1c_3c_t) + \ldots + N(c_{t-2}c_{t-1}c_t)]$$

$$+ \ldots$$

$$+ (-1)^t N(c_1c_2\ldots c_t) \quad \ldots \text{Eq(1)}$$

$$= N - \sum_{1 \le i \le t} N(c_i) + \sum_{1 \le i < j \le t} N(c_ic_j) - \sum_{1 \le i < j < k \le t} N(c_ic_jc_k) + \ldots + (-1)^t N(c_1c_2\ldots c_t) \quad \ldots \text{Eq(2)}$$

Proof. (1/3)

1) By induction on *t*.

$$N - \sum_{1\le i\le t} N(c_i) + \sum_{1\le i<j\le t} N(c_ic_j) - \sum_{1\le i<j<k\le t} N(c_ic_jc_k) + \dots + (-1)^t N(c_1c_2\dots c_t)$$

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Proof. (2/3)

2) Combinatorial argument:

$\forall\, x \in S$:

<u>case 1</u>: x satisfy none of the condition:

$$\Rightarrow x \in S - \bigcup_{i=1}^{t} N_i \wedge x \in S$$

$\therefore$ x is counted once in N and $\overline{N}$, and not in any of the terms in Eq(2).

$\therefore$ x contributes a count of 1 to each side of the equation.

<u>case 2</u>: x satisfy exactly r of the conditions; when $1 \le r \le t$.

$$\Rightarrow x \notin S - \bigcup_{i=1}^{t} N_i$$

$\therefore$ x is contributes nothing to $\overline{N}$,

§ 8.1 The Principle of Inclusion and Exclusion

Proof. (3/3)

On the right hand side of Eq(2), x is counted:

$$(1)\ N \Rightarrow 1$$

$$(2)\ \sum_{1 \le i \le t} N(c_i) \Rightarrow r$$

$$(3)\ \sum_{1 \le i < j \le t} N(c_i c_j) \Rightarrow \binom{r}{2}$$

$$(4)\ \sum_{1 \le i < j < k \le t} N(c_i c_j c_k) \Rightarrow \binom{r}{3}$$

$$\vdots$$

$$(r+1)\ \sum N(c_{i_1} c_{j_2} \ldots c_{i_r}) \Rightarrow \binom{r}{r}$$

$$\Rightarrow 1 - r + \binom{r}{2} - \binom{r}{3} + \ldots + (-1)^r \binom{r}{r} = [1 + (-1)]^r = 0^r = 0$$

$\therefore$ x contributes 0 to each side of the equation.

$\therefore$ the equality is verified.

§ 8.1 The Principle of Inclusion and Exclusion

Corollary 8.1: The number of elements in S that satisfy at least one of the condition c_i, where $1 \le i \le t$, is given by

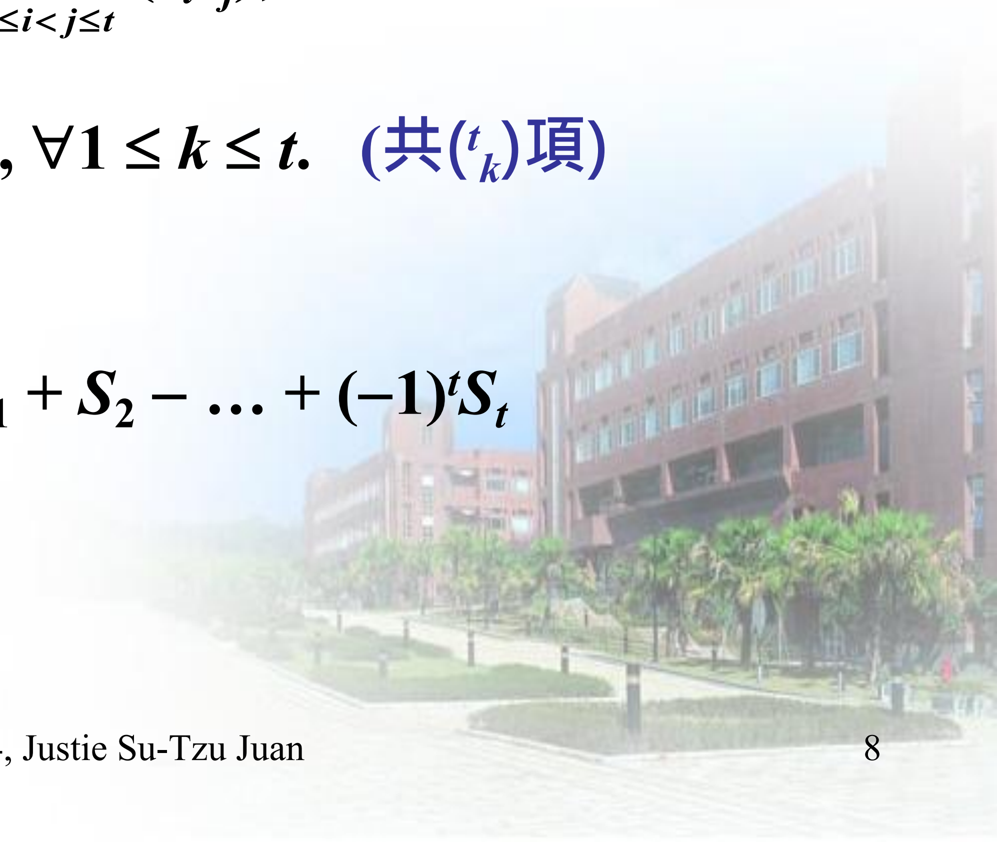
$$N(c_1 \text{ or } c_2 \text{ or } \dots \text{ or } c_t) = N - \overline{N}.$$

Def: $S_0 = N$; $S_1 = \sum_{1 \le i \le t} N(c_i)$; $S_2 = \sum_{1 \le i < j \le t} N(c_i c_j)$; …

$$S_k = \sum_{1 \le i_1 < i_2 < \dots < i_k \le t} N(c_{i_1} c_{i_2} \dots c_{i_k}),\ \forall 1 \le k \le t. \quad (\text{共}\tbinom{t}{k}\text{項})$$

Note: By Thm 8.1, $\overline{N} = S_0 - S_1 + S_2 - \dots + (-1)^t S_t$

$$= \sum_{k=0}^{t} (-1)^k S_k$$



§ 8.1 The Principle of Inclusion and Exclusion

Ex 8.4: $1 \leq n \leq 100$, n is not divisible by 2, 3, or 5. How many such n ?

Sol.

Let $S = \{1, 2, \ldots, 100\}$, $N = 100$.

Let condition c_1: if n is divisible by 2;

Let condition c_2: if n is divisible by 3;

Let condition c_3: if n is divisible by 5.

By Thm 8.1,

$$\bar{N} = N(\bar{c}_1\bar{c}_2\bar{c}_3) = N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$$

$$= 100 - \{\lfloor 100/2 \rfloor + \lfloor 100/3 \rfloor + \lfloor 100/5 \rfloor\} + \{\lfloor 100/6 \rfloor + \lfloor 100/10 \rfloor + \lfloor 100/15 \rfloor\} - \lfloor 100/30 \rfloor$$

$$= 26$$

§ 8.1 The Principle of Inclusion and Exclusion

Ex 8.5: $\begin{cases} x_1 + x_2 + x_3 + x_4 = 18; \\ 0 \le x_i \le 7 \text{ and } x_i \in \mathbb{N};\ \forall\ 1 \le i \le 4 \end{cases}$

Sol.

Let condition c_i: $x_i > 7$ (or $x_i \ge 8$), $i = 1, 2, 3, 4$

$$\therefore \overline{N} = N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4)$$

$$= N - [N(c_1) + N(c_2) + N(c_3) + N(c_4)]$$

$$+ [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)]$$

$$- [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] + N(c_1c_2c_3c_4)$$

$$= S_0 - S_1 + S_2 - S_3 + S_4$$

$$= C(18 + 4 - 1, 18) - \binom{4}{1}C(10 + 4 - 1, 10)$$

$$+ \binom{4}{2}C(2 + 4 - 1, 2) - \binom{4}{3} \cdot 0 + \binom{4}{4} \cdot 0$$

$$= 246$$

§ 8.1 The Principle of Inclusion and Exclusion

Ex 8.6: counting onto function:

$A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_n\}$

$S = \{ f \mid f: A \to B\}$, $\mathbb{N} = |S| = n^m$

Let condition c_i: if b_i is not in the range of f for $1 \le i \le n$.

$\therefore \overline{N} = N(\bar{c}_1\bar{c}_2\ldots\bar{c}_n)$ = the number of onto function $f: A \to B$.

Sol.

$$S_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} N(c_{i_1}, c_{i_2}, \ldots, c_{i_k}) = \binom{n}{k}(n-k)^m$$

$$\therefore \overline{N} = N - S_1 + S_2 - \ldots + (-1)^n S_n$$

$$= n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m + \ldots + (-1)^n(n-n)^m$$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i}(n-i)^m = n!\, S(m, n)$$

Note: If $n > m \Rightarrow \overline{N} = 0$

$\therefore \forall m, n \in \mathbb{Z}^+$, if $n > m$, then $\sum_{i=0}^{n} (-1)^i \binom{n}{i}(n-i)^m = 0$

§ 8.1 The Principle of Inclusion and Exclusion

<u>Ex 8.7</u>: In how many ways can the 26 letters of the alphabet be permuted so that none of the patterns *car*, *dog*, *pun*, or *byte* occurs?

Sol.

Let $S = \{a \mid a \text{ is a permutation of the 26 letters}\}$.

$\therefore N = |S| = 26!$

Let condition c_1: contains *car*;

Let condition c_2: contains *dog*;

Let condition c_3: contains *pun*;

Let condition c_4: contains *byte*;

$$\therefore \overline{N} = N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = N - S_1 + S_2 - S_3 + S_4$$

$$= 26! - [24! + 24! + 24! + 23!] + [22! + 22! + 22! + 21! + 21! + 21!] - [20! + 3(19!)] + 17!$$

§ 8.1 The Principle of Inclusion and Exclusion

Ex 8.8: (**Euler's phi function**)

For $n \in \mathbb{Z}^+$, $n \geq 2$, Let $\phi(n) = |\{m \in \mathbb{Z}^+ \mid \gcd(m, n) = 1, 1 \leq m < n\}|$

$$\phi(n) = n \prod_{p|n,\ p \text{ is a prime}} (1 - (1/p))$$

Sol. $\forall n \geq 2$, $n = p_1^{e_1} p_2^{e_2} \ldots p_t^{e_t}$, where p_i: prime, $e_i \in \mathbb{Z}^+$, $\forall 1 \leq i \leq t$.

Let $S = \{1, 2, \ldots, n\}$, $N = n$.

(say $t = 4$: condition c_i: if divisible by p_i, $i = 1, 2, 3, 4$.

$\therefore \phi(n) = N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = S_0 - S_1 + S_2 - S_3 + S_4$

$$= n - [\frac{n}{p_1} + \frac{n}{p_2} + \frac{n}{p_3} + \frac{n}{p_4}] + [\frac{n}{p_1p_2} + \frac{n}{p_1p_3} + \frac{n}{p_1p_4} + \frac{n}{p_2p_3} + \frac{n}{p_2p_4} + \frac{n}{p_3p_4}]$$

$$- [\frac{n}{p_1p_2p_3} + \frac{n}{p_1p_2p_4} + \frac{n}{p_1p_3p_4} + \frac{n}{p_2p_3p_4}] + \frac{n}{p_1p_2p_3p_4}$$

$$= \frac{n}{p_1p_2p_3p_4}\{p_1p_2p_3p_4 - [p_2p_3p_4 + p_1p_3p_4 + p_1p_2p_4 + p_1p_2p_3]$$

$$+ [p_3p_4 + p_2p_4 + p_2p_3 + p_1p_4 + p_1p_3 + p_1p_2] - [p_4 + p_3 + p_2 + p_1] + 1\}$$

$$= n\,(\frac{p_1-1}{p_1} \cdot \frac{p_2-1}{p_2} \cdot \frac{p_3-1}{p_3} \cdot \frac{p_4-1}{p_4}) = n \prod_{i=1}^{4} (1 - \frac{1}{p_i})\ )$$

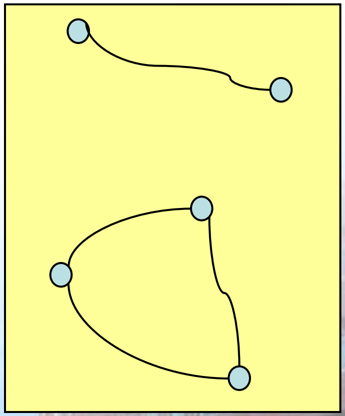
In general, $\phi(n) = n \prod_{p|n,\ p \text{ is a prime}} (1 - (1/p))$

§ 8.1 The Principle of Inclusion and Exclusion

ex: $\phi(23100) = \phi(2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11)$
$= 23100\,(1 - 1/2)(1 - 1/3)(1 - 1/5)(1 - 1/7)(1 - 1/11)$
$= 4800$

Ex 8.10: five villages a, b, c, d, e devise a system of 2-way road, s.t. no village will be isolated.

Sol.



Let $S = \{G \mid V(G) = \{a, b, c, d, e\}\}$.
$\therefore N = |S| = 2^{C(5, 2)} = 2^{10}$
Let condition c_i: a (b, c, d, e) be isolated.
$\therefore N(c_i) = 2^{C(4, 2)} = 2^6, \forall 1 \le i \le 5.$
$\Rightarrow S_1 = \binom{5}{1} \cdot 2^6$
$\Rightarrow \overline{N} = 2^{10} - \binom{5}{1} \cdot 2^6 + \binom{5}{2} \cdot 2^3 - \binom{5}{3} \cdot 2^1 + \binom{5}{4} \cdot 2^0 - \binom{5}{5} \cdot 2^0$
$= 768$

§ 8.1 The Principle of Inclusion and Exclusion

Discussion:

Ex 8.1.8: Determine the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 19$ where $-5 \leq x \leq 10$ for $1 \leq i \leq 4$.

§ 8.1 The Principle of Inclusion and Exclusion

Checklist:

1. Symbols

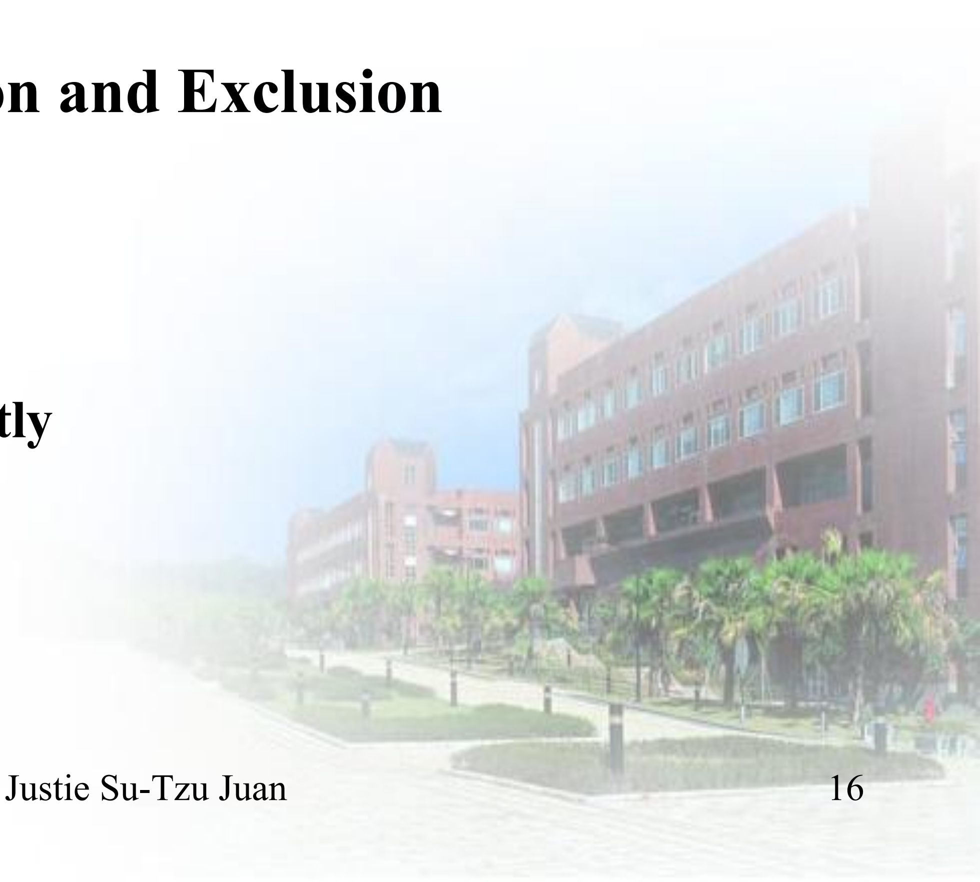
- ☐ N_i, $N(c_i)$, $N(c_ic_j)$, $N(c_ic_jc_k)$, $N(\bar{c}_i)$, $N(\bar{c}_i\bar{c}_j)$, $\overline{N}$
- ☐ $N(c_1$ or c_2 or … or $c_t)$, S_0, S_1, …, S_k

2. The Principle of Inclusion and Exclusion

- ☐ Proof

3. Application

- ☐ Define "conditions" correctly
- ☐ Euler's phi function $\phi(n)$



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Chapter 8 The Principle of Inclusion and Exclusion

§ 8.2 Generalizations of the Principle

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§ 8.2 Generalizations of the Principle

Outline

1. **Symbols**
2. **Thm. 8.2**
3. **Cor. 8.2**

§ 8.2 Generalizations of the Principle

Def: $m \in \mathbb{Z}^+$, $1 \leq m \leq t$.

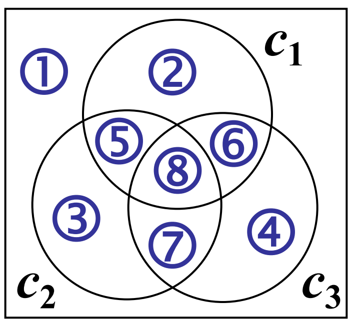
E_m: # of elements in S that satisfy exactly m of the t conditions.

$E_0 = \bar{N}$

$E_1 = N(c_1\bar{c}_2\bar{c}_3\ldots\bar{c}_t) + N(\bar{c}_1c_2\bar{c}_3\ldots\bar{c}_t) + \ldots + N(\bar{c}_1\bar{c}_2\bar{c}_3\ldots c_t)$

$E_2 = N(c_1c_2\bar{c}_3\ldots\bar{c}_t) + N(\bar{c}_1c_2c_3\ldots\bar{c}_t) + \ldots + N(\bar{c}_1\bar{c}_2\bar{c}_3\ldots\bar{c}_{t-2}c_{t-1}c_t)$

ex: $t = 3$



②+③+④

‖

$E_1 = S_1 - 2S_2 + 3S_3 = S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3$

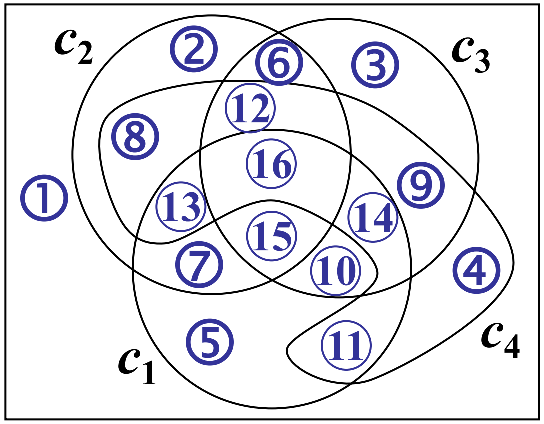
= (②+⑤+⑥+⑧)+(③+⑤+⑧+⑦)+(④+⑥+⑦+⑧)

−2 (⑤+⑧+⑥+⑧+⑦+⑧)+3 ⑧

$E_2 = S_2 - 3S_3 = S_2 - \binom{3}{1}S_3$

§ 8.2 Generalizations of the Principle

ex: $t = 4$



$$E_1 = S_1 - 2S_2 + 3S_3 - 4S_4$$
$$= S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3 - \binom{4}{3}S_4$$

$$E_2 = S_2 - 3S_3 + 6S_4 = S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4$$

§ 8.2 Generalizations of the Principle

Thm 8.2: $\forall\ 1 \le m \le t,\ E_m = \sum_{i=0}^{t-m} (-1)^i \binom{m+i}{i} S_{m+i}$

$$(E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \ldots + (-1)^{t-m} \binom{t}{t-m} S_t)$$

Proof. (1/2) skip

Let $x \in S$:

a) x satisfy fewer than m condition:

left: **0**;

right: $S_m, S_{m+1}, \ldots, S_t = 0$ $\Rightarrow$ **0**

b) x satisfy exactly m of the condition:

left: **1**;

right: $S_m = 1;\ S_{m+1}, \ldots, S_t = 0$ $\Rightarrow$ **1**

$$E_m = S_m - \binom{m+1}{1}S_{m+1} + \binom{m+2}{2}S_{m+2} - \ldots + (-1)^{t-m}\binom{t}{t-m}S_t$$

§ 8.2 Generalizations of the Principle

Proof. (2/2) skip

c) x satisfy r of the conditions, where $m < r \le t$:

left: **0**;

right: $S_m = \binom{r}{m}$, $S_{m+1} = \binom{r}{m+1}$; …, $S_r = \binom{r}{r}$; $S_{r+1} = \ldots = S_t = 0$

$$\Rightarrow \binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \binom{m+2}{2}\binom{r}{m+2} - \ldots + (-1)^{r-m}\binom{r}{r-m}\binom{r}{r}$$

$$\forall 0 \le k \le r-m: \binom{m+k}{k}\binom{r}{m+k} = \frac{(m+k)!}{m!\,k!}\frac{r!}{(m+k)!(r-m-k)!}$$

$$= \frac{r!}{m!}\frac{1}{k!(r-m-k)!} = \frac{r!}{m!(r-m)!}\frac{(r-m)!}{k!(r-m-k)!}$$

$$= \binom{r}{m}\binom{r-m}{k}$$

$$\Rightarrow \binom{r}{m}\binom{r-m}{0} - \binom{r}{m}\binom{r-m}{1} + \binom{r}{m}\binom{r-m}{2} - \ldots + (-1)^{r-m}\binom{r}{m}\binom{r-m}{r-m}$$

$$= \binom{r}{m}\left[\binom{r-m}{0} - \binom{r-m}{1} + \binom{r-m}{2} + \ldots + (-1)^{r-m}\binom{r-m}{r-m}\right]$$

$$= \binom{r}{m}(1-1)^{r-m} = \binom{r}{m}\cdot 0 = 0. \qquad \Rightarrow \mathbf{0}$$

∴ the formula is verified.

$E_m = S_m - \binom{m+1}{1}S_{m+1} + \binom{m+2}{2}S_{m+2} - \ldots + (-1)^{t-m}\binom{t}{t-m}S_t$

§ 8.2 Generalizations of the Principle

Def: $L_m = |\{x \in S \mid x \text{ satisfy at least } m \text{ of the } t \text{ conditions}\}| = \sum_{i=m}^{t} E_i$

Corollary 8.2:

$$L_m = S_m - \binom{m}{m-1}S_{m+1} + \binom{m+1}{m-1}S_{m+2} - \ldots + (-1)^{t-m}\binom{t-1}{m-1}S_t$$

Proof. exercises

Note: If $m = 1$, $L_1 = S_1 - S_2 + S_3 - \ldots + (-1)^{t-1}S_t = N - \overline{N} = |S| - \overline{N}$

Ex 8.11: 在 Ex 8.10 中: E_2 及 L_2?

Sol.

$E_2 = S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4 - \binom{5}{3}S_5 = 80 - 3(20) + 6(5) - 10(1) = 40$

$L_2 = S_2 - \binom{2}{1}S_3 + \binom{3}{1}S_4 - \binom{4}{1}S_5 = 80 - 2(20) + 3(5) - 4(1) = 51$

Ex 8.10: five villages *a*, *b*, *c*, *d*, *e*. devise a system of 2-way road, s.t. no village will be isolated.

§ 8.2 Generalizations of the Principle

Discussion:

Ex 8.2.7: If 13 cards are dealt from a standard deck of 52, what is the probability that these 13 cards include

(a) at least one card from each suit;

(b) exactly one void (for example, no clubs);

(c) exactly two voids?

§ 8.2 Generalizations of the Principle

Checklist:

1. Symbols
   - E_m
   - L_m

2. Thm. 8.2
   - $E_m = S_m - \binom{m+1}{1}S_{m+1} + \binom{m+2}{2}S_{m+2} - \ldots + (-1)^{t-m}\binom{t}{t-m}S_t$

3. Cor. 8.2
   - $L_m = S_m - \binom{m}{m-1}S_{m+1} + \binom{m+1}{m-1}S_{m+2} - \ldots + (-1)^{t-m}\binom{t-1}{m-1}S_t$

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Chapter 8 The Principle of Inclusion and Exclusion

§ 8.3 Derangements: Nothing Is in Its Right Place

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§ 8.3 Derangements: Nothing Is in Its Right Place

<u>Outline</u>

1. **Maclaurin series for e^x**
2. **The definition of derangement**
3. **Application**

§ 8.3 Derangements: Nothing Is in Its Right Place

Recall: Maclaurin series for the exponential function:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Note: $\therefore e^{-1} = \sum_{i=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots$

$\because e^{-1} \fallingdotseq 0.3678794411714423215955237\,7$

and $1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots - (\frac{1}{7!}) \fallingdotseq 0.36786$

$\Rightarrow \forall k \in \mathbb{Z}^+$, if $k \geq 7$ then

$\sum_{n=0}^{k} \frac{(-1)^n}{n!}$ is a very good approximation to e^{-1}.

§ 8.3 Derangements: Nothing Is in Its Right Place

Def: The **derangement** of 1, 2, …, n is an arrangement of 1, 2, …, n so that i is not in ith place for i = 1, 2, …, n; denoted by d_n.

Ex 8.12: $n = 10$. How many derangements? probability?

Sol.

Let condition c_i: integer i is in the ith place, for $1 \le i \le 10$.

$$\therefore d_{10} = N(\bar{c}_1\bar{c}_2\ldots\bar{c}_{10})$$

$$= 10! - \binom{10}{1}9! + \binom{10}{2}8! - \binom{10}{3}7! + \ldots + \binom{10}{10}0!$$

$$= 10![1 - \binom{10}{1}\left(\frac{9!}{10!}\right) + \binom{10}{2}\left(\frac{8!}{10!}\right) + \ldots + \binom{10}{10}\left(\frac{0!}{10!}\right)]$$

$$= 10![1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{1}{10!}] \doteqdot (10!)(e^{-1})$$

$\therefore$ the probability $\doteqdot (10!)(e^{-1})/(10!) = e^{-1}$

(n = 11, 12, … the same race. !!)

§ 8.3 Derangements: Nothing Is in Its Right Place

Note: $\forall n \geq 10$, the probability that our gambler wins at least one of his bet is approximately $1 - e^{-1} \doteqdot 0.63212$.

Ex 8.13: $d_4 = ?$

Sol.

$d_4 = 4![1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}] = 4![\frac{1}{2} - \frac{1}{6} + \frac{1}{24}] = 12 - 4 + 1 = 9$

$\Rightarrow$ 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321

Ex8.14: seven books, seven people to review them.
First week; second week: each persons read one book.
In how many ways can she make these two distributions so that she gets two reviews (by different people) of each books?

Sol. first week: 7! second week: $d_7 \doteqdot (7!)(e^{-1})$

$\Rightarrow (7!) \cdot d_7 \doteqdot (7!)^2(e^{-1})$

§ 8.3 Derangements: Nothing Is in Its Right Place

Discussion:

Ex 8.3.14:

(a) In how many ways can the integers 1, 2, 3, …, n be arranged in a line so that none of the patterns 12, 23, 34, …, $(n - 1)n$ occurs?

(b) Show that the result in part (a) equals $d_{n-1} + d_n$.

§ 8.3 Derangements: Nothing Is in Its Right Place

Checklist

1. **Maclaurin series for e^x**
   - $e^{-1} = \sum_{i=0}^{\infty} \frac{(-1)^n}{n!}$
2. **The definition of derangement**
   - $d_{10} \fallingdotseq (10!)(e^{-1})$
3. **Application**