Computer Science and Information Engineering
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Combinatorial Mathematics

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Chapter 7 Relations: The Second Time Around

§ 7.3 Partial Orders: Hasse Diagrams

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

§ 7.3 Partial Orders: Hasse Diagrams

Outline

1. **Hasse diagram**
   - **Topological Sorting Algorithm**
2. **Special Elements**
3. **Special Poset**

§ 7.3 Partial Orders: Hasse Diagrams

$\mathbb{N} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{R} \longrightarrow \mathbb{C}$

- $\mathbb{N}$: **closed under +, ·** **but not –**, $x + 5 = 2$?
- $\mathbb{Z}$: $2x + 3 = 4$?
- $\mathbb{Q}$: $x^2 - 2 = 0$
- $\mathbb{R}$: $\forall r_1 \neq r_2 \Rightarrow$ either $r_1 < r_2$ or $r_1 > r_2$
- $\mathbb{C}$: $x^2 + 1 = 0$; ? ×

Def: 1) $(A, \mathcal{R})$ is called a poset (partially ordered set)
≡ A relation $\mathcal{R}$ on A is a partial order.

2) A is called a poset ≡ ∃ a relation $\mathcal{R}$ on A s.t. $(A, \mathcal{R})$ is a poset.

§ 7.3 Partial Orders: Hasse Diagrams

EX 7.34: Let $A = \{x \mid x$ is a course offered at a college$\}$

Define $\mathcal{R}$ on A by $x\mathcal{R}y$ if x, y are the same course or if x is a prerequisite for y.

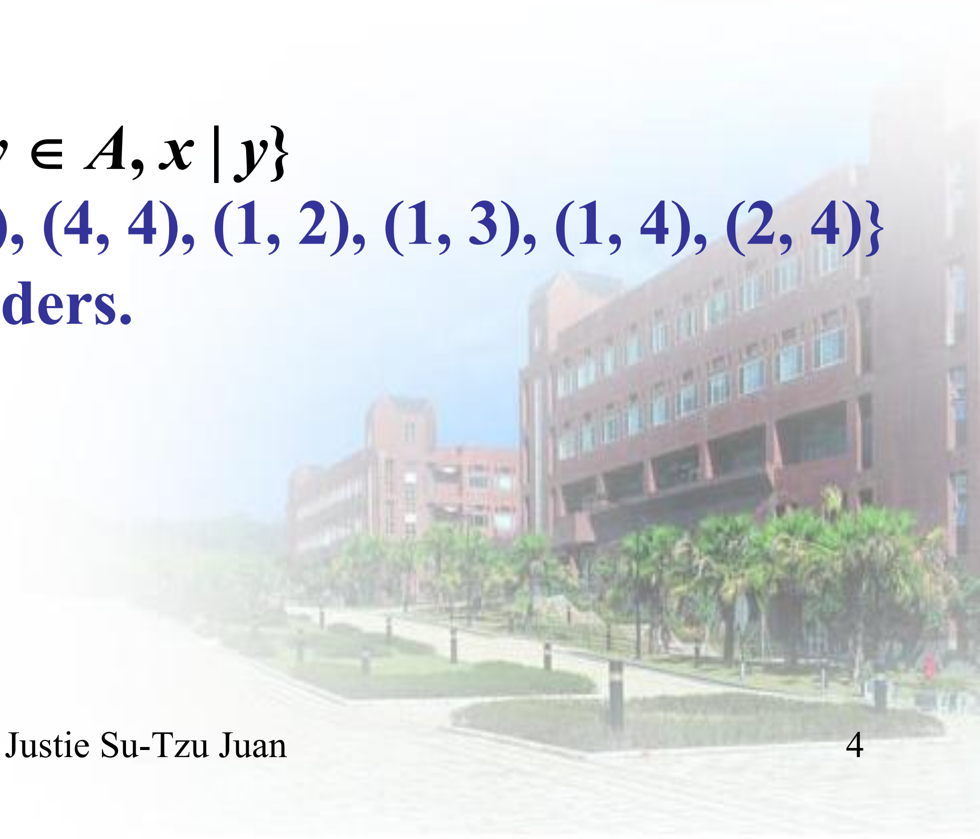
$\Rightarrow$ $\mathcal{R}$ makes A into a poset.

Ex 7.35: Let $A = \{1, 2, 3, 4\}$

Define $\mathcal{R} = \{(x, y) \mid x, y \in A, x \mid y\}$

$\mathcal{R} = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 4)\}$ is a partial orders.

$\therefore$ $(A, \mathcal{R})$ is a poset.



§ 7.3 Partial Orders: Hasse Diagrams

<u>Ex 7.36</u>:

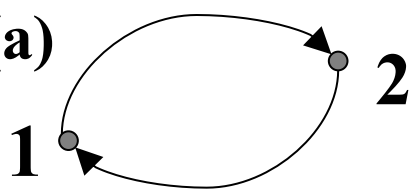
A = a set of tasks that must be performed in building a house

$\mathcal{R}$ on A by $x\mathcal{R}y$ if x, y denote the same task or

if task x must be performed before the start of task y.

$\Rightarrow$ A is a poset

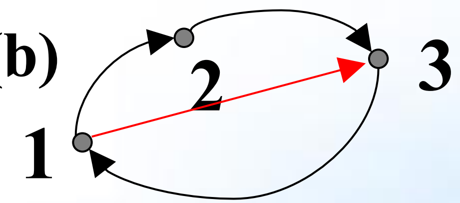
ex: (a)



$\because$ (1, 2), (2, 1) $\in \mathcal{R}$

with 1 $\neq$ 2 : ~~A~~

(b)



$\because$ (1, 2), (2, 3) $\in \mathcal{R} \Rightarrow$ (1, 3) $\in \mathcal{R}$

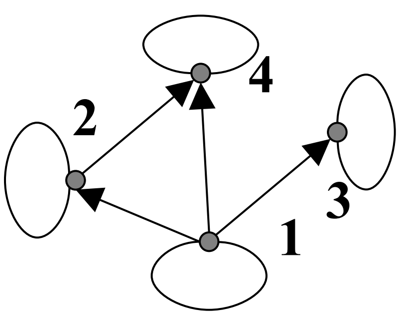
but (3, 1) $\in \mathcal{R}$ and 1 $\neq$ 3 : ~~A~~

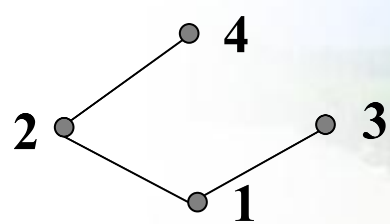
§ 7.3 Partial Orders: Hasse Diagrams

Note: In a digraph $G = (A, \mathcal{R})$, when
(1) $\exists\ a \neq b \in A$, $(a, b), (b, a) \in \mathcal{R}$, or
(2) $\exists$ a directed cycle
then $\mathcal{R}$ cannot be transitive and antisymmetric.
$\therefore$ $(A, \mathcal{R})$ is not a poset.

Ex 7.37: Hasse diagram for $\mathcal{R}$: Give $G = (A, \mathcal{R})$
step 1: eliminate the loops at x, $\forall\ x \in A$.
step 2: eliminate the edges is enough to in sure the existence by transitive. (if $\exists\ (x, y), (y, z) \in \mathcal{R}$, eliminate (x, z))
step 3: eliminate the directions : the directions are assumed to go from the bottom to the top.

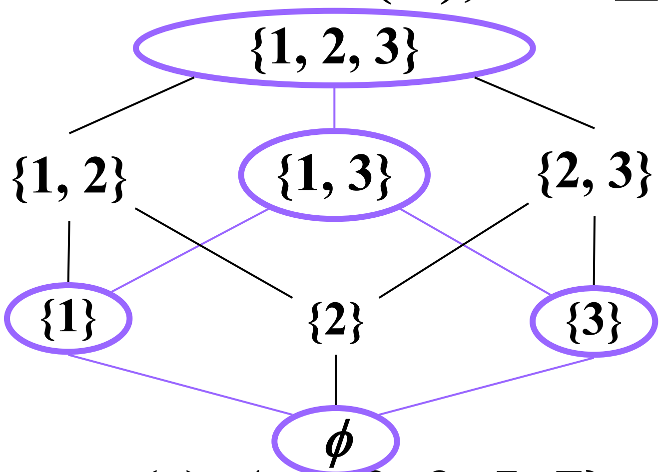
ex:



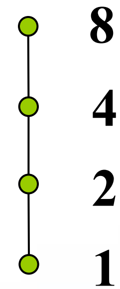


§ 7.3 Partial Orders: Hasse Diagrams

Ex 7.38: (a) $\mathcal{U} = \{1, 2, 3\}$ back
$A = \mathcal{P}(\mathcal{U})$, $\mathcal{R} = \subseteq$



(b) $A = \{1, 2, 4, 8\}$ back
$\mathcal{R} = \{(x, y) \mid x, y \in A, x \mid y\}$



(c) $A = \{2, 3, 5, 7\}$
$\mathcal{R} = \{(x, y) \mid x, y \in A, x \mid y\}$

back

back

(d) $A = \{2, 3, 5, 6, 7, 11, 12, 35, 385\}$
$\mathcal{R} = \{(x, y) \mid x, y \in A, x \mid y\}$

back

§ 7.3 Partial Orders: Hasse Diagrams

Ex 7.39: Let $A = \{1, 2, 3, 4, 5\}$, $\mathcal{R}$ on A defined by $x\mathcal{R}y$ if $x \leq y$
A is a poset, denoted by $(A, \leq)$.
$B = \{1, 2, 4\} \subset A$; $B \times B \cap \mathcal{R}$ is a partial order on B
$= \{(1, 1), (2, 2), (4, 4), (1, 2), (1, 4), (2, 4)\}$

Note: If $\mathcal{R}$ is a partial order on A, then $\forall\ B \subset A$, $(B, (B\times B)\cap\mathcal{R})$ is a poset.

ex: $\{\phi, \{1\}, \{3\}, \{1, 3\}, \{1, 2, 3\}\} = B$. see

Def 7.16: 1) A partial order $\mathcal{R}$ on A is called a **total order** if $\forall\ x, y \in A$, either $x\mathcal{R}y$ or $y\mathcal{R}x$.
2) $\mathcal{R}$ is a total order on A, then A is called **totally ordered**.

§ 7.3 Partial Orders: Hasse Diagrams

Ex 7.40: (a) $(\mathbb{N}, \leq)$ is a total order.

(b) $\mathcal{U} = \{1, 2, 3\}$, $(\mathcal{P}(\mathcal{U}), \subseteq)$ is not a total order.

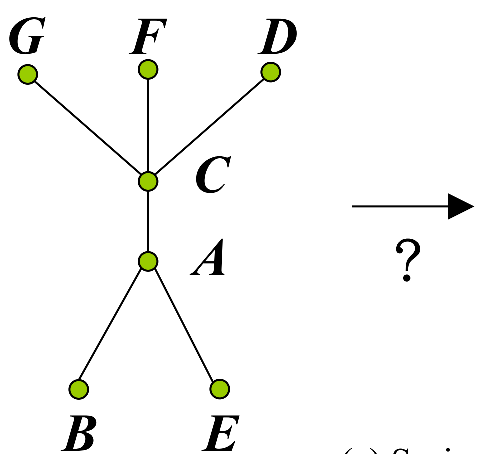
$\because$ $\{1, 2\}, \{1, 3\} \in \mathcal{P}(\mathcal{U})$, but $\{1, 2\} \not\mathcal{R} \{1, 3\}$, $\{1, 3\} \not\mathcal{R} \{1, 2\}$.

(c) Ex 7.38 (b) shows a total order. see

Ex 7.41: 請自己看！

Q: Whether we can take the partial order $\mathcal{R}$, given by the Hasse diagram, and fine a total order $\mathcal{T}$ on these tasks for which $\mathcal{R} \subseteq \mathcal{T}$?

ex:

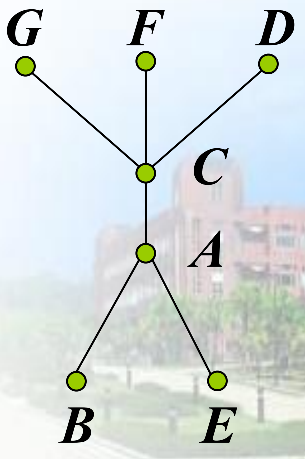


§ 7.3 Partial Orders: Hasse Diagrams

Topological Sorting Algorithm (for a poset $(A, \mathcal{R})$ with $|A| = n$)

Step 1: Let $k = 1$. Let H_1 = the Hasse diagram for $(A, \mathcal{R})$
Step 2: Select $v_k \in V(H_k)$ s.t. no edge in H_k starts at v_k
Step 3: If $k = n$, output $\mathcal{T}$: $v_n < v_{n-1} < \ldots < v_2 < v_1$ and STOP
else $(k < n)$ { $H_{k+1} := H_k - v_k$; $k := k + 1$;
go to Step2.}

ex: $E < B < A < C < G < F < D$
$\Rightarrow$ 12 possible answers



§ 7.3 Partial Orders: Hasse Diagrams

<u>Def 7.17</u>: $(A, \mathcal{R})$ is a poset:

1) $x \in A$ is called a **maximal** element of A
   $\equiv \forall\ a \in A, a \neq x \Rightarrow x \not\mathcal{R} a \equiv \forall\ a \in A, x\mathcal{R}a \Rightarrow x = a.$
2) $y \in A$ is called a **minimal** element of A
   $\equiv \forall\ b \in A, b \neq y \Rightarrow b \not\mathcal{R} y \equiv \forall\ b \in A, b\mathcal{R}y \Rightarrow y = b.$

<u>Ex 7.42</u>: Let $\mathcal{U} = \{1, 2, 3\}$, $A = \mathcal{P}(\mathcal{U})$

(a) For the poset $(A, \subseteq)$, the maximal element = <u>$\mathcal{U}$</u>, and
the minimal element = <u>ϕ</u>

(b) Let $B = A - \{\{1, 2, 3\}\}$, In $(B, \subseteq)$:
the maximal elements = <u>$\{1, 2\}, \{1, 3\}, \{2, 3\}$</u>;
the minimal element = <u>ϕ</u>.

§ 7.3 Partial Orders: Hasse Diagrams

Ex 7.43: 1) $(\mathbb{Z}, \leq)$ is a poset: the maximal element = None;
the minimal element = None.
2) $(\mathbb{N}, \leq)$ is a poset: the minimal element = 0;
the maximal element = None (empty set).

Ex 7.44: In Ex 7.38 (b), (c), (d): see

	minimal element	maximal element
(b)	1	8
(c)	2, 3, 5, 7	2, 3, 5, 7
(d)	2, 3, 5, 7, 11	12, 385

§ 7.3 Partial Orders: Hasse Diagrams

Thm 7.3: If $(A, \mathcal{R})$ is a poset and A is finite, then A has both a maximal and a minimal element.

Proof. maximal:

Let $a_1 \in A$, If $\forall\ a \in A, a \neq a_1, a_1 \not\mathcal{R} a \quad \Rightarrow a_1$ is maximal

else $\exists\ a_2 \in A, a_2 \neq a_1, a_1 \mathcal{R} a_2$:

If $\forall\ a \in A, a \neq a_2, a_2 \not\mathcal{R} a \quad \Rightarrow a_2$ is maximal

else $\exists\ a_3 \in A, a_3 \neq a_2, a_2 \mathcal{R} a_3$:

$\because \mathcal{R}$ is antisymmetric and $a_1 \mathcal{R} a_2 \ \therefore\ a_3 \neq a_1$

$\because a_1 \mathcal{R} a_2$ and $a_2 \mathcal{R} a_3 \quad \therefore\ a_1 \mathcal{R} a_3$

If $\forall\ a \in A, a \neq a_3, a_3 \not\mathcal{R} a \quad \Rightarrow a_3$ is maximal

else …

Continuing in this manner, $\because A$ is finite

$\therefore$ We get $a_n \in A$ with $\forall\ a \in A, a \neq a_n, a_n \not\mathcal{R} a$

$\Rightarrow a_n$ is maximal.

minimal element can be proved in a similar way.

§ 7.3 Partial Orders: Hasse Diagrams

Note: In the topological sorting algorithm: Step2 selecting a maximal element from $(A, \mathcal{R})$ or $(B, \mathcal{R}')$, where $B \subseteq A$; $\mathcal{R}' = (B \times B) \cap \mathcal{R}$.
$\Rightarrow$ By Thm 7.3, $\exists$ at least one such element!

Def 7.18: $(A, \mathcal{R})$ is a poset:
1) $x \in A$ is called a least element $\equiv \forall\ a \in A, x\mathcal{R}a$.
2) $y \in A$ is called a greatest element $\equiv \forall\ a \in A, a\mathcal{R}y$.

Ex 7.45: Let $\mathcal{U} = \{1, 2, 3\}$, $\mathcal{R} = \subseteq$, the subset relation
(a) $A = \mathcal{P}(\mathcal{U})$: $(A, \subseteq)$: least element $= \phi$; greatest element $= \mathcal{U}$
(b) $B = \mathcal{P}(\mathcal{U}) - \{\phi\}$: $(B, \subseteq)$: greatest element $= \mathcal{U}$;
no least element,
(but $\exists$ 3 minimal element.)

§ 7.3 Partial Orders: Hasse Diagrams

Ex 7.46: In Ex 7.38: see

	least element	greatest element
(b)	1	8
(c)	no	no
(d)	no	no

Thm 7.4: If the poset (A, $\mathcal{R}$) has a greatest (least) element, then the element is unique.

Proof. Suppose $\exists\, x, y \in A$ and both are greatest elements

$\because$ x is a greatest element $\therefore$ $y\mathcal{R}x$

$\because$ y is a greatest element $\therefore$ $x\mathcal{R}y$

$\Rightarrow$ $\because$ $\mathcal{R}$ is antisymmetric $\therefore$ $x = y$

The proof for the least element is similar.

§ 7.3 Partial Orders: Hasse Diagrams

Def 7.19: Let $(A, \mathcal{R})$ be a poset with $B \subseteq A$:

1) $x \in A$ is called a **lower bound** of $B \equiv x\mathcal{R}b, \forall\ b \in B$.
2) $y \in A$ is called a **upper bound** of $B \equiv b\mathcal{R}y, \forall\ b \in B$.
3) A lower bound of B, $x' \in A$ is called a **greatest lower bound (glb)** of $B \equiv \forall$ lower bounds x'' $(\neq x')$ of B, $x''\mathcal{R}x'$.
4) A upper bound of B, $y' \in A$ is called a **least upper bound (lub)** of $B \equiv \forall$ upper bounds y'' $(\neq y')$ of B, $y'\mathcal{R}y''$.

Ex 7.47: $\mathcal{U} = \{1, 2, 3, 4\}$, $A = \mathcal{P}(\mathcal{U})$, $B = \{\{1\}, \{2\}, \{1, 2\}\}$:

In $(B, \subseteq)$: upper bounds: $\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}$

lub: $\{1, 2\}$ $(\in B)$

glb: ϕ $(\notin B)$

§ 7.3 Partial Orders: Hasse Diagrams

Ex 7.48: $\mathcal{R} = \leq$ ("less than or equal to")

a) $A = \mathbb{R}$, $B = [0, 1]$: B has glb: 0 ($\in B$) lub: 1 ($\in B$)

$A = \mathbb{R}$, $C = (0, 1]$: C has glb: 0 ($\notin C$) lub: 1 ($\in C$)

b) $A = \mathbb{R}$, $B = \{q \in \mathbb{Q} \mid q^2 < 2\}$: B has glb: $-\sqrt{2}$ ($\notin B$) lub: $\sqrt{2}$ ($\notin B$)

c) $A = \mathbb{Q}$, $B = \{q \in \mathbb{Q} \mid q^2 > 2\}$: B has no glb or lub.

Thm 7.5: If $(A, \mathcal{R})$ is a poset and $B \subseteq A$, then B has at most one lub (glb).

Def 7.20: The poset $(A, \mathcal{R})$ is called a **lattice**

$\equiv \forall\, x, y \in A$, lub$\{x, y\}$ and glb$\{x, y\}$ both exist in A

($\exists\, a, b \in A$, which $a =$ lub$\{x, y\}$, $b =$ glb$\{x, y\}$)

§ 7.3 Partial Orders: Hasse Diagrams

Ex 7.49: $A = \mathbb{N}$, define $\mathcal{R}$ on A by $x\mathcal{R}y$ if $x \leq y$: $(\mathbb{N}, \leq)$:

lub$\{x, y\}$ = max$\{x, y\}$; glb$\{x, y\}$ = min$\{x, y\}$

$\Rightarrow$ $(\mathbb{N}, \leq)$ is a lattice.

Ex 7.50: $\mathcal{U} = \{1, 2, 3\}$ in $(\mathcal{P}(\mathcal{U}), \subseteq)$: $\forall\ S, T \in \mathcal{P}(\mathcal{U})$

lub$\{S, T\} = S \cup T$ $(\in \mathcal{P}(\mathcal{U}))$; glb$\{x, y\} = S \cap T$ $(\in \mathcal{P}(\mathcal{U}))$

$\Rightarrow$ $(\mathcal{P}(\mathcal{U}), \subseteq)$ is a lattice.

Ex 7.51: In Ex 7.38 (d): see

lub{2, 3} = 6; lub{3, 6} = 6; lub{5, 7} = 35; lub{7, 11} = 385; …

glb{3, 6} = 3; glb{2, 12} = 2; glb{35, 385} = 35; …

but $\nexists$ glb{2, 3} $\in A$, $\nexists$ glb{5, 7} …

$\Rightarrow$ this poset is not a lattice.

Q3:

§ 7.3 Partial Orders: Hasse Diagrams

Checklist

1. **Hasse diagram**
   - ☐ **Topological Sorting Algorithm**
2. **Special Elements**
   - ☐ **Maximal, minimal**
   - ☐ **Least, greatest**
   - ☐ **Lower bound, upper bound**
   - ☐ **glb, lub**
4. **Special Poset**
   - ☐ **Total Order**
   - ☐ **Lattice**

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Chapter 7 Relations: The Second Time Around

§ 7.4 Equivalence Relations and Partitions

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§ 7.4 Equivalence Relations and Partitions

Outline

1. **Partition**
2. **Equivalence Relations: Equivalence Class**
3. **Thm 7.6**
4. **Equivalence Relations vs. Partition**
5. **Counting**

§ 7.4 Equivalence Relations and Partitions

Note: 1) If $A \neq \phi$, $\mathcal{R}$ = the equality relation.

$\Rightarrow$ (A, $\mathcal{R}$) is a equivalence relation.

$\Rightarrow$ establishes the property of "sameness" on A.

2) If $A = \mathbb{Z}$, $\mathcal{R}$ defined by $x\mathcal{R}y$ if $2 \mid (x - y)$.

$\Rightarrow$ ($\mathbb{Z}$, $\mathcal{R}$) is a equivalence relation.

$\Rightarrow$ splits $\mathbb{Z}$ into two subsets consisting of the odd and even integers.

Def 7.21: A: set; I: index set; $\forall\ i \in I$, $\phi \neq A_i \subseteq A$

1) $\{A_i\}_{i \in I}$ is a partition of A if

(a) $A = \bigcup_{i \in I} A_i$ and (b) $\forall\ i, j \in I$ where $i \neq j$, $A_i \cap A_j = \phi$.

2) Each A_i is called a cell (or block) of the partition.

§ 7.4 Equivalence Relations and Partitions

Ex 7.52: $A = \{1, 2, 3, \ldots, 10\}$, each of (a), (b), (c) determines a partition of A:

a) $A_1 = \{1, 2, 3, 4, 5\}$, $A_2 = \{6, 7, 8, 9, 10\}$.
b) $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 6, 7, 9\}$, $A_3 = \{5, 8, 10\}$.
c) $A_i = \{i, i + 5\}$, $1 \le i \le 5$.

Note: Each element of A belongs to exactly one cell in each partition.
($\forall\, x \in A, \exists!\, i^* \in I$, s.t. $x \in A_{i^*}$ for any partition $\{A_i\}_{i \in I}$)

Ex 7.53: Let $A = \mathbb{R}$, $\forall\, i \in \mathbb{Z}$, let $A_i = [i, i+1)$
$\Rightarrow \{A_i\}_{i \in \mathbb{Z}}$ is a partition of $\mathbb{R}$

§ 7.4 Equivalence Relations and Partitions

Def 7.22: Let $\mathcal{R}$ be an equivalence relation on a set A. $\forall\, x \in A$, the **equivalence class** of x, denoted by $[x] \equiv \{y \in A \mid y\mathcal{R}x\}$

Ex 7.54: Define $\mathcal{R}$ on $\mathbb{Z}$ by $x\mathcal{R}y$ if $4 \mid (x - y)$

$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\} = \{4k \mid k \in \mathbb{Z}\}$

$[1] = \{4k+1 \mid k \in \mathbb{Z}\}$; $[2] = \{4k+2 \mid k \in \mathbb{Z}\}$; $[3] = \{4k+3 \mid k \in \mathbb{Z}\}$;

$[4] = [0] = [8] = \ldots$; $[5] = [1] = [9] = \ldots$;

$[6] = [2] = [10] = \ldots$; …

e.g.: $[6] = [2] = [-2]$; $[51] = [3]$, …

$\Rightarrow \{[0], [1], [2], [3]\}$ provides a partition of $\mathbb{Z}$.

Note: The index set for the partition is implicit.

§ 7.4 Equivalence Relations and Partitions

Ex 7.55: Define $\mathscr{R}$ on $\mathbb{Z}$ by $a\mathscr{R}b$ if $a^2 = b^2$ ($a = \pm b$).

1) $\mathscr{R}$ is an equivalence relation:

1. $\forall\, a \in \mathbb{Z},\ a^2 = a^2 \quad \Rightarrow\ a\mathscr{R}a$
2. $\forall\, a, b \in \mathbb{Z}$, if $a\mathscr{R}b \Rightarrow\ a^2 = b^2 \ \Rightarrow\ b^2 = a^2 \ \Rightarrow b\mathscr{R}a$
3. $\forall\, a, b, c \in \mathbb{Z}$, if $a\mathscr{R}b$ and $b\mathscr{R}c \Rightarrow$

   $a^2 = b^2$ and $b^2 = c^2 \ \Rightarrow\ a^2 = c^2 \qquad \Rightarrow a\mathscr{R}c$

2) What can we say about the corresponding partition of $\mathbb{Z}$?

$\because\ [0] = \{0\}, [1] = \{1, -1\}, [2] = \{2, -2\}, \ldots$

In general, $\forall\, n \in \mathbb{Z}^+$, $[n] = \{-n, n\}$

$\therefore$ The partition: $\mathbb{Z} = \bigcup_{n=0}^{\infty}[n] = \bigcup_{n\in N}[n]$

$= \{0\} \cup \bigcup_{n=1}^{\infty}\{-n, n\}$

$= \{0\} \cup \bigcup_{n\in Z^+}[n]$

§ 7.4 Equivalence Relations and Partitions

Thm 7.6: $\mathcal{R}$ is an equivalence relation on A, and $x, y \in A$, then

(a) $x \in [x]$

(b) $x\mathcal{R}y \Leftrightarrow [x] = [y]$

(c) $[x] = [y]$ or $[x] \cap [y] = \phi$ back

Proof. (1/2)

(a) $\because$ $\mathcal{R}$ is reflexive

(b) $(\Rightarrow)$ $\forall\ w \in [x], w\mathcal{R}x$

$\because x\mathcal{R}y$ and $\mathcal{R}$ is transitive $\Rightarrow w\mathcal{R}y$

$\Rightarrow w \in [y]$ $\quad \therefore\ [x] \subseteq [y]$ …(1)

$\forall\ t \in [y], t\mathcal{R}y$

$\because$ $\mathcal{R}$ is symmetric $\quad \therefore\ x\mathcal{R}y \quad \Rightarrow y\mathcal{R}x$

$\Rightarrow \because t\mathcal{R}y$ and $y\mathcal{R}x$ and $\mathcal{R}$ is transitive $\quad \therefore\ t\mathcal{R}x$

$\Rightarrow t \in [x]$ $\quad \therefore\ [y] \subseteq [x]$…(2)

by (1), (2), $[x] = [y]$.

§ 7.4 Equivalence Relations and Partitions

Thm 7.6: $\mathcal{R}$ is an equivalence relation on A, and $x, y \in A$, then

(a) $x \in [x]$

(b) $x\mathcal{R}y \Leftrightarrow [x] = [y]$

(c) $[x] = [y]$ or $[x] \cap [y] = \phi$

Proof. (2/2)

(b) ($\Leftarrow$) If $[x] = [y]$, by (a), $x \in [x] \Rightarrow x \in [y] \Rightarrow x\mathcal{R}y$

(c) $(p \vee q) \Leftrightarrow (\neg p \wedge \neg q \rightarrow F_0)$ (Prove by contradiction)

If $[x] \neq [y]$ and $[x] \cap [y] \neq \phi$,

then let $v \in A$, s.t. $v \in [x] \cap [y]$

that is, $v \in [x]$ and $v \in [y]$

$\Rightarrow v\mathcal{R}x$ and $v\mathcal{R}y$

$\because$ $\mathcal{R}$ is symmetric $\therefore$ $v\mathcal{R}x \Rightarrow x\mathcal{R}v$

$\because$ $x\mathcal{R}v$ and $v\mathcal{R}y \Rightarrow x\mathcal{R}y$

By (b), $x\mathcal{R}y \Rightarrow [x] = [y] \rightarrow\leftarrow$

§ 7.4 Equivalence Relations and Partitions

Ex 7.56:

(a) $A = \{1, 2, 3, 4, 5\}$,

$\mathcal{R} = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$

$\Rightarrow$ $\mathcal{R}$ **is an equivalence relation on** A**:**

Sol.

$[1] = \{1\}$; $[2] = \{2, 3\} = [3]$; $[4] = \{4, 5\} = [5]$

$A = [1] \cup [2] \cup [4]$ $([1] \cap [2] = \phi = [1] \cap [4] = [2] \cap [4])$

$\therefore$ $\{[1], [2], [4]\}$ **determines a partition of** A

Ex 7.56:

(b) In Ex 7.16 (d), $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{x, y, z\}$,
$f: A \to B$ is the onto function,
$f = \{(1, x), (2, z), (3, x), (4, y), (5, z), (6, y), (7, x)\}$
Define $\mathcal{R}$ on A by $a\mathcal{R}b$ if $f(a) = f(b)$ was shown to be an equivalence relation:

Sol.

$f^{-1}(x) = \{1, 3, 7\} = [1]$ $(= [3] = [7])$

$f^{-1}(y) = \{4, 6\} = [4]$ $(= [6])$

$f^{-1}(z) = \{2, 5\} = [2]$ $(= [5])$

$A = [1] \cup [4] \cup [2] = f^{-1}(x) \cup f^{-1}(y) \cup f^{-1}(z)$

$\therefore$ $\{f^{-1}(x), f^{-1}(y), f^{-1}(z)\}$ **determines a partition of A.**

§ 7.4 Equivalence Relations and Partitions

Note: $\forall$ nonempty sets A, B if f: $A \to B$ is an onto function
$\Rightarrow A = \cup_{b\in B} f^{-1}(b)$ and $\{f^{-1}(b) \mid b \in B\}$ determines a partition of A.

Ex 7.57: In C++:

```
union
{
  int a;
  int c;
  int p;
};
```

```
union
{
  int up;
  int down;
};
```

$\equiv$ *a*, *c*, *p* share one memory location;
up, *down* share another memory location
$\therefore$ all variable is partitioned by the equivalence relation $\mathcal{R}$,
where $v_1 \mathcal{R} v_2$ if v_1, v_2 share the same memory location.

§ 7.4 Equivalence Relations and Partitions

<u>Ex 7.58</u>: $A = \{1, 2, 3, 4, 5, 6, 7\}$. If $\mathcal{R}$ induces the partition of $A = \{1, 2\} \cup \{3\} \cup \{4, 5, 7\} \cup \{6\}$. What is $\mathcal{R}$?

Sol.

$\because \{1, 2\} \Rightarrow [1] = \{1, 2\} = [2]$

$\Rightarrow (1, 1), (1, 2), (2, 1), (2, 2) \in \mathcal{R}$

$\{4, 5, 7\} \Rightarrow [4] = [5] = [7] = \{4, 5, 7\}$

$\Rightarrow \{4, 5, 7\} \times \{4, 5, 7\} \subseteq \mathcal{R}$

$\vdots$

$\therefore \mathcal{R} = (\{1, 2\}\times\{1, 2\}) \cup (\{3\}\times\{3\}) \cup (\{4, 5, 7\}\times\{4, 5, 7\}) \cup (\{6\}\times\{6\})$

$|\mathcal{R}| = 2^2 + 1^2 + 3^2 + 1^2 = 15$

§ 7.4 Equivalence Relations and Partitions

Thm 7.7: A is a set:

(a) Any equivalence relation $\mathcal{R}$ on A induces a partition of A.

(b) Any partition of A gives rise to an equivalence relation $\mathcal{R}$ on A.

Proof.

(a) By Thm 7.6 (a), (c). see

(b) For any partition $\{A_i\}_{i \in I}$ of A,
Define $\mathcal{R}$ on A by $x\mathcal{R}y$ if x and y are in the same cell of the partition ($\mathcal{R} = \{(x, y) \mid \exists\ i \in I, \text{ s.t. } x \in A_i \text{ and } y \in A_i\}$)
need to verify $\mathcal{R}$ is an equivalence relation → reader

$$S(m,n)=\frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}(n-k)^m$$

§ 7.4 Equivalence Relations and P

Thm 7.8: ∀ set A: ∃ 1-1 correspondence between the set of equivalence relations on A and the set of partition of A.

Proof.

EXERCISE 7.4 (16)

Ex 7.59:

(a) A = {1, 2, 3, 4, 5, 6}. How many relations on A are equivalence relation?

Sol.

(a) From Sec 5.3, using the Stirling numbers of the second kind,

∃ $\sum_{i=1}^{6} S(6,i) = 203$ different partition of A.

⇒ ∃ 203 equivalence relation on A.

$$S(m,n)=\frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}(n-k)^m$$

§ 7.4 Equivalence Relations and Pa

Ex 7.59:

(b) How many of the equivalence relation in (a) satisfy 1, 2 ∈ [4]?

Sol.

(b) Identifying 1, 2, 4 as the "same".

⇒ Let B = {1, 3, 5, 6}.

∴ ∃ $\sum_{i=1}^{4} S(4,i)=15$ equivalence relation on A for which [1] =[2] = [4].

Note: If A is finite with $|A| = n$, then ∀ $n \le r \le n^2$,

∃ an equivalence relation $\mathcal{R}$ on A with $|\mathcal{R}| = r$

⇔ ∃ $n_1, n_2, \ldots, n_k \in \mathbb{Z}^+$ s.t. $\sum_{i=1}^{k} n_i = n$ And $\sum_{i=1}^{k} n_i^2 = r$.

§ 7.4 Equivalence Relations and Partitions

Discussion:

Ex 7.4.11: How many of the equivalence relations on $A = \{a, b, c, d, e, f\}$ have

(a) exactly two equivalence classes of size 3?

(b) exactly one equivalence class of size 3?

(c) one equivalence class of size 4?

(d) at least one equivalence class with three or more elements?

§ 7.4 Equivalence Relations and Partitions

Checklist

1. **Partition - Judge**
2. **Equivalence Relations: Equivalence Class**
   - Definition and Symbol
3. **Thm 7.6 - proof**
4. **Equivalence Relations vs. Partition**
   - 1-1 correspondence
5. **Counting**