Chapter 7 Relations: The Second Time Around

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs (2)

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics (5th Edition)
by Ralph P. Grimaldi
§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Def 7.14 : \( V \): finite nonempty. A directed graph (or digraph) \( G \) ≡

- \( G = (V, E) \), where \( V \) is called the vertex set, \( E \subseteq V \times V \) is called the edge set.
- \( v \in V \) is called the vertices or nodes of \( G \)
- \( (a, b) \in E \) is called the (directed) edges or arcs of \( G \)
- \( a \) is called the origin or source of \( (a, b) \)
- \( b \) is called the terminus or terminating vertex of \( (a, b) \)
- \( a \) is adjacent to \( b \); \( b \) is adjacent from \( a \)
- \( (a, a) \) is called a loop at \( a \)

Ex 7.25 : \( V = \{1, 2, 3, 4, 5\} \),
\[
E = \{(1, 1), (1, 2), (1, 4), (3, 2)\}
\]
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Def: If \((a, b), (b, a) \in E, (a \neq b)\), then use \(\{a, b\} = \{b, a\}\) to represent. \(a\) and \(b\) are called adjacent vertices.

Ex 7.26: precedence graph for the computer program

(S1) \(b := 3\)
(S2) \(c := b + 2\)
(S3) \(a := 1\)
(S4) \(d := a \times b + 5\)
(S5) \(e := d - 1\)
(S6) \(f := 7\)
(S7) \(e := c + d\)
(S8) \(g := b \times f\)
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Ex 7.27: $A = \{1, 2, 3, 4\}$, $\mathcal{R} = \{(1, 1), (1, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 2)\}$. The directed graph associated with $\mathcal{R}$ is $G = (A, \mathcal{R})$, where undirected edge $\{x, y\} = (x, y)$ and $(y, x)$.

The associated undirected graph: replace all edges $(x, y)$ by undirected edges $\{x, y\}$. back
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**Def**: For an (directed) undirected graph $G = (V, E)$:

1) A (directed) $x$-$y$ path starting at $x$ and ending at $y \equiv$ a finite sequence of (directed) undirected edges with no repeat vertex.

2) The *length* of a path $\equiv$ the edge on the path

3) A path is *closed* $\equiv x = y$

4) A closed (directed) path $\equiv$ (directed) *cycle* ($\geq 3$ edges)

5) A undirected graph is *connected* $\equiv \forall x \neq y \in V, \exists x$-$y$ path
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Note: (1) loops \( \subseteq \) cycles; (2) loops have no bearing on graph connectivity

Ex:

(1) \{a, b\}, \{b, e\}, \{e, f\}, \{f, b\}, \{b, a\} is not a path
(2) \((b, f), (f, e), (e, d), (d, c), (c, b) = a \text{ directed cycle of length 5}
(3) \((b, f), (f, e), (e, b), (b, d), (d, c), (c, b) \neq \text{directed cycle} \)
### § 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

**Def 7.15**: A directed graph $G = (V, E)$ is called **strongly connected**

$\equiv \forall x, y \in V$, where $x \neq y$, $\exists x$-$y$ directed path

i.e. $(x, y) \in E$ or $\exists v_1, v_2, \ldots, v_n \in V$

s.t. $(x, v_1), (v_1, v_2), \ldots, (v_n, y) \in E$

**Ex**: In **Ex 7.27**, $G$ is not strongly connected.

$\therefore$ no 3-1 directed path

**Def**: **loop-free** $\equiv$ no loop

**Ex**: 1) 上上**Ex** 中，$D$ 為 strongly connected and loop-free.

2) $G$ is strongly connected and loop-free.
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Ex 7.29:
- Complete graphs on \( n \) vertices, \( K_n \) = an undirected graph that are loop-free and have an edge for every pair of distinct vertices.

\[
\begin{align*}
K_1 & \quad K_2 & \quad K_3 & \quad K_4 & \quad K_5 \\
\end{align*}
\]

- The adjacency matrix for \( G = (A, R) \) = the relation matrix for \( R \).

Ex 7.30: \( A = \{1, 2, 3\} \), \( R = \{(1, 1), (1, 2), (2, 2), (3, 3), (3, 1)\} \)

\( R \) : reflexive, antisymmetric; not symmetric, not transitive.
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**Note**: \( R \) is reflexive \( \iff \) in \( G = (A, R) : \forall x \in V(G), \exists \) loop at \( x \).

**Ex 7.31**: \( A = \{1, 2, 3\}, R = \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 2)\}:

\( R \) : symmetric; not reflexive, not antisymmetric, not transitive.

**Note**: \( R \) is symmetric \( \iff \) in \( G = (A, R) :

\( E(G) = \) loops \( \cup \) undirected edges

**Ex 7.32**: \( A = \{1, 2, 3\}, R = \{(1, 1), (1, 2), (2, 3), (1, 3)\}:

\( R \) : transitive, antisymmetric; but not reflexive, not symmetric.
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**Note**: \( R \) is antisymmetric \( \iff \) For the associated graph \( G = (A, R) \), \( E(G) = \text{loops} \cup \text{directed edges} \)

**Note**: \( R \) is transitive \( \iff \) For the associated graph \( G = (A, R) \), \( \forall x, y \in A, \exists x-y \text{ directed path in } G \Rightarrow \exists (x, y) \in R. \)

**Ex 7.33**: \( A = \{1, 2, 3, 4, 5\} \),

\( R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\} \),

\( R_2 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\} \)

\( R_1, R_2 \) : equivalence relations
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Note: $\mathcal{R}$ is equivalence relation $\Leftrightarrow$

in its associated graph $G = (A, \mathcal{R})$, $G = (A, \{(a, a) \mid a \in A\} \cup \bigcup_{j=1}^{k} E(K_{i_j}),$ where $\sum_{j=1}^{k} i_j = |A|$, $i_j \in \mathbb{Z}^+$, $\forall 1 \leq j \leq k$.

i.e. $G$ is one complete graph augmented by loops at every vertex or consists of the disjoint union of complete graphs augmented by loops at every vertex.
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§ 7.3 Partial Orders: Hasse Diagrams

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\[ \begin{array}{cccccc}
N & \longrightarrow & Z & \longrightarrow & Q & \longrightarrow & R & \longrightarrow & C \\
\text{closed under } +, \cdot & 2x + 3 = 4? & x^2 - 2 = 0 & x^2 + 1 = 0 & \text{but not } - & x + 5 = 2? \\
\end{array} \]

Def: 1) \((A, R)\) is called a \textbf{poset (partially ordered set)}

\[ \equiv \text{A relation } R \text{ on } A \text{ is a partial order.} \]

2) \(A\) is called a \textbf{poset} \(\equiv \exists\) \text{ a relation } \(R\) on \(A\)

s.t. \((A, R)\) is a poset.

\[
\forall r_1 \neq r_2 \implies \text{either } r_1 < r_2 \text{ or } r_1 > r_2
\]
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**EX 7.34**: Let $A = \{ x \mid x \text{ is a course offered at a college} \}$
Define $R$ on $A$ by $xRy$ if $x, y$ are the same course or
if $x$ is a prerequisite for $y$.

$\Rightarrow R$ makes $A$ into a poset.

**Ex 7.35**: Let $A = \{ 1, 2, 3, 4 \}$
Define $R = \{ (x, y) \mid x, y \in A, x \mid y \}$
$R = \{ (1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 4) \}$
is a partial orders.

$\therefore (A, R)$ is a poset.
Ex 7.36:

A set of tasks that must be performed in building a house \( R \) on \( A \) by \( xRy \) if \( x, y \) denote the same task or if task \( x \) must be performed before the start of task \( y \).

\[ \Rightarrow A \text{ is a poset} \]

ex: (a)

\[
\begin{align*}
\therefore (1, 2), (2, 1) & \in R \\
\text{with } 1 \neq 2 : \times
\end{align*}
\]

(b)

\[
\begin{align*}
\therefore (1, 2), (2, 3) & \in R \Rightarrow (1, 3) \in R \\
\text{but } (3, 1) & \in R \text{ and } 1 \neq 3 : \times
\end{align*}
\]
Note: In a digraph $G = (A, R)$, when
(1) $\exists a \neq b \in A, (a, b), (b, a) \in R$
(2) $\exists$ a directed cycle
then $R$ cannot be transitive and antisymmetric.
\[\therefore (A, R) \text{ is not a poset.}\]

Ex 7.37: Hasse diagram for $R$ : Give $G = (A, R)$
step 1: eliminate the loops at $x$, $\forall x \in A$.
step 2: eliminate the edges is enough to in sure the existence
by transitive.  (if $\exists (x, y), (y, z) \in R$, eliminate $(x, z)$)
step 3: eliminate the directions : the directions are assumed
to go from the bottom to the top.

ex: 

\begin{center}
\begin{tikzpicture}
\node at (0,0) [circle,draw] (1) [label= accommodate] {1};
\node at (4,0) [circle,draw] (2) [label= accommodate] {2};
\node at (4,-4) [circle,draw] (3) [label= accommodate] {3};
\node at (0,-4) [circle,draw] (4) [label= accommodate] {4};
\node at (2,-2) [circle,draw] (5) [label= accommodate] {5};
\node at (2,-6) [circle,draw] (6) [label= accommodate] {6};
\draw (1) -- (2); \draw (2) -- (3); \draw (3) -- (4);
\end{tikzpicture}
\end{center}
Ex 7.38: (a) \( U = \{1, 2, 3\} \)
\[ A = \mathcal{P}(U), \quad R = \subseteq \]
\[ \{1, 2, 3\} \]
\[ \{1, 2\} \]
\[ \{1, 3\} \]
\[ \{2, 3\} \]
\[ \{1\} \]
\[ \{2\} \]
\[ \{3\} \]
\[ \emptyset \]

(b) \( A = \{1, 2, 4, 8\} \)
\[ R = \{ (x, y) \mid x, y \in A, x \mid y \} \]

(d) \( A = \{2, 3, 5, 6, 7, 11, 12, 35, 385\} \)
\[ R = \{ (x, y) \mid x, y \in A, x \mid y \} \]
§ 7.3 Partial Orders: Hasse Diagrams

**Ex 7.39:** Let $A = \{1, 2, 3, 4, 5\}$, $\mathcal{R}$ on $A$ defined by $x \mathcal{R} y$ if $x \leq y$.

$A$ is a poset, denoted by $(A, \leq)$.

$B = \{1, 2, 4\} \subset A$; $B \times B \cap \mathcal{R}$ is a partial order on $B$

$= \{(1, 1), (2, 2), (4, 4), (1, 2), (1, 4), (2, 4)\}$

**Note:** If $\mathcal{R}$ is a partial order on $A$, then $\forall B \subset A$, $(B, (B \times B) \cap \mathcal{R})$ is a poset.

**ex:** $\{\emptyset, \{1\}, \{3\}, \{1, 3\}, \{1, 2, 3\}\} = B$. [see]

**Def 7.16:** 1) A partial order $\mathcal{R}$ on $A$ is called a **total order** if $\forall x, y \in A$, either $x \mathcal{R} y$ or $y \mathcal{R} x$.

2) $\mathcal{R}$ is a total order on $A$, then $A$ is called **totally ordered**.

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**Ex 7.40:** (a) \((\mathbb{N}, \leq)\) is a total order.
(b) \(\mathcal{U} = \{1, 2, 3\}, (\mathcal{P}(\mathcal{U})), \subseteq\) is not a total order.
   \[\therefore \{1, 2\}, \{1, 3\} \in \mathcal{P}(\mathcal{U}), \text{ but } \{1, 2\} \not\equiv \{1, 3\}, \{1, 3\} \not\equiv \{1, 2\}.\]
(c) **Ex 7.38** (b) shows a total order. see

**Ex 7.41:** 請自己看！

**Q:** Whether we can take the partial order \(\mathcal{R}\), given by the Hasse diagram, and fine a total order \(\mathcal{I}\) on these tasks for which \(\mathcal{R} \subseteq \mathcal{I}\)?

ex:

```
\(\mathcal{R}\):
\(\mathcal{G} \rightarrow \mathcal{F} \rightarrow \mathcal{D} \rightarrow \mathcal{C} \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{E}\)
```

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Topological Sorting Algorithm (for a poset \((A, \mathcal{R})\) with \(|A| = n\))

**Step 1:** Let \(k = 1\). Let \(H_1\) = the Hasse diagram for \((A, \mathcal{R})\)

**Step 2:** Select \(v_k \in V(H_k)\) s.t. no edge in \(H_k\) starts at \(v_k\)

**Step 3:** If \(k = n\), output \(T: v_n < v_{n-1} < \ldots < v_2 < v_1\) and STOP
else \((k < n)\) \{ \(H_{k+1} := H_k - v_k; k := k + 1;\) \}
  go to **Step 2.**

**ex:** \(E < B < A < C < G < F < D\)
\(\Rightarrow 12\) possible answers

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**Def 7.17:** $(A, \mathcal{R})$ is a poset:

1) $x \in A$ is called a **maximal** element of $A$
   \[ \equiv \forall a \in A, a \neq x \Rightarrow x \mathcal{R} a \equiv \forall a \in A, x \mathcal{R} a \Rightarrow x = a. \]

2) $y \in A$ is called a **minimal** element of $A$
   \[ \equiv \forall b \in A, b \neq y \Rightarrow b \mathcal{R} y \equiv \forall b \in A, b \mathcal{R} y \Rightarrow y = b. \]

**Ex 7.42:** Let $\mathcal{U} = \{1, 2, 3\}$, $A = \mathcal{P}(\mathcal{U})$

(a) $\mathcal{U}$ is maximal and $\emptyset$ is minimal for the poset $(A, \subseteq)$

(b) $\forall B = A - \{\{1, 2, 3\}\}$, In $(B, \subseteq)$:
    
    $\{1, 2\}, \{1, 3\}, \{2, 3\}$ are all maximal elements;
    $\emptyset$ is the minimal element.
§ 7.3 Partial Orders: Hasse Diagrams

Ex 7.43: 1) \((\mathbb{Z}, \leq)\) is a poset: **neither a maximal nor a minimal element.**

2) \((\mathbb{N}, \leq)\) is a poset: **minimal element = 0; no maximal element.**

Ex 7.44: In Ex 7.38 (b), (c), (d): see

<table>
<thead>
<tr>
<th></th>
<th>minimal element</th>
<th>maximal element</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>(c)</td>
<td>2, 3, 5, 7</td>
<td>2, 3, 5, 7</td>
</tr>
<tr>
<td>(d)</td>
<td>2, 3, 5, 7, 11</td>
<td>12, 385</td>
</tr>
</tbody>
</table>

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§ 7.3 Partial Orders: Hasse Diagrams

Thm 7.3: If \((A, \mathcal{R})\) is a poset and \(A\) is finite, then \(A\) has both a maximal and a minimal element.

Proof. maximal:

Let \(a_1 \in A\), If \(\forall a \in A, a \neq a_1, a_1 \mathcal{R} a \Rightarrow a_1\) is maximal

else \(\exists a_2 \in A, a_2 \neq a_1, a_1 \mathcal{R} a_2:\)

If \(\forall a \in A, a \neq a_2, a_2 \mathcal{R} a \Rightarrow a_2\) is maximal

else \(\exists a_3 \in A, a_3 \neq a_2, a_2 \mathcal{R} a_3:\)

\[\therefore \mathcal{R}\) is antisymmetric and \(a_1 \mathcal{R} a_2 \therefore a_3 \neq a_1\]

\[\therefore a_1 \mathcal{R} a_2 \) and \(a_2 \mathcal{R} a_3 \therefore a_1 \mathcal{R} a_3\]

If \(\forall a \in A, a \neq a_3, a_3 \mathcal{R} a \Rightarrow a_3\) is maximal

else ...

Continuing in this manner, \(\therefore A\) is finite

\[\therefore \text{We get } a_n \in A \text{ with } \forall a \in A, a \neq a_n, a_n \mathcal{R} a\]

\[\Rightarrow a_n \text{ is maximal.}\]

minimal element can be proved in a similar way.
§ 7.3 Partial Orders: Hasse Diagrams

Note: In the topological sorting algorithm: Step 2 selecting a maximal element from \((A, \mathcal{R})\) or \((B, \mathcal{R}')\), where \(B \subseteq A\); 
\[\mathcal{R}' = (B \times B) \cap \mathcal{R}.\]
\[\Rightarrow\] By Thm 7.3, \(\exists\) at least one such element!

Def 7.18: \((A, \mathcal{R})\) is a poset:
1) \(x \in A\) is called a least element \(\equiv \forall a \in A, x \mathcal{R} a.\)
2) \(y \in A\) is called a greatest element \(\equiv \forall a \in A, a \mathcal{R} y.\)

Ex 7.45: Let \(\mathcal{U} = \{1, 2, 3\}, \mathcal{R} = \subseteq, \) the subset relation
(a) \(A = \mathcal{P}(\mathcal{U})\): \((A, \subseteq)\): least element = \(\emptyset\); greatest element = \(\mathcal{U}\)
(b) \(B = \mathcal{P}(\mathcal{U}) - \{\emptyset\}\): \((B, \subseteq)\): greatest element = \(\mathcal{U}\); no least element, but \(\exists\) 3 minimal element.
§ 7.3 Partial Orders: Hasse Diagrams

Ex 7.46: In Ex 7.38: see

<table>
<thead>
<tr>
<th></th>
<th>least element</th>
<th>greatest element</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>(c)</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>(d)</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>

Thm 7.4: If the poset \((A, R)\) has a greatest (least) element, then the element is unique.

Proof. Suppose \(\exists x, y \in A\) and both are greatest elements

\[\therefore x\text{ is a greatest element} \quad \therefore y \mathcal{R} x\]

\[\therefore y\text{ is a greatest element} \quad \therefore x \mathcal{R} y\]

\[\Rightarrow \therefore \mathcal{R}\text{ is antisymmetric} \quad \therefore x = y\]

The proof for the least element is similar.
Def 7.19: Let \((A, \mathcal{R})\) be a poset with \(B \subseteq A\):
1) \(x \in A\) is called a **lower bound** of \(B \equiv x \mathcal{R} b, \forall b \in B\).
2) \(y \in A\) is called a **upper bound** of \(B \equiv b \mathcal{R} y, \forall b \in B\).
3) A lower bound of \(B\), \(x' \in A\) is called a **greatest lower bound** (\(\text{glb}\)) of \(B \equiv \forall\) lower bounds \(x'' (\neq x')\) of \(B\), \(x'' \mathcal{R} x'\).
4) A upper bound of \(B\), \(y' \in A\) is called a **least upper bound** (\(\text{lub}\)) of \(B \equiv \forall\) upper bounds \(y'' (\neq y')\) of \(B\), \(y'' \mathcal{R} y'\).

Ex 7.47: \(\mathcal{U} = \{1, 2, 3, 4\}, A = \mathcal{P}(\mathcal{U}), B = \{\{1\}, \{2\}, \{1, 2\}\}:
\)
In \((B, \subseteq)\): upper bounds: \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}
\quad lub: \{1, 2\} (\in B)
\quad glb: \emptyset (\notin B)
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**Ex 7.48:** \( R = \leq \) ("less than or equal to")
- a) \( A = \mathbb{R}, B = [0, 1] \): \( B \) has \( \text{glb}: 0 \ (\in B) \) \( \text{lub}: 1 \ (\in B) \)
- A = \mathbb{R}, C = (0, 1]: C has \( \text{glb}: 0 \ (\not\in C) \) \( \text{lub}: 1 \ (\in C) \)
- b) \( A = \mathbb{R}, B = \{ q \in \mathbb{Q} \mid q^2 < 2 \} \): \( B \) has \( \text{glb}: -\sqrt{2} \ (\not\in B) \) \( \text{lub}: \sqrt{2} \ (\not\in B) \)
- c) \( A = \mathbb{Q}, B = \{ q \in \mathbb{Q} \mid q^2 > 2 \} \): \( B \) has \( \text{no} \) \( \text{glb} \) or \( \text{lub} \).

**Thm 7.5:** If \((A, R)\) is a poset and \( B \subseteq A \), then \( B \) has at most one \( \text{lub} \) (\( \text{glb} \)).

**Def 7.20:** The poset \((A, R)\) is called a **lattice**

\[ \equiv \forall x, y \in A, \text{lub}\{x, y\} \text{ and } \text{glb}\{x, y\} \text{ both exist in } A \]

\[ \exists a, b \in A, \text{ which } a = \text{lub}\{x, y\}, b = \text{glb}\{x, y\} \]
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Ex 7.49: \( A = \mathbb{N}, \) define \( \mathcal{R} \) on \( A \) by \( x\mathcal{R}y \) if \( x \leq y: (\mathbb{N}, \leq) \):
\[
lub\{x, y\} = \max \{x, y\}; \quad \text{glb}\{x, y\} = \min \{x, y\}
\]
\( \Rightarrow (\mathbb{N}, \leq) \) is a lattice.

Ex 7.50: \( \mathcal{U} = \{1, 2, 3\} \) in \( (\mathcal{P}(\mathcal{U}), \subseteq) \):
\( \forall S, T \in \mathcal{P}(\mathcal{U}) \)
\[
lub\{S, T\} = S \cup T (\in \mathcal{P}(\mathcal{U})); \quad \text{glb}\{x, y\} = S \cap T (\in \mathcal{P}(\mathcal{U}))
\]
\( \Rightarrow (\mathcal{P}(\mathcal{U}), \subseteq) \) is a lattice.

Ex 7.51: In Ex 7.38 (d):
\[
lub\{2, 3\} = 6; \quad \text{lub}\{3, 6\} = 6; \quad \text{lub}\{5, 7\} = 35; \quad \text{lub}\{7, 11\} = 385; \ldots
\]
\[
\text{glb}\{3, 6\} = 3; \quad \text{glb}\{2, 12\} = 2; \quad \text{glb}\{35, 385\} = 35; \ldots
\]
but \( \not\exists \) \( \text{glb}\{2, 3\} \in A, \not\exists \) \( \text{glb}\{5, 7\} \ldots \)
\( \Rightarrow \) this poset is not a lattice.