§ 7.1 Relations Revisited: Properties of Relations

Checklist

1. **The Properties of Relations**
   - ***Reflexive***
   - ***Symmetric***
   - ***Transitive***
   - ***Antisymmetric***
2. **Special Relations**
   - ***Partial Ordering Relation***
   - ***Equivalence Relation***
3. **Counting**

Computer Science and Information Engineering
National Chi Nan University

Discrete Mathematics

Dr. Justie Su-Tzu Juan

Chapter 7 Relations: The Second Time Around

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5^{th} Edition)**
by Ralph P. Grimaldi

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Outline

1. Composite Relation
2. Relation Matrices
3. The Directed Graph Associated with a Relation

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Def 7.8 : A, B, C: sets, $\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2 \subseteq B \times C$. The *composite relation* $\mathcal{R}_1 \circ \mathcal{R}_2 \subseteq A \times C$ defined by $\mathcal{R}_1 \circ \mathcal{R}_2 = \{(x, z) \mid x \in A, z \in C, \text{ and } \exists\, y \in B \text{ with } (x, y) \in \mathcal{R}_1, (y, z) \in \mathcal{R}_2\}$.

Ex 7.17: $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$, $C = \{5, 6, 7\}$ back

$\mathcal{R}_1 = \{(1, x), (2, x), (3, y), (3, z)\} \subseteq A \times B$

$\mathcal{R}_2 = \{(w, 5), (x, 6)\} \subseteq B \times C$

$\mathcal{R}_3 = \{(w, 5), (w, 6)\} \subseteq B \times C$

$\mathcal{R}_1 \circ \mathcal{R}_2 = \{(1, 6), (2, 6)\}$ $\qquad$ $\mathcal{R}_1 \circ \mathcal{R}_3 = \phi$

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Ex 7.18 : A: employees, B: programming languages,

$C = \{p_1, p_2, \ldots, p_8\}$: projects.

$\mathcal{R}_1 \subseteq A \times B$: $(x, y) \in \mathcal{R}_1$ means x is proficient in y,

$\mathcal{R}_2 \subseteq B \times C$: $(y, z) \in \mathcal{R}_2$ means z need y.

$\Rightarrow \mathcal{R}_1 \circ \mathcal{R}_2$ has been used to set up a matching process between employees and projects on the basis of employee knowledge of specific programming languages.

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Thm 7.1 : A, B, C, D : sets, $\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2 \subseteq B \times C$, $\mathcal{R}_3 \subseteq C \times D$.
The $\mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3$.

Proof.

1. $\mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) \subseteq A \times D$, $(\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3 \subseteq A \times D$.
2. $\forall\ (a, d) \in \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3)$

   $\Rightarrow \exists\ b \in B$ s.t. $(a, b) \in \mathcal{R}_1 \wedge (b, d) \in \mathcal{R}_2 \circ \mathcal{R}_3$

   $\Rightarrow \exists\ c \in C$ s.t. $(b, c) \in \mathcal{R}_2 \wedge (c, d) \in \mathcal{R}_3$

   $\because (a, b) \in \mathcal{R}_1 \wedge (b, c) \in \mathcal{R}_2 \quad \Rightarrow (a, c) \in \mathcal{R}_1 \circ \mathcal{R}_2$

   $\because (a, c) \in \mathcal{R}_1 \circ \mathcal{R}_2 \wedge (c, d) \in \mathcal{R}_3$

   $\Rightarrow (a, d) \in (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3$

   $\therefore \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) \subseteq (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3$

   Similar, $(\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3 \subseteq \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3)$

   $\Rightarrow \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3$

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Def 7.9 : A: sets, $\mathcal{R} \subseteq A \times A$. The *power of* $\mathcal{R}$ defined recursively:

(a) $\mathcal{R}^1 = \mathcal{R}$;

(b) $\mathcal{R}^{n+1} = \mathcal{R} \circ \mathcal{R}^n$, $\forall\, n \in Z^+$.

Ex 7.19: $A = \{1, 2, 3, 4\}$, $\mathcal{R} = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$ back

$\Rightarrow \mathcal{R}^2 = \{(1, 4), (1, 2), (3, 4)\}$

$\Rightarrow \mathcal{R}^3 = \{(1, 4)\}$

$\Rightarrow \mathcal{R}^n = \phi$, $\forall\, n \geq 4$.

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Def 7.10: 1) An *$m \times n$ zero-one matrix $E = (e_{ij})_{m \times n}$*, *(0, 1)-matrix*:
$\equiv$ m rows, n columns, each entry is 0 or 1.
2) e_{ij} $\equiv$ the entry in the ith row and the jth column of E, $\forall$ $1 \le i \le m$ and $1 \le j \le n$.

Ex 7.20 : $E = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ is a 3×4 (0, 1)-matrix.

1) $e_{11} = 1$ 2) $e_{23} = 0$ 3) $e_{31} = 1$

Note : Use the standard operations of matrix addition and multiplication with the stipulation that $1 + 1 = 1$ (*Boolean addition*).

§ 7.2 Computer Recogni… Directed Graphs

$\mathcal{R}_1 = \{(1, x), (2, x), (3, y), (3, z)\} \subseteq A \times B$

$\mathcal{R}_2 = \{(w, 5), (x, 6)\} \subseteq B \times C$

Ex 7.21: The *relation matrices* for $\mathcal{R}_1$, $\mathcal{R}_2$ of Ex 7.17: see

$$M(\mathcal{R}_1) = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array} \begin{array}{c} \begin{array}{cccc} (w) & (x) & (y) & (z) \end{array} \leftarrow B \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad M(\mathcal{R}_2) = \begin{array}{c} \\ (w) \\ (x) \\ (y) \\ (z) \end{array} \begin{array}{c} \begin{array}{ccc} (5) & (6) & (7) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\uparrow$ A

$$M(\mathcal{R}_1) \cdot M(\mathcal{R}_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = M(\mathcal{R}_1 \circ \mathcal{R}_2)$$

Note : $M(\mathcal{R}_1) \cdot M(\mathcal{R}_2) = M(\mathcal{R}_1 \circ \mathcal{R}_2)$

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Ex 7.22 : $A = \{1, 2, 3, 4\}$, $\mathcal{R} = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$, as in **Ex 7.19**. see

Define the *relation matrix* for $\mathcal{R}$: $M(\mathcal{R})$ is the 4×4 (0, 1)-matrix whose entries m_{ij}, for $1 \le i, j \le 4$, are given by $m_{ij} = \begin{cases} 1, \text{ if } (i, j) \in \mathcal{R}, \\ 0, \text{ otherwise.} \end{cases}$

$$M(\mathcal{R}) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (M(\mathcal{R}))^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M(\mathcal{R}^2)$$

$$(M(\mathcal{R}))^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathcal{R}^4 = \phi$$

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

<u>In general</u> : A: set, $|A| = n$, $\mathcal{R} \subseteq A \times A$, $M(\mathcal{R})$ is the relation matrix for $\mathcal{R}$:

(a) $M(\mathcal{R}) = \mathbb{0}$ (all 0's) iff $\mathcal{R} = \phi$

(b) $M(\mathcal{R}) = \mathbb{1}$ (all 1's) iff $\mathcal{R} = A \times A$

(c) $M(\mathcal{R}^m) = [M(\mathcal{R})]^m$, for $m \in \mathrm{Z}^+$.

<u>Def 7.11</u> : $E = (e_{ij})_{m\times n}$, $F = (f_{ij})_{m\times n}$: 2 $m\times n$ (0, 1)-matrices.

E ***precedes*** (or ***is less than***) F, $E \leq F$,

$\equiv e_{ij} \leq f_{ij}$, $\forall$ $1 \leq i \leq m$, $1 \leq j \leq n$.

<u>Ex 7.23</u> : $E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\Rightarrow E \leq F$

$\Rightarrow \exists$ 8 (0, 1)-matrices G for which $E \leq G$.

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Def 7.12 : For $n \in \mathbb{Z}^+$, $I_n = (\delta_{ij})_{n\times n}$ is the $n\times n$ (0, 1)-matrix, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Def 7.13 : Let $A = (a_{ij})_{m\times n}$. The ***transpose*** of A, $A^{tr} = (a^*_{ji})_{n\times m}$, where $a^*_{ji} = a_{ij}$, for all $1 \le j \le n$, $1 \le i \le m$.

Ex 7.24 : $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$, $A^{tr} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Def : 1) $0 \cap 0 = 0 \cap 1 = 1 \cap 0 = 0$, $1 \cap 1 = 1$ (usual multiplication)

2) $E \cap F = (x_{ij})_{m\times n}$, where $x_{ij} = e_{ij} \cap f_{ij}$.

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Thm 7.2 : A: set, $|A| = n$, $\mathcal{R} \subseteq A \times A$, let M denote the relation matrix for $\mathcal{R}$. Then

(a) $\mathcal{R}$ is reflexive iff

(b) $\mathcal{R}$ is symmetric iff

(c) $\mathcal{R}$ is transitive iff

(d) $\mathcal{R}$ is antisymmetric iff

Kahoot!: https://play.kahoot.it/v2/?quizId=2e4a70da-637b-4ab9-8fec-fa99c7679b8f&hostId=e3b5c5c7-c22d-4353-a580-53c46d132332

Discussion (5+5 min):

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Thm 7.2 : A: set, $|A| = n$, $\mathcal{R} \subseteq A \times A$, let M denote the relation matrix for $\mathcal{R}$. Then
(c) $\mathcal{R}$ is transitive iff $M \cdot M = M^2 \leq M$.

Proof. (1/2)

Let $M = (a_{ij})_{n \times n}$.

(c) ($\Leftarrow$) Let $M^2 \leq M$. If $(x, y), (y, z) \in \mathcal{R}$.

$\Rightarrow m_{xy} = m_{yz} = 1$

(m_{xy} means the entry of M in row (x), column (y))

$\Rightarrow s_{xz} = 1$

(s_{xz} means the entry of M^2 in row (x), column (z))

$\because M^2 \leq M \quad \therefore m_{xz} = 1$

$\Rightarrow (x, z) \in \mathcal{R}$ and $\mathcal{R}$ is transitive.

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

<u>Thm 7.2</u> : A: set, $|A| = n$, $\mathcal{R} \subseteq A \times A$, let M denote the relation matrix for $\mathcal{R}$. Then

(c) $\mathcal{R}$ is transitive iff $M \cdot M = M^2 \leq M$.

Proof. (2/2)

(c) ($\Rightarrow$) If $\mathcal{R}$ is transitive

Let $s_{xz} \equiv$ the entry in row (x) and column (z) of $M^2 = 1$

$\because s_{xz} = 1 \quad \therefore \exists\, y \in A$ s.t. $m_{xy} = m_{yz} = 1$

$\Rightarrow (x, y) \in \mathcal{R} \wedge (y, z) \in \mathcal{R}$

$\Rightarrow (x, z) \in \mathcal{R} \quad (\because \mathcal{R}$ is transitive)

$\Rightarrow m_{xz} = 1$

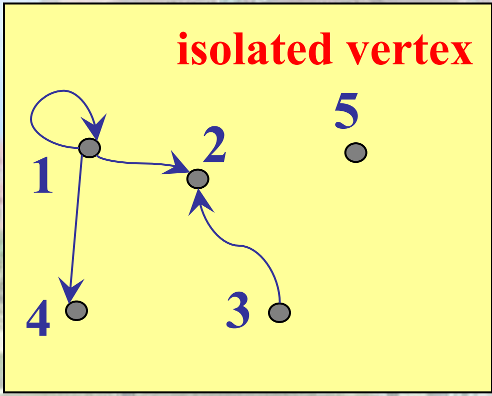
$\therefore M^2 \leq M$.

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

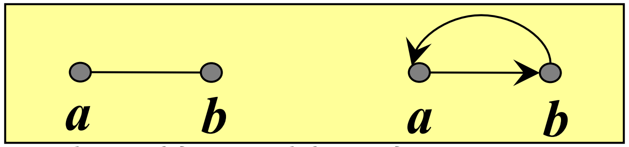
Def 7.14 : V: finite nonempty. *A* ***directed graph*** (or ***digraph***) $G \equiv$

- $G = (V, E)$, where V is called the ***vertex set***, $E \subseteq V \times V$ is called the ***edge set***.
- $v \in V$ is called the ***vertices*** or ***nodes*** of G
- $(a, b) \in E$ is called the ***(directed) edges*** or ***arcs*** of G
- a is called the ***origin*** or ***source*** of (a, b)
- b is called the ***terminus*** or ***terminating vertex*** of (a, b)
- a is ***adjacent to*** b; b is ***adjacent from*** a
- (a, a) is called a ***loop*** at a

Ex 7.25 : $V = \{1, 2, 3, 4, 5\}$,
$E = \{(1, 1), (1, 2), (1, 4), (3, 2)\}$



§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs



<u>Def</u> : If $(a, b), (b, a) \in E$, $(a \neq b)$, then use $\{a, b\} = \{b, a\}$ to represent. a and b are called *adjacent* vertices.

<u>Ex 7.26</u> : *precedence graph* for the computer program

(S1) $b := 3$

(S2) $c := b + 2$

(S3) $a := 1$

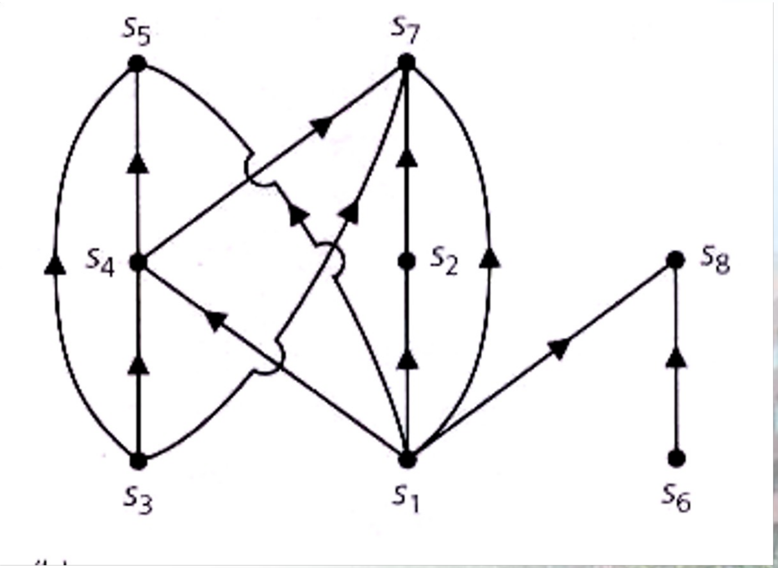
(S4) $d := a \times b + 5$

(S5) $e := d - 1$

(S6) $f := 7$

(S7) $e := c + d$

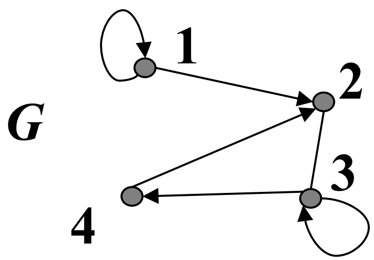
(S8) $g := b \times f$

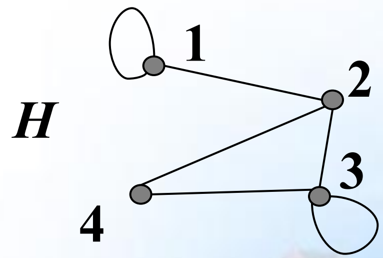


§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Ex 7.27 : $A = \{1, 2, 3, 4\}$, $\mathcal{R} = \{(1, 1), (1, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 2)\}$. The **directed graph associated with $\mathcal{R}$** is $G = (A, \mathcal{R})$, where undirected edge $\{x, y\} = (x, y)$ and (y, x).

The ***associated undirected graph*** ： replace all edges (x, y) by undirected edges $\{x, y\}$. back





§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Def : For an undirected graph $G = (V, E)$:

1) A *x-y* **path** starting at x and ending at $y \equiv$ a finite sequence of undirected edges with no repeat vertex.
2) The ***length*** of a path $\equiv$ the edge on the path
3) A path is ***closed*** $\equiv x = y$
4) A closed path $\equiv$ ***cycle*** ($\geq$ 3 edges) (a finite sequence of undirected edges with no repeat vertex except $x = y$.)
5) A undirected graph is ***connected*** $\equiv \forall\, x \neq y \in V, \exists\, x$-$y$ path

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

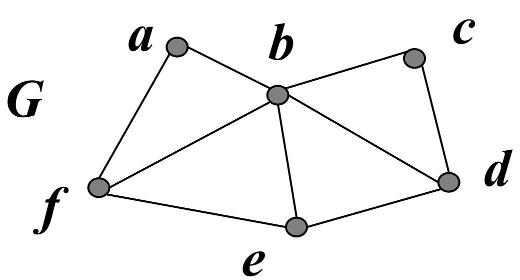
Def : **For an directed graph $G = (V, E)$:**

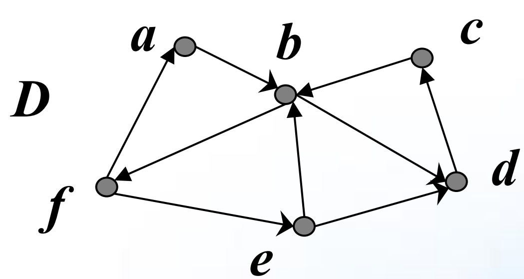
1) **A directed x-y path starting at x and ending at y ≡ a finite sequence of directed edges with no repeat vertex.**
2) **A closed directed path ≡ directed *cycle* (≥ 3 edges) (a finite sequence of directed edges with no repeat vertex except $x = y$.)**

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

<u>Note</u> : (1) loops $\not\subseteq$ cycles;
(2) loops have no bearing on graph connectivity

<u>Ex</u> :





back

(1) $\{a, b\}, \{b, e\}, \{e, f\}, \{f, b\}, \{b, a\}$ is not a path
(2) $(b, f), (f, e), (e, d), (d, c), (c, b)$ = a directed cycle of length 5
(3) $(b, f), (f, e), (e, b), (b, d), (d, c), (c, b) \neq$ directed cycle

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

<u>Def 7.15</u> : A directed graph $G = (V, E)$ is called ***strongly connected***

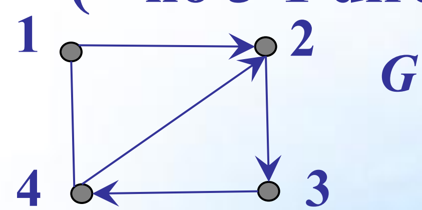
$\equiv \forall\, x, y \in V$, where $x \neq y$, $\exists$ x-y directed path

i.e. $(x, y) \in E$ or $\exists\, v_1, v_2, \ldots, v_n \in V$

s.t. $(x, v_1), (v_1, v_2), \ldots, (v_n, y) \in E$

<u>Ex</u> : In <u>Ex 7.27</u>, G is not strongly connected.

($\because$ no 3-1 directed path) see



<u>Def</u> : ***loop-free*** $\equiv$ no loop

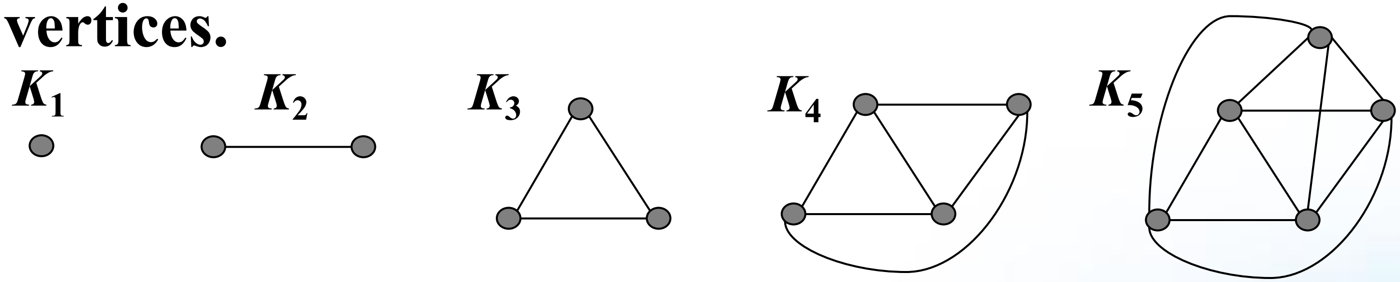
<u>Ex</u> : 1) 上上<u>Ex</u>中，D為strongly connected and loop-free. see

2) G is strongly connected and loop-free.

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Ex 7.29 :

- ***Complete graphs*** **on *n* vertices, K_n ≡ an undirected graph that are loop-free and have an edge for every pair of distinct vertices.**



- **The *adjacency matrix* for $G = (A, \mathcal{R})$ ≡ the relation matrix for $\mathcal{R}$.**

Quiz:
https://play.kahoot.it/v2/?quizId=a3ca3070-b05f-438e-a69c-67f34fb55991&hostId=e3b5c5c7-c22d-4353-a580-53c46d132332

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Note : $\mathscr{R}$ is reflexive $\Leftrightarrow$ in $G = (A, \mathscr{R})$: $\forall\, x \in V(G)$, $\exists$ loop at x.

Note : $\mathscr{R}$ is symmetric $\Leftrightarrow$ in $G = (A, \mathscr{R})$:
$E(G)$ = loops $\cup$ undirected edges

Note : $\mathscr{R}$ is antisymmetric $\Leftrightarrow$ For the associated graph $G = (A, \mathscr{R})$, $E(G)$ = loops $\cup$ directed edges

Note : $\mathscr{R}$ is transitive $\Leftrightarrow$ For the associated graph $G = (A, \mathscr{R})$, $\forall\, x, y \in A$, $\exists$ x-y directed path in $G \Rightarrow \exists\, (x, y) \in \mathscr{R}$.

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Note : $\mathcal{R}$ is equivalence relation $\Leftrightarrow$

in its associated graph $G = (A, \mathcal{R})$, $G = (A, \{(a, a) \mid a \in A\} \cup \bigcup_{j=1}^{k} E(K_{i_j}))$, where $\sum_{j=1}^{k} i_j = |A|$, $i_j \in Z^+$, $\forall\ 1 \leq j \leq k$.

i.e. G is one complete graph augmented by loops at every vertex or consists of the disjoint union of complete graphs augmented by loops at every vertex.

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

<u>**Checklist**</u>

1. **Composite Relation**
   - **Associativity**
   - **The *power of* $\mathcal{R}$**
2. **Relation Matrix**
   - **Definitions**
   - **Thm 7.2: Use the relation matrix to find the properties of the relation.**
3. **The Directed Graph Associated with a Relation**
   - **Definitions**
   - **Use the associated digraph to find the properties of the relation.**
   - **Find a equivalence relation quickly by its associated digraph.**

Computer Science and Information Engineering
National Chi Nan University

Combinatorial Mathematics

Dr. Justie Su-Tzu Juan

Chapter 7 Relations: The Second Time Around

§ 7.3 Partial Orders: Hasse Diagrams

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

§ 7.3 Partial Orders: Hasse Diagrams

Outline

1. **Hasse diagram**
2. **Topological Sorting Algorithm**
3. **Special Elements**
4. **Special Poset**

§ 7.3 Partial Orders: Hasse Diagrams

$\mathbb{N} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{R} \longrightarrow \mathbb{C}$

closed under +, · | $2x + 3 = 4?$ | $x^2 - 2 = 0$ | $x^2 + 1 = 0$

but not –

$x + 5 = 2?$

$\forall r_1 \neq r_2 \Rightarrow$ **either** $r_1 < r_2$ **or** $r_1 > r_2$ ↔ ? ×

Def: 1) (A, $\mathcal{R}$) is called a poset (partially ordered set)
≡ A relation $\mathcal{R}$ on A is a partial order.

2) A is called a poset ≡ ∃ a relation $\mathcal{R}$ on A s.t. (A, $\mathcal{R}$) is a poset.

§ 7.3 Partial Orders: Hasse Diagrams

<u>EX 7.34</u>: Let $A = \{x \mid x$ is a course offered at a college$\}$
Define $\mathcal{R}$ on A by $x\mathcal{R}y$ if x, y are the same course or if x is a prerequisite for y.
$\Rightarrow \mathcal{R}$ makes A into a poset.

<u>Ex 7.35</u>: Let $A = \{1, 2, 3, 4\}$
Define $\mathcal{R} = \{(x, y) \mid x, y \in A, x \mid y\}$
$\mathcal{R} = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 4)\}$ is a partial orders.
$\therefore (A, \mathcal{R})$ is a poset.

§ 7.3 Partial Orders: Hasse Diagrams

<u>**Ex 7.36**</u>**:**

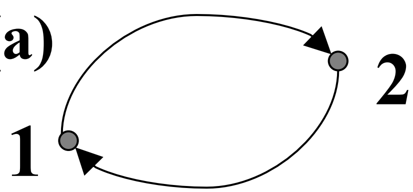
A = a set of tasks that must be performed in building a house

$\mathcal{R}$ on A by $x\mathcal{R}y$ if x, y denote the same task or

if task x must be performed before the start of task y.

$\Rightarrow$ A is a poset

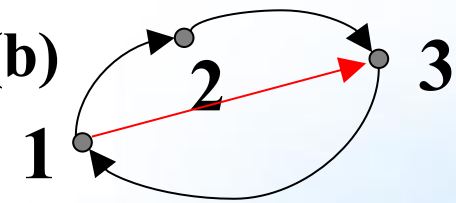
ex: (a)



$\because$ (1, 2), (2, 1) $\in \mathcal{R}$

with 1 $\neq$ 2 : ~~A~~

(b)



$\because$ (1, 2), (2, 3) $\in \mathcal{R}$ $\Rightarrow$ (1, 3) $\in \mathcal{R}$

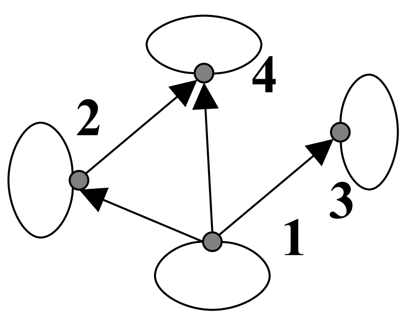
but (3, 1) $\in \mathcal{R}$ and 1 $\neq$ 3 : ~~A~~

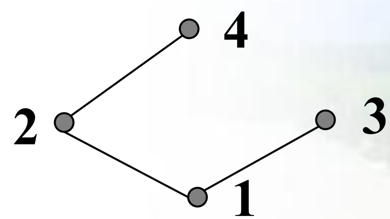
§ 7.3 Partial Orders: Hasse Diagrams

Note: In a digraph $G = (A, \mathcal{R})$, when
(1) $\exists\ a \neq b \in A$, $(a, b), (b, a) \in \mathcal{R}$, or
(2) $\exists$ a directed cycle
then $\mathcal{R}$ cannot be transitive and antisymmetric.
$\therefore$ $(A, \mathcal{R})$ is not a poset.

Ex 7.37: Hasse diagram for $\mathcal{R}$: Give $G = (A, \mathcal{R})$
step 1: eliminate the loops at x, $\forall\ x \in A$.
step 2: eliminate the edges is enough to in sure the existence by transitive. (if $\exists\ (x, y), (y, z) \in \mathcal{R}$, eliminate (x, z))
step 3: eliminate the directions : the directions are assumed to go from the bottom to the top.

ex:

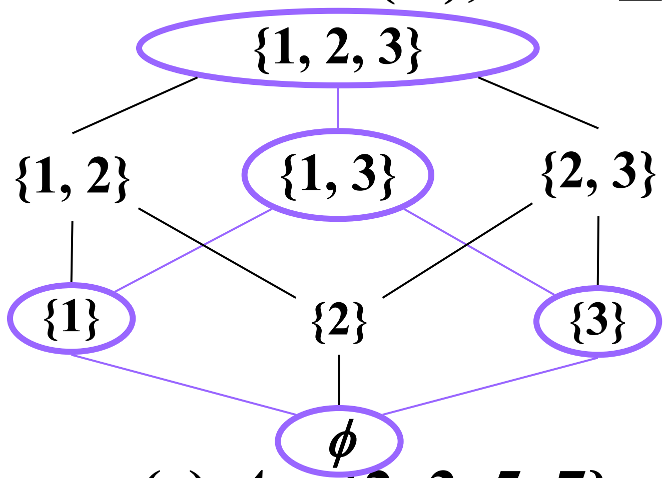




§ 7.3 Partial Orders: Hasse Diagrams

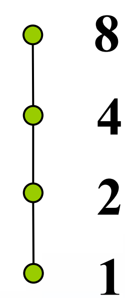
Ex 7.38: (a) $\mathcal{U} = \{1, 2, 3\}$ back

$A = \mathcal{P}(\mathcal{U})$, $\mathcal{R} = \subseteq$



(b) $A = \{1, 2, 4, 8\}$ back

$\mathcal{R} = \{(x, y) \mid x, y \in A, x \mid y\}$



(c) $A = \{2, 3, 5, 7\}$

$\mathcal{R} = \{(x, y) \mid x, y \in A, x \mid y\}$

back

back

(d) $A = \{2, 3, 5, 6, 7, 11, 12, 35, 385\}$

$\mathcal{R} = \{(x, y) \mid x, y \in A, x \mid y\}$

back

§ 7.3 Partial Orders: Hasse Diagrams

Ex 7.39: Let $A = \{1, 2, 3, 4, 5\}$, $\mathcal{R}$ on A defined by $x\mathcal{R}y$ if $x \le y$

A is a poset, denoted by $(A, \le)$.

$B = \{1, 2, 4\} \subset A$; $B \times B \cap \mathcal{R}$ is a partial order on B

$= \{(1, 1), (2, 2), (4, 4), (1, 2), (1, 4), (2, 4)\}$

Note: If $\mathcal{R}$ is a partial order on A, then $\forall\ B \subset A$, $(B, (B\times B)\cap\mathcal{R})$ is a poset.

ex: $\{\phi, \{1\}, \{3\}, \{1, 3\}, \{1, 2, 3\}\} = B$. see

Def 7.16: 1) A partial order $\mathcal{R}$ on A is called a **total order** if $\forall\ x, y \in A$, either $x\mathcal{R}y$ or $y\mathcal{R}x$.

2) $\mathcal{R}$ is a total order on A, then A is called **totally ordered**.

§ 7.3 Partial Orders: Hasse Diagrams

Ex 7.40: (a) $(\mathbb{N}, \leq)$ is a total order.

(b) $\mathcal{U} = \{1, 2, 3\}$, $(\mathcal{P}(\mathcal{U})), \subseteq)$ is not a total order.

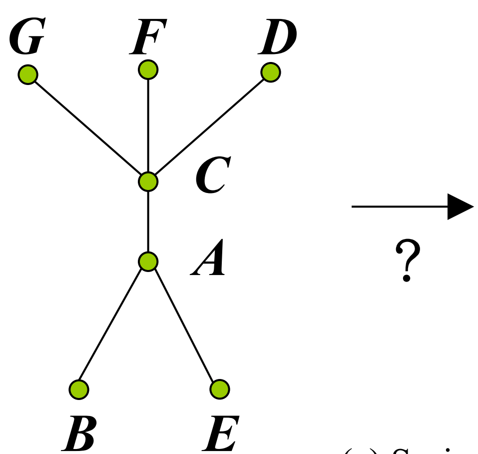
$\because$ $\{1, 2\}, \{1, 3\} \in \mathcal{P}(\mathcal{U})$, but $\{1, 2\} \not\mathcal{R} \{1, 3\}$, $\{1, 3\} \not\mathcal{R} \{1, 2\}$.

(c) Ex 7.38 (b) shows a total order. see

Ex 7.41: 請自己看！

Q: Whether we can take the partial order $\mathcal{R}$, given by the Hasse diagram, and fine a total order $\mathcal{T}$ on these tasks for which $\mathcal{R} \subseteq \mathcal{T}$?

ex:

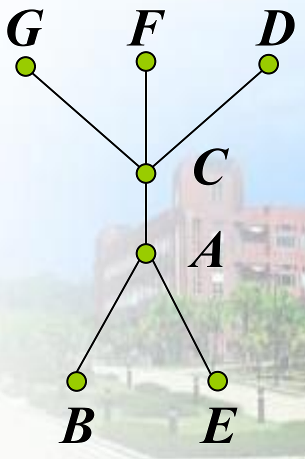


§ 7.3 Partial Orders: Hasse Diagrams

Topological Sorting Algorithm (for a poset $(A, \mathcal{R})$ with $|A| = n$)

Step 1: Let $k = 1$. Let H_1 = the Hasse diagram for $(A, \mathcal{R})$
Step 2: Select $v_k \in V(H_k)$ s.t. no edge in H_k starts at v_k
Step 3: If $k = n$, output $\mathcal{T}$: $v_n < v_{n-1} < \ldots < v_2 < v_1$ and STOP
else $(k < n)$ { $H_{k+1} := H_k - v_k$; $k := k + 1$;
go to Step2.}

ex: $E < B < A < C < G < F < D$
$\Rightarrow$ 12 possible answers



§ 7.3 Partial Orders: Hasse Diagrams

Def 7.17: $(A, \mathcal{R})$ is a poset:

1) $x \in A$ is called a **maximal** element of A

   $\equiv \forall\, a \in A, a \neq x \Rightarrow x \not\mathcal{R} a \equiv \forall\, a \in A, x\mathcal{R}a \Rightarrow x = a.$

2) $y \in A$ is called a **minimal** element of A

   $\equiv \forall\, b \in A, b \neq y \Rightarrow b \not\mathcal{R} y \equiv \forall\, b \in A, b\mathcal{R}y \Rightarrow y = b.$

Ex 7.42: Let $\mathcal{U} = \{1, 2, 3\}$, $A = \mathcal{P}(\mathcal{U})$

(a) $\mathcal{U}$ is maximal and ϕ is minimal for the poset $(A, \subseteq)$

(b) $\forall\, B = A - \{\{1, 2, 3\}\}$, In $(B, \subseteq)$:

   $\{1, 2\}, \{1, 3\}, \{2, 3\}$ are all maximal elements;

   ϕ is the minimal element.

§ 7.3 Partial Orders: Hasse Diagrams

Ex 7.43: 1) $(\mathbb{Z}, \leq)$ is a poset: neither a maximal nor a minimal element.

2) $(\mathbb{N}, \leq)$ is a poset: minimal element = 0; no maximal element.

Ex 7.44: In Ex 7.38 (b), (c), (d): see

	minimal element	maximal element
(b)	1	8
(c)	2, 3, 5, 7	2, 3, 5, 7
(d)	2, 3, 5, 7, 11	12, 385

§ 7.3 Partial Orders: Hasse Diagrams

Thm 7.3: If $(A, \mathcal{R})$ is a poset and A is finite, then A has both a maximal and a minimal element.

Proof. maximal:

Let $a_1 \in A$, If $\forall\, a \in A, a \neq a_1, a_1 \not\mathcal{R} a \Rightarrow a_1$ is maximal

else $\exists\, a_2 \in A, a_2 \neq a_1, a_1 \mathcal{R} a_2$:

If $\forall\, a \in A, a \neq a_2, a_2 \not\mathcal{R} a \Rightarrow a_2$ is maximal

else $\exists\, a_3 \in A, a_3 \neq a_2, a_2 \mathcal{R} a_3$:

$\because \mathcal{R}$ is antisymmetric and $a_1 \mathcal{R} a_2 \ \therefore\ a_3 \neq a_1$

$\because a_1 \mathcal{R} a_2$ and $a_2 \mathcal{R} a_3 \quad \therefore\ a_1 \mathcal{R} a_3$

If $\forall\, a \in A, a \neq a_3, a_3 \not\mathcal{R} a \Rightarrow a_3$ is maximal

else …

Continuing in this manner, $\because A$ is finite

$\therefore$ We get $a_n \in A$ with $\forall\, a \in A, a \neq a_n, a_n \not\mathcal{R} a$

$\Rightarrow a_n$ is maximal.

minimal element can be proved in a similar way.

§ 7.3 Partial Orders: Hasse Diagrams

Note: In the topological sorting algorithm: Step2 selecting a maximal element from $(A, \mathcal{R})$ or $(B, \mathcal{R}')$, where $B \subseteq A$; $\mathcal{R}' = (B \times B) \cap \mathcal{R}$.
$\Rightarrow$ By Thm 7.3, $\exists$ at least one such element!

Def 7.18: $(A, \mathcal{R})$ is a poset:
1) $x \in A$ is called a least element $\equiv \forall\ a \in A, x\mathcal{R}a$.
2) $y \in A$ is called a greatest element $\equiv \forall\ a \in A, a\mathcal{R}y$.

Ex 7.45: Let $\mathcal{U} = \{1, 2, 3\}$, $\mathcal{R} = \subseteq$, the subset relation
(a) $A = \mathcal{P}(\mathcal{U})$: $(A, \subseteq)$: least element $= \phi$; greatest element $= \mathcal{U}$
(b) $B = \mathcal{P}(\mathcal{U}) - \{\phi\}$: $(B, \subseteq)$: greatest element $= \mathcal{U}$; no least element, but $\exists$ 3 minimal element.

§ 7.3 Partial Orders: Hasse Diagrams

Ex 7.46: In Ex 7.38: see

	least element	greatest element
(b)	1	8
(c)	no	no
(d)	no	no

Thm 7.4: If the poset $(A, \mathcal{R})$ has a greatest (least) element, then the element is unique.

Proof. Suppose $\exists\ x, y \in A$ and both are greatest elements

$\because x$ is a greatest element $\quad \therefore y\mathcal{R}x$

$\because y$ is a greatest element $\quad \therefore x\mathcal{R}y$

$\Rightarrow \because \mathcal{R}$ is antisymmetric $\quad \therefore x = y$

The proof for the least element is similar.

§ 7.3 Partial Orders: Hasse Diagrams

Def 7.19: Let $(A, \mathcal{R})$ be a poset with $B \subseteq A$:

1) $x \in A$ is called a **lower bound** of $B \equiv x\mathcal{R}b, \forall\ b \in B$.
2) $y \in A$ is called a **upper bound** of $B \equiv b\mathcal{R}y, \forall\ b \in B$.
3) A lower bound of B, $x' \in A$ is called a **greatest lower bound (*glb*)** of $B \equiv \forall$ lower bounds x'' $(\neq x')$ of B, $x''\mathcal{R}x'$.
4) A upper bound of B, $y' \in A$ is called a **least upper bound (*lub*)** of $B \equiv \forall$ upper bounds y'' $(\neq y')$ of B, $y'\mathcal{R}y''$.

Ex 7.47: $\mathcal{U} = \{1, 2, 3, 4\}$, $A = \mathcal{P}(\mathcal{U})$, $B = \{\{1\}, \{2\}, \{1, 2\}\}$:

In $(B, \subseteq)$: upper bounds: $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 3, 4\}$

lub: $\{1, 2\}$ $(\in B)$

glb: ϕ $(\notin B)$

§ 7.3 Partial Orders: Hasse Diagrams

<u>Ex 7.48</u>: $\mathscr{R} = \leq$ ("less than or equal to")

a) $A = \mathbb{R}$, $B = [0, 1]$: B has glb: <u>0 ($\in B$)</u> lub: <u>1 ($\in B$)</u>

$A = \mathbb{R}$, $C = (0, 1]$: C has glb: <u>0 ($\notin C$)</u> lub: <u>1 ($\in C$)</u>

b) $A = \mathbb{R}$, $B = \{q \in \mathbb{Q} \mid q^2 < 2\}$: B has glb: <u>$-\sqrt{2}$ ($\notin B$)</u> lub: <u>$\sqrt{2}$ ($\notin B$)</u>

c) $A = \mathbb{Q}$, $B = \{q \in \mathbb{Q} \mid q^2 > 2\}$: B has <u>no</u> glb or lub.

<u>Thm 7.5</u>: If $(A, \mathscr{R})$ is a poset and $B \subseteq A$, then B has at most one lub (glb).

<u>Def 7.20</u>: The poset $(A, \mathscr{R})$ is called a lattice

$\equiv \forall\, x, y \in A$, lub$\{x, y\}$ and glb$\{x, y\}$ both exist in A

($\exists\, a, b \in A$, which $a =$ lub$\{x, y\}$, $b =$ glb$\{x, y\}$)

§ 7.3 Partial Orders: Hasse Diagrams

Ex 7.49: $A = \mathbb{N}$, define $\mathcal{R}$ on A by $x\mathcal{R}y$ if $x \le y$: $(\mathbb{N}, \le)$:

lub$\{x, y\}$ = max$\{x, y\}$; glb$\{x, y\}$ = min$\{x, y\}$

$\Rightarrow$ $(\mathbb{N}, \le)$ is a lattice.

Ex 7.50: $\mathcal{U} = \{1, 2, 3\}$ in $(\mathcal{P}(\mathcal{U}), \subseteq)$: $\forall\ S, T \in \mathcal{P}(\mathcal{U})$

lub$\{S, T\}$ = $S \cup T$ ($\in \mathcal{P}(\mathcal{U})$); glb$\{x, y\}$ = $S \cap T$ ($\in \mathcal{P}(\mathcal{U})$)

$\Rightarrow$ $(\mathcal{P}(\mathcal{U}), \subseteq)$ is a lattice.

Ex 7.51: In Ex 7.38 (d): see

lub{2, 3} = 6; lub{3, 6} = 6; lub{5, 7} = 35; lub{7, 11} = 385; …

glb{3, 6} = 3; glb{2, 12} = 2; glb{35, 385} = 35; …

but $\nexists$ glb{2, 3} $\in A$, $\nexists$ glb{5, 7} …

$\Rightarrow$ this poset is not a lattice.

§ 7.3 Partial Orders: Hasse Diagrams

Checklist

1. **Hasse diagram**
2. **Topological Sorting Algorithm**
3. **Special Elements**
   - ☐ **Maximal, minimal**
   - ☐ **Least, greatest**
   - ☐ **Lower bound, upper bound**
   - ☐ **glb, lub**
4. **Special Poset**
   - ☐ **Total Order**
   - ☐ **Lattice**