Computer Science and Information Engineering
National Chi Nan University

Combinatorial Mathematics

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Chapter 7 Relations: The Second Time Around

§ 7.1 Relations Revisited: Properties of Relations

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

§ 7.1 Relations Revisited: Properties of Relations

<u>Def 7.1</u> : A, B: sets,

$\mathcal{R} \subseteq A \times B$: $\mathcal{R}$ is called a **relation from A to B**;

$\mathcal{R} \subseteq A \times A$: $\mathcal{R}$ is called a **relation on A.**

e.q. $A = \{1, 2\}$, $B = \{x, y, z\}$

$A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z)\}$

$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

$\mathcal{R}_1 = \{(2, x), (2, y)\} \subseteq A \times B$

$\mathcal{R}_2 = \{(1, 1), (2, 1), (2, 2)\} \subseteq A \times A$

<u>Ex 7.5</u> : If $|A| = n$, $|A \times A| = n^2$,

there are 2^{n^2} relations on A.

<u>Q1</u>: If $|A| = n$, $|B| = m$,

$|A \times B| =$ <u>(1)</u>, and there are <u>(2)</u> relations from A to B.

§ 7.1 Relations Revisited: Properties of Relations

Ex 7.1 :

a) Defined $\mathcal{R}$ on $\mathbb{Z}$ by $a \mathcal{R} b$ or $(a, b) \in \mathcal{R}$, if $a \le b$:
$\mathcal{R}$ is the ordinary "less than or equal to" relation on $\mathbb{Z}$.
($\mathbb{Z}$ 可改成 $\mathbb{Q}$, $\mathbb{R}$, but not on $\mathbb{C}$)

b) Let $n \in \mathbb{Z}^+$, Define $\mathcal{R}$ on $\mathbb{Z}$ by $x \mathcal{R} y$, if $n \mid (x - y)$:
$\mathcal{R}$ is the modulo n relation on $\mathbb{Z}$.
ex. $n = 7$: $9 \mathcal{R} 2$, $-3 \mathcal{R} 11$, $(14, 0) \in \mathcal{R}$, $3 \not\mathcal{R} 7$ (3 is not related to 7).

c) Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $C \subseteq U$, $C = \{1, 2, 3, 6\}$
Define $\mathcal{R}$ on $\mathcal{P}(U)$ by $A \mathcal{R} B$, if $A \cap C = B \cap C$
ex: $\{1, 2, 4, 5\}$ and $\{1, 2, 5, 7\}$ are related,
$X = \{4, 5\}$ and $Y = \{7\}$ are related;
$S = \{1, 2, 3, 4, 5\}$ and $T = \{1, 2, 3, 6, 7\}$ are not related ($S \not\mathcal{R} T$)

§ 7.1 Relations Revisited: Properties of Relations

Def 7.2 : A relation $\mathcal{R}$ on A is called **reflexive** $\equiv$

$$\forall x \in A, (x, x) \in \mathcal{R}.$$

Ex 7.5 : If $|A| = n$, $|A \times A| = n^2$, there are 2^{n^2} relations on A.

How many of these are reflexive?

$2^{(n^2-n)}$

Def 7.3 : A relation $\mathcal{R}$ on A is called **symmetric**

$$\equiv \forall x, y \in A, (x, y) \in \mathcal{R} \Rightarrow (y, x) \in \mathcal{R}.$$

Note : Let $|A| = n$

1) How many relations on A are symmetric? $2^{(n^2+n)/2}$
2) Both reflexive and symmetric? $2^{(n^2-n)/2}$

§ 7.1 Relations Revisited: Properties of Relations

Def 7.4 : A relation $\mathcal{R}$ on A is called **transitive**

$$\equiv \forall\, x, y, z \in A,\ (x, y), (y, z) \in \mathcal{R} \Rightarrow (x, z) \in \mathcal{R}.$$

Def 7.5 : A relation $\mathcal{R}$ on A is called **antisymmetric**

$$\equiv \forall\, a, b \in A,\ (a\ \mathcal{R}\ b \text{ and } b\ \mathcal{R}\ a) \Rightarrow a = b.$$

Note : How many relations of A are antisymmetric? ($|A| = n$) ?

$(2^n)(3^{(n^2-n)/2})$

Discussion (5 min):

§ 7.1 Relations Revisited: Properties of Relations

Def 7.6 : A relation $\mathscr{R}$ on A is called a ***partial order*** or a ***partial ordering relation***, if $\mathscr{R}$ is reflexive, antisymmetric, and transitive.

Def 7.7 : An ***equivalence relation*** $\mathscr{R}$ on a set A is a relation that is reflexive, symmetric, and transitive.

Quiz:

https://play.kahoot.it/v2/?quizId=6585b9dd-9928-4ad2-ab9b-24ed182a3eb2&hostId=e3b5c5c7-c22d-4353-a580-53c46d132332

Q2:

§ 7.1 Relations Revisited: Properties of Relations

b) Let $A = \{1, 2, 3\}$, then

$\mathcal{R}_1 = \{(1, 1), (2, 2), (3, 3)\}$

$\mathcal{R}_2 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$

$\mathcal{R}_3 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$

$\mathcal{R}_4 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

are all equivalence relations on A?

c) For a given finite set $A = \{a_1, a_2, \ldots, a_n\}$,

$A \times A$: **the largest equivalence relation on A.**

$\mathcal{R} = \{(a_i, a_i) \mid 1 \le i \le n\}$: **the smallest equivalence relation on A.**

(equality relation)

e) If $\mathcal{R}$ is a relation on A, then $\mathcal{R}$ is both an equivalent relation and a partial order on A if and only if $\mathcal{R}$ is the equality relation on A.

§ 7.1 Relations Revisited: Properties of Relations

Discussion (10 min): Exercises 7.1

5. For each of the following relations, determine whether the relation is reflexive, symmetric, antisymmetric, or transitive.
   (d) On the set A of all lines in R^2, define the relation $\mathcal{R}$ for two lines l_1, l_2 by $l_1 \mathcal{R} l_2$ if l_1 is perpendicular to l_2.
   (f) $\mathcal{R}$ is the relation on Z where $x \mathcal{R} y$ if $x - y$ is even.

10. If $A = \{w, x, y, z\}$, determine the number of relations on A that are (a) reflexive; (b) symmetric; (c) reflexive and symmetric; (d) reflexive and contain (x, y); (e) symmetric and contain (x, y); (f) antisymmetric; (g) antisymmetric and contain (x, y); (h) symmetric and antisymmetric; and (i) reflexive, symmetric, and antisymmetric.

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Chapter 7 Relations: The Second Time Around

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs (1)

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§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

<u>Def 7.8</u> : A, B, C: sets, $\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2 \subseteq B \times C$. The ***composite relation*** $\mathcal{R}_1 \circ \mathcal{R}_2 \subseteq A \times C$ defined by

$\mathcal{R}_1 \circ \mathcal{R}_2 = \{(x, z) \mid x \in A, z \in C, \text{ and } \exists\, y \in B \text{ with } (x, y) \in \mathcal{R}_1, (y, z) \in \mathcal{R}_2\}$.

<u>Ex 7.17</u>: $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$, $C = \{5, 6, 7\}$ back

$\mathcal{R}_1 = \{(1, x), (2, x), (3, y), (3, z)\} \subseteq A \times B$

$\mathcal{R}_2 = \{(w, 5), (x, 6)\} \subseteq B \times C$

$\mathcal{R}_3 = \{(w, 5), (w, 6)\} \subseteq B \times C$

$\mathcal{R}_1 \circ \mathcal{R}_2 = \{(1, 6), (2, 6)\}$ $\qquad$ $\mathcal{R}_1 \circ \mathcal{R}_3 = \phi$

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

<u>**Ex 7.18**</u> **:** ***A*****: employees,** ***B*****: programming languages,**

$C = \{p_1, p_2, \ldots, p_8\}$**: projects.**

$\mathcal{R}_1 \subseteq A \times B : (x, y) \in \mathcal{R}_1$ **means** x **is proficient in** y**,**

$\mathcal{R}_2 \subseteq B \times C : (y, z) \in \mathcal{R}_2$ **means** z **need** y**.**

$\Rightarrow \mathcal{R}_1 \circ \mathcal{R}_2$ **has been used to set up a matching process between employees and projects on the basis of employee knowledge of specific programming languages.**

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Thm 7.1 : A, B, C, D : sets, $\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2 \subseteq B \times C$, $\mathcal{R}_3 \subseteq C \times D$.
The $\mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3$.

Proof.

1. $\mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) \subseteq A \times D$, $(\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3 \subseteq A \times D$.
2. $\forall (a, d) \in \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3)$
   $\Rightarrow \exists\, b \in B$ s.t. $(a, b) \in \mathcal{R}_1 \wedge (b, d) \in \mathcal{R}_2 \circ \mathcal{R}_3$
   $\Rightarrow \exists\, c \in C$ s.t. $(b, c) \in \mathcal{R}_2 \wedge (c, d) \in \mathcal{R}_3$
   $\because (a, b) \in \mathcal{R}_1 \wedge (b, c) \in \mathcal{R}_2 \quad \Rightarrow (a, c) \in \mathcal{R}_1 \circ \mathcal{R}_2$
   $\because (a, c) \in \mathcal{R}_1 \circ \mathcal{R}_2 \wedge (c, d) \in \mathcal{R}_3$
   $\Rightarrow (a, d) \in (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3$
   $\therefore \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) \subseteq (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3$
   Similar, $(\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3 \subseteq \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3)$
   $\Rightarrow \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3$

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Def 7.9 : A: sets, $\mathcal{R} \subseteq A \times A$. The *power of $\mathcal{R}$* defined recursively:

(a) $\mathcal{R}^1 = \mathcal{R}$;

(b) $\mathcal{R}^{n+1} = \mathcal{R} \circ \mathcal{R}^n$, $\forall\, n \in Z^+$.

Ex 7.19: $A = \{1, 2, 3, 4\}$, $\mathcal{R} = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$ back

$\Rightarrow \mathcal{R}^2 = \{(1, 4), (1, 2), (3, 4)\}$

$\Rightarrow \mathcal{R}^3 = \{(1, 4)\}$

$\Rightarrow \mathcal{R}^n = \phi, \forall\, n \geq 4.$

Def 7.10: 1) An *$m \times n$ zero-one matrix $E = (e_{ij})_{m\times n}$*, *(0, 1)-matrix*:

$\equiv$ m rows, n columns, each entry is 0 or 1.

2) $e_{ij} \equiv$ the entry in the ith row and the jth column of E, $\forall\, 1 \leq i \leq m$ and $1 \leq j \leq n$.

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Ex 7.20 : $E = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ **is a 3 × 4 (0, 1)-matrix.**

1) $e_{11} = 1$ **2)** $e_{23} = 0$ **3)** $e_{31} = 1$

Note : Use the standard operations of matrix addition and multiplication with the stipulation that 1 + 1 = 1 (*Boolean addition*).

§ 7.2 Computer Recogni
Directed Graphs

$\mathcal{R}_1 = \{(1, x), (2, x), (3, y), (3, z)\} \subseteq A \times B$

$\mathcal{R}_2 = \{(w, 5), (x, 6)\} \subseteq B \times C$

Ex 7.21: The ***relation matrices*** for $\mathcal{R}_1$, $\mathcal{R}_2$ of **Ex 7.17**: see

$$M(\mathcal{R}_1) = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array} \begin{array}{c} (w)\ (x)\ (y)\ (z) \leftarrow B \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad M(\mathcal{R}_2) = \begin{array}{c} \\ (w) \\ (x) \\ (y) \\ (z) \end{array} \begin{array}{c} (5)\ (6)\ (7) \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\uparrow$
A

$$M(\mathcal{R}_1) \cdot M(\mathcal{R}_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = M(\mathcal{R}_1 \circ \mathcal{R}_2)$$

Note : $M(\mathcal{R}_1) \cdot M(\mathcal{R}_2) = M(\mathcal{R}_1 \circ \mathcal{R}_2)$

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Ex 7.22 : $A = \{1, 2, 3, 4\}$, $\mathcal{R} = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$, as in **Ex 7.19**. see

Define the *relation matrix* for $\mathcal{R}$: $M(\mathcal{R})$ is the 4×4 (0, 1)-matrix whose entries m_{ij}, for $1 \le i, j \le 4$, are given by $m_{ij} = \begin{cases} 1, & \text{if } (i, j) \in \mathcal{R}, \\ 0, & \text{otherwise.} \end{cases}$

$$M(\mathcal{R}) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (M(\mathcal{R}))^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M(\mathcal{R}^2)$$

$$(M(\mathcal{R}))^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathcal{R}^4 = \phi$$

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

In general : A: set, $|A| = n$, $\mathcal{R} \subseteq A \times A$, $M(\mathcal{R})$ is the relation matrix for $\mathcal{R}$:

(a) $M(\mathcal{R}) = \mathbb{0}$ (all 0's) iff $\mathcal{R} = \phi$

(b) $M(\mathcal{R}) = \mathbb{1}$ (all 1's) iff $\mathcal{R} = A \times A$

(c) $M(\mathcal{R}^m) = [M(\mathcal{R})]^m$, for $m \in \mathbb{Z}^+$.

Def 7.11 : $E = (e_{ij})_{m\times n}$, $F = (f_{ij})_{m\times n}$: 2 $m\times n$ (0, 1)-matrices.

E ***precedes*** (or ***is less than***) F, $E \leq F$,

$\equiv e_{ij} \leq f_{ij}$, $\forall\ 1 \leq i \leq m$, $1 \leq j \leq n$.

Ex 7.23 : $E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\Rightarrow E \leq F$

$\Rightarrow \exists$ 8 (0, 1)-matrices G for which $E \leq G$.

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Def 7.12 : For $n \in \mathbb{Z}^+$, $I_n = (\delta_{ij})_{n\times n}$ is the $n\times n$ (0, 1)-matrix, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Def 7.13 : Let $A = (a_{ij})_{m\times n}$. The ***transpose*** of A, $A^{tr} = (a^*_{ji})_{n\times m}$, where $a^*_{ji} = a_{ij}$, for all $1 \le j \le n$, $1 \le i \le m$.

Ex 7.24 : $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$, $A^{tr} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Def : 1) $0 \cap 0 = 0 \cap 1 = 1 \cap 0 = 0$, $1 \cap 1 = 1$ (usual multiplication)

2) $E \cap F = (x_{ij})_{m\times n}$, where $x_{ij} = e_{ij} \cap f_{ij}$.

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Thm 7.2 : A: set, $|A| = n$, $\mathcal{R} \subseteq A \times A$, let M denote the relation matrix for $\mathcal{R}$. Then

(a) $\mathcal{R}$ is reflexive iff $I_n \leq M$.

(b) $\mathcal{R}$ is symmetric iff $M = M^{tr}$.

(c) $\mathcal{R}$ is transitive iff $M \cdot M = M^2 \leq M$.

(d) $\mathcal{R}$ is antisymmetric iff $M \cap M^{tr} \leq I_n$.

Proof. (1/2)

Let $M = (a_{ij})_{n \times n}$.

(c) ($\Leftarrow$) Let $M^2 \leq M$. If $(x, y), (y, z) \in \mathcal{R}$.

$\Rightarrow m_{xy} = m_{yz} = 1$

(m_{xy} means the entry of M in row (x), column (y))

$\Rightarrow s_{xz} = 1$

(s_{xz} means the entry of M^2 in row (x), column (z))

$\because M^2 \leq M \quad \therefore m_{xz} = 1$

$\Rightarrow (x, z) \in \mathcal{R}$ and $\mathcal{R}$ is transitive.

§ 7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

<u>Thm 7.2</u> : A: set, $|A| = n$, $\mathcal{R} \subseteq A \times A$, let M denote the relation matrix for $\mathcal{R}$. Then

(c) $\mathcal{R}$ is transitive iff $M \cdot M = M^2 \leq M$.

Proof. (2/2)

(c) ($\Rightarrow$) If $\mathcal{R}$ is transitive

Let $s_{xz} \equiv$ the entry in row (x) and column (z) of $M^2 = 1$

$\because s_{xz} = 1 \quad \therefore \exists\, y \in A$ s.t. $m_{xy} = m_{yz} = 1$

$\Rightarrow (x, y) \in \mathcal{R} \wedge (y, z) \in \mathcal{R}$

$\Rightarrow (x, z) \in \mathcal{R} \quad (\because \mathcal{R}$ is transitive)

$\Rightarrow m_{xz} = 1$

$\therefore M^2 \leq M$.