Computer Science and Information Engineering
National Chi Nan University

Discrete Mathematics

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Chap 5 Relations and Functions

§ 5.4 Special Functions (2)

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

§ 5.4 Special Functions

Def 5.12 : Let $f : A \times A \to B$, (i.e. f is a binary operation on A)

ⓐ f is said to be ***commutative*** $\equiv$

$$\forall (a, b) \in A \times A, f(a, b) = f(b, a).$$

ⓑ $B \subseteq A$, f is said to be ***associative*** $\equiv$

$$\forall a, b, c \in A, f(f(a, b), c) = f(a, f(b, c)).$$

§ 5.4 Special Functions

<u>EX 5.32</u> : ⓑ $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, **where** $h(a, b) = a\,|b|$.

(i) $h(3, -2) = 3\,|-2| = 3\,(2) = 6$
$h(-2, 3) = -2\,|3| = -2 \cdot 3 = -6$
} **h is <u>not</u> commutative**

(ii) $\forall\, a, b, c \in \mathbb{Z}$, $h(h(a, b), c) = h(a, b)\,|c| = a\,|b|\,|c|$,
$h(a, h(b, c)) = a\,|h(b, c)| = a\,|b\,|c|| = a\,|b|\,|c|$

$\Rightarrow$ ***h* is associative**

<u>EX 5.33</u> : If $A = \{a, b, c, d\}$, **then** $|A \times A| = 16$.

① $\exists\ 4^{16}$ **function** $f : A \times A \to A$ **(closed binary operation)**

② $\exists$ **? Commutative closed binary operations** g **on** A?

$\because\ \forall\, a \neq b$, $g(a, b) = g(b, a)$, **and** $(4 \times 4) - 4 = 12$, $12 / 2 = 6$

$\therefore$ **the number of commutative closed binary operations** g **on** A
$= 4^4 \cdot 4^6 = 4^{10}$.

§ 5.4 Special Functions

Def 5.13 : Let $f : A \times A \to B$ be a binary operation on A.
An element $x \in A$ is called an *identity* (or *identity element*) for $f \equiv \forall\, a \in A, f(a, x) = f(x, a) = a$.

EX 5.34 : (a) Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, where $f(a, b) = a + b$,
0 is the identity since
$f(a, 0) = a + 0 = a = 0 + a = f(0, a), \forall\, a \in \mathbb{Z}$.

(b) Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, defined by $f(a, b) = a - b$. $\nexists$ identity.
If f had an identity x, then
$\forall\, a \in \mathbb{Z}, f(a, x) = a \Rightarrow a - x = a \Rightarrow x = 0$
But $f(x, a) = f(0, a) = 0 - a \neq a$, unless $a = 0$ $\rightarrow\leftarrow$

(c) Let A ={1, 2, 3, 4, 5, 6, 7}, let $g = A \times A \to A$ be defined by $g(a, b) = min\{a, b\} \equiv$ the minimum (or smallest) of a, b.

(i) $g(a, b) = \min\{a, b\} = \min\{b, a\} = g(b, a)$

Hence, g is commutative.

(ii) $g(g(a, b), c) = \min\{\min\{a, b\}, c\} = \min\{a, b, c\}$

$= \min\{a, \min\{b, c\}\} = g(a, g(b, c))$

Hence, g is associative.

(iii) $\forall\ a \in A$, $g(a, 7) = \min\{a, 7\} = a = \min\{7, a\} = g(7, a)$

$\therefore$ 7 is an identity element for g.

§ 5.4 Special Functions

<u>Thm 5.4</u> : Let $f : A \times A \to B$ be a binary operation.

If f has an identity, then that identity is unique.

Proof.

Let $x_1, x_2 \in A$ are identities of A :

$\therefore$ ① $f(a, x_1) = a = f(x_1, a), \forall a \in A$ and

② $f(a, x_2) = a = f(x_2, a), \forall a \in A$.

$\because x_1 \in A$, by ② : $f(x_1, x_2) = x_1$;

$\because x_2 \in A$, by ① : $f(x_1, x_2) = x_2$,

$\Rightarrow x_1 = x_2$

$\therefore f$ has at most one identity.

<u>EX 5.35</u> : If $A = \{x, a, b, c, d\}$, how many closed binary operations on A as

Sol.

① **closed binary operations on A where x is the identity** : 5^{16}.

Let $f : A \times A \rightarrow A$ with $f(x, y) = y = f(y, x), \forall\, y \in A$.

② **and commutative** :

$5^{10} = 5^4 \cdot 5^{(4^2 - 4)/2}$.

③ **close binary operations on A have an identity** : $5^{17} = \binom{5}{1} 5^{16}$

$= \binom{5}{1} 5^{5^2 - [2(5) - 1]} = \binom{5}{1} 5^{(5-1)^2}$.

④ **and commutative** : $5^{11} = \binom{5}{1} 5^{10}$

$= \binom{5}{1} 5^4 \cdot 5^{(4^2 - 4)/2}$.

Table 5.2

f	x	a	b	c	d
x	x	a	b	c	d
a	a	—	—	—	—
b	b	—	—	—	—
c	c	—	—	—	—
d	d	—	—	—	—

§ 5.4 Special Functions

Def 5.14 : For sets A, B, if $D \subseteq A \times B$, then

① $\Pi_A : D \to A$ defined by $\Pi_A(a, b) = a$, is called the *projection* on the first coordinate.

② $\Pi_B : D \to B$ defined by $\Pi_B(a, b) = b$, is called the *projection* on the second coordinate.

Note : If $D = A \times B$, then Π_A, Π_B are both onto.

EX 5.36 : $A = \{w, x, y\}$, $B = \{1, 2, 3, 4\}$, $D = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 4)\}$

① $\Pi_A : D \to A$ satisfies $\Pi_A(x, 1) = \Pi_A(x, 2) = \Pi_A(x, 3) = x$

$\Pi_A(y, 1) = \Pi_A(y, 4) = y$

$\because \Pi_A(D) = \{x, y\} \subset A$, $\therefore \Pi_A$ is not onto

§ 5.4 Special Functions

EX 5.36 : $A = \{w, x, y\}$, $B = \{1, 2, 3, 4\}$, $D = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 4)\}$

② $\Pi_B : D \rightarrow B$ satisfies $\Pi_B(x, 1) = \Pi_B(y, 1) = 1$, $\Pi_B(x, 2) = 2$

$\Pi_B(x, 3) = 3$, $\Pi_B(y, 4) = 4$

$\because \Pi_B(D) = \{1, 2, 3, 4\} = B$, $\therefore \Pi_B$ is an onto function.

EX 5.37 : Let $A = B = \mathbb{R}$, $D \subseteq A \times B$ where $D = \{(x, y) \mid y = x^2\}$.

ex : $(3, 9) \in D$, $\Pi_A(3, 9) = 3$, $\Pi_B(3, 9) = 9$.

$\Pi_A(D) = \mathbb{R} = A$. $\therefore \Pi_A$ is onto. (also one - to - one)

$\Pi_B(D) = [0, +\infty) \subset \mathbb{R}$, $\therefore \Pi_B$ is not onto. (nor one - to - one)

§ 5.4 Special Functions

<u>Def</u> : Let $A_1, A_2, \ldots, A_n$ be sets, and $\{i_1, i_2, \ldots, i_m\} \subseteq \{1, 2, \ldots, n\}$ with $i_1 < i_2 < \ldots < i_m$ and $m \le n$. $D \subseteq A_1 \times A_2 \times \ldots \times A_n$ $(= \times_{i=1}^{n} A_i)$,

① $\Pi : D \to A_{i_1} \times A_{i_2} \times \ldots \times A_{i_m}$ defined by

$$\Pi(a_1, a_2, \ldots, a_n) = (a_{i_1}, a_{i_2}, \ldots, a_{i_m})$$

is the *projection* of D on the i_1th, i_2th, …, i_mth coordinates.

② The element of D are called (*ordered*) *n-tuples* ;

③ An element in $\Pi(D)$ is an (*ordered*) *m-tuples*.

<u>EX 5.38</u> : A_1 = the set of course # for courses offered in math.
A_2 = the set of course titles offered in math.
A_3 = the set of math faculty.
A_4 = the set of letters of the alphabet.

Consider the *table* (or relation $D \subseteq A_1 \times A_2 \times A_3 \times A_4$) given :

Table 5.3

§ 5.4 Special Functions

Table 5.3

Course Number	Course Title	Professor	Section Letter
MA 111	Calculus I	P. Z. Chinn	A
MA 111	Calculus I	V. Larney	B
MA 112	Calculus II	J. Kinney	A
MA 112	Calculus II	A. Schmidt	B
MA 112	Calculus II	R. Mines	C
MA 113	Calculus III	J. Kinney	A

Def : ① **The sets A_1, A_2, A_3, A_4 are called the *domains of the relational data base*.**

② ***Table D* is said to have *degree* 4.**

③ **Each element of *D* is often called *a list*.**

§ 5.4 Special Functions

Note :

① The projection of D on $A_1 \times A_3 \times A_4$; $A_1 \times A_2$ = Table 5.4; 5.5.

② Table 5.4, 5.5 are another way of representing the same data that appear in Table 5.3.

③ Given Table 5.4; 5.5, one can recapture Table 5.3.

Table 5.4

Course Number	Professor	Section Letter
MA 111	P. Z. Chinn	A
MA 111	V. Larney	B
MA 112	J. Kinney	A
MA 112	A. Schmidt	B
MA 112	R. Mines	C
MA 113	J. Kinney	A

Table 5.5

Course Number	Course Title
MA 111	Calculus I
MA 112	Calculus II
MA 113	Calculus III

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Chap 5 Relations and Functions

§ 5.5 The Pigeonhole Principle

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§ 5.5 The Pigeonhole Principle

The pigeonhole Principle **: If m pigeons occupy n pigeonholes and $m > n$ then ≥ 1 pigeonhole has ≥ 2 pigeons roosting in it.**

Proof.

If not, each pigeonhole has ≤ 1 pigeons roosting in it.

$\Rightarrow$ total $\leq n$ pigeons. $\because n < m \rightarrow\leftarrow$

EX 5.39 : An office employs 13 file clerks, ≥ 2 of them must have birthdays during the same month.

Sol.

13 pigeons and 12 pigeonholes.

(the file clerks) (the months)

§ 5.5 The Pigeonhole Principle

<u>EX 5.40</u> : Drawing the socks from a bag which contains 12 pairs of socks (each pair a different color) randomly.

⇒ at most 13 of them to get a matched pair.

<u>EX 5.41</u> : In 500000 "words" of four or fewer lowercase letters. Can it be true that the 500000 words are all distinct?

Sol.

the total number of different possible words using ≤ 4 letter

$= 26^4 + 26^3 + 26^2 + 26 = 475254 < 500000$

(pigeonholes) (pigeons)

∴ at least one word is repeated.

§ 5.5 The Pigeonhole Principle

<u>EX 5.42</u> : Let $S \subseteq Z^+$, where $|S| = 37$. Then S contains two elements that have the same remainder upon division by 36.

Proof.

By division algorithm : $\forall\ n \in Z^+, \exists\ !\ q, r \in Z^+$ such that

$$n = 36 \cdot q + r,\ 0 \leq r < 36.$$

r : 36 possible values : pigeonholes

n : 37 positive integers : pigeons

$\therefore$ By the pigeonhole principle, $\exists \geq 2$ elements in S that have the same remainder upon division by 36.

EX 5.43 : Prove : 101 integers from $S \subseteq \{1, 2, 3, \ldots, 200\}$, $\exists\, a, b \in S$ such that $a \mid b$ or $b \mid a$.

Proof.

By the Fundamental Theorem of Arithmetic:

$\forall\, x \in S, x = 2^k y$, with $k \geq 0$ and $\gcd(2, y) = 1$

$\Rightarrow y \in T = \{1, 3, 5, \ldots, 199\}$

$|T| = 100$. (pigeonholes)

$\because$ 101 integers are selected from S. (pigeons)

By the pigeonhole principle,

$\exists\, a \neq b \in S$ s.t. $a = 2^m y$, $b = 2^n y$ for some $y \in T$.

If $m < n$, then $a \mid b$ ($\because 2^m y \mid 2^n y$),

else $m > n$, then $b \mid a$ ($\because 2^n y \mid 2^m y$).

§ 5.5 The Pigeonhole Principle

<u>EX 5.44</u> : Any subsets of size 6 from $S = \{1, 2, 3, \ldots, 9\}$ must contain two elements whose sum is 10.

Sol.

Let $T = \{\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5\}\}$, $|T| = 5$ (pigeonholes)

∵ 6 element are selected from S (pigeons)

By the pigeonhole Principle :

∃ At least one of the two-element subsets of T whose sum to 10 be complete selected.

i.e. ∃ two elements whose sum is 10.

§ 5.5 The Pigeonhole Principle

EX 5.45 : ΔACE is equilateral with $|\overline{AC}| = 1$.
If 5 points are selected from the interior of ΔACE, then $\exists \geq 2$ of them whose distance < ½.

Proof.

We break up the interior of ΔACE into the following 4 regions, which are mutually disjoint in pairs with $|\overline{AB}| = ½$:

R_1 : the interior of $\Delta BCD \cup \overline{BD} - \{B, D\}$

R_2 : the interior of ΔABF

R_3 : the interior of $\Delta BDF \cup \overline{BF} \cup \overline{DF} - \{B, D, F\}$

R_4 : the interior of ΔFDE

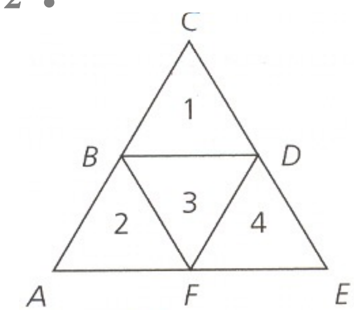

Figure 5.8

∴ By the pigeonhole principle, five points (the pigeons) in the interior of ΔACE must be s. t. at least 2 of them are in one of the four regions R_i (pigeonholes) $1 \leq i \leq 4$, where any two points are separated by a distance less than ½.

<u>EX 5.46</u> : Let S be a set of six positive integers whose maximum is at most 14. Show that the sums of the elements in all the nonempty subsets of S cannot all be distinct.

Proof.

Let $S = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $1 \le x_i \le 14$, $x_i \in Z^+$, $\forall\ i = 1, \ldots, 6$.

$\forall\ A \subseteq S$, let the sum of the element in $A = S_A$,

then $1 \le S_A \le 9 + 10 + \ldots + 14 = 69$

$\therefore$ there are 69 possible sums : pigeonholes

$\exists\ 2^6 - 1 = 63$ nonempty subsets of S : pigeons (< 69, wrong !!)

$\forall\ A \subseteq S$, s.t. $|A| \le 5$, $1 \le S_A \le 10 + 11 + \ldots + 14 = 60$ (pigeonholes)

$\exists\ 2^6 - 2 = 62$ possible subset $A \subseteq S$, s.t. $A \ne \phi$, $A \ne S$. (pigeons)

By the pigeonhole principle : the elements of at least two of the 62 subsets must yield the same sum.

§ 5.5 The Pigeonhole Principle

<u>EX 5.47</u> : Let $m \in Z^+$ with m is odd.

Prove : $\exists\, n \in Z^+$ such that $m \mid (2^n - 1)$.

Proof.

Let $T = \{2^i - 1 \mid i = 1, 2, \ldots, m + 1\}$, $|T| = m + 1$ (Pigeon)

By the division algorithm and the pigeonhole principle :

$\exists\, s, t \in Z^+$ with $1 \le s < t \le m + 1$, where $\exists\, q_1, q_2 \in N$ such that $2^s - 1 = q_1 m + r$, $2^t - 1 = q_2 m + r$.

$\Rightarrow (2^t - 1) - (2^s - 1) = (q_2 m + r) - (q_1 m + r) = (q_2 - q_1) m$

but $(2^t - 1) - (2^s - 1) = 2^t - 2^s = 2^s (2^{t-s} - 1)$

i.e. $(q_2 - q_1) m = 2^s (2^{t-s} - 1)$

$\Rightarrow m \mid 2^s (2^{t-s} - 1)$

$\because \gcd(m, 2^s) = 1$

$\therefore m \mid (2^{t-s} - 1)$

$\Rightarrow$ Let $n = t - s \in Z^+$, $m \mid (2^n - 1)$.

EX 5.48 : On a 4-week vacation : ≥ 1 set of tennis each day,
≤ 40 set total during this time.
Prove : $\exists$ consecutive days which play 15 sets.

Proof.

Let x_i = the total number of sets from the first day, $1 \leq i \leq 28$.

$\Rightarrow 1 \leq x_1 < x_2 < \ldots < x_{28} \leq 40$

$\therefore x_1 + 15 < x_2 + 15 < \ldots < x_{28} + 15 \leq 55$

Let $T = \{x_1, x_2, \ldots, x_{28}, x_1 + 15, x_2 + 15, \ldots, x_{28} + 15\}$,

$|T| = 56$ (pigeons)

And $\forall\, x \in T$, $1 \leq x \leq 55$: 55 possible values (pigeonholes)

$\therefore$ By the Pigeonhole Principle : $\exists\, x, y \in T$ are equal.

$\because x_1, x_2, \ldots, x_{28}$ are distinct, $x_1+15, x_2+15, \ldots, x_{28}+15$ are distinct

$\Rightarrow \exists\, 1 \leq j < i < 28$ with $x = x_i = x_j + 15 = y$

i.e. from the start of day $j + 1$ to the end of day i, Herbert will play exactly 15 sets of tennis.

§ 5.5 The Pigeonhole Principle

EX 5.49 : (1935. Paul Erdős and George Szekeres)

① 6, 5, 8, 3, 7 contains the decreasing subsequence 6, 5, 3

② 11, 8, 7, 1, 9, 6, 5, 10, 3, 12 (length 10) contains the increasing subsequence 8, 9, 10, 12 (length 4)

Thm : For each $n \in Z^+$, a sequence of $n^2 + 1$ distinct real numbers contains a decreasing or increasing subsequence of length $n + 1$.

Proof.(1/2)

Let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of $n^2 + 1$ distinct real numbers

$\forall\ 1 \le k \le n^2 + 1$, let

x_k = the max. length of a decreasing subsequence that ends with a_k

y_k = the max. length of a increasing subsequence that ends with a_k

ex: EX 5.49 ② :

§ 5.5 The Pigeonhole Principle

Proof.(2/2)

k	1	2	3	4	5	6	7	8	9	10
a_k	11	8	7	1	9	6	5	10	3	12
x_k	1	2	3	4	2	4	5	2	6	1
y_k	1	1	1	1	2	2	2	3	2	4

If $\nexists$ decreasing or increasing subsequence of length $n + 1$,
then $1 \leq x_k \leq n$ and $1 \leq y_k \leq n$ $\forall$ $1 \leq k \leq n^2 + 1$.

$\because \exists \leq n^2$ distinct ordered pairs (x_k, y_k),

but $\because 1 \leq k \leq n^2 + 1$, we have $n^2 + 1$ ordered pairs (x_k, y_k).

$\therefore$ by the Pigeonhole Principle,
$\exists\, i \neq j \in$ N with $1 \leq i < j \leq n^2 + 1$, s.t. $(x_i, y_i) = (x_j, y_j)$.

But $a_1, a_2, \ldots, a_{n^2+1}$ are distinct :
If $a_i < a_j$ then $y_i < y_j$; else if $a_i > a_j$ then $x_i < x_j$ $\rightarrow\leftarrow$

$\therefore x_k = n + 1$ or $y_k = n + 1$ for some $n + 1 \leq k \leq n^2 + 1$

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Chap 5 Relations and Functions

§ 5.6 Function Composition and Inverse Functions

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§ 5.6 Function Composition and Inverse Functions

Recall : ① $\forall\, c \in \mathbb{Z}, \exists\, d \in \mathbb{Z}$ s.t. $c + d = d + c = 0$

we call d the ***additive inverse*** of c

② $\forall\, t \in \mathbb{R}, t \neq 0, \exists\, u \in \mathbb{R}$ s.t. $tu = ut = 1$

we call u the ***multiplicative inverse*** of t

Def 5.15 : If $f : A \rightarrow B$, then f is said to be ***bijective***, (or to be a ***one - to - one correspondence***) $\equiv f$ is one - to - one and onto.

EX 5.50 : $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$

① $f = \{(1, w), (2, x), (3, y), (4, z)\}$ is a 1 - 1 correspondence from A onto B.

② $g = \{(w, 1), (x, 2), (y, 3), (z, 4)\}$ is a 1 - 1 correspondence from B onto A.

§ 5.6 Function Composition and Inverse Functions

Def 5.16 : The function $1_A : A \to A$, defined by $1_A(a) = a$, $\forall\, a \in A$, is called the ***identity function*** for A.

Def 5.17 : If $f, g : A \to B$, f and g are ***equal***, write $f = g \equiv f(a) = g(a), \forall\, a \in A$.

EX 5.51 : Let $f : \mathbb{Z} \to \mathbb{Z}$ and
$g : \mathbb{Z} \to \mathbb{Q}$ where $f(x) = x = g(x)$, $\forall\, x \in \mathbb{Z}$.

$\forall\, a \in \mathbb{Z}$, $f(a) = g(a)$ but $f \neq g$!

$\because$ f is 1 - 1 correspondence ;

g is 1 - 1 but not onto!

§ 5.6 Function Composition and Inverse Functions

EX 5.52 : $f, g : \mathbb{R} \to \mathbb{Z}$ defined by :

$$f(x) = \begin{cases} x & \text{, if } x \in \mathbb{Z} ; \\ \lfloor x \rfloor + 1, & \text{if } x \in \mathbb{R} - \mathbb{Z} . \end{cases} \qquad g(x) = \lceil x \rceil, \forall x \in \mathbb{R}$$

Sol.

$\forall x \in \mathbb{Z}, f(x) = x = \lceil x \rceil = g(x).$

$\forall x \in \mathbb{R} - \mathbb{Z}$, let $x = n + r$, where $n \in \mathbb{Z}$ and $0 < r < 1$.

Then $f(x) = \lfloor x \rfloor + 1 = n + 1 = \lceil x \rceil = g(x).$

$\therefore f(x) = g(x), \forall x \in \mathbb{R}$ (the domain).

$\therefore f = g$, are the *same* function.

Def 5.18 : $f : A \to B$ and $g : B \to C$, the ***composite function***,

$g \circ f : A \to C \equiv (g \circ f)(a) = g(f(a)), \forall a \in A.$

§ 5.6 Function Composition and Inverse Functions

EX 5.53 : Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, $C = \{w, x, y, z\}$ with $f : A \to B$, $g : B \to C$ given by $f = \{(1, a), (2, a), (3, b), (4, c)\}$, $g = \{(a, x), (b, y), (c, z)\}$. $g \circ f = ?$

Sol.

$\forall$ element of A : $(g \circ f)(1) = g(f(1)) = g(a) = x$,

$(g \circ f)(2) = g(f(2)) = g(a) = x$,

$(g \circ f)(3) = g(f(3)) = g(b) = y$,

$(g \circ f)(4) = g(f(4)) = g(c) = z$.

$\therefore g \circ f = \{(1, x), (2, x), (3, y), (4, z)\}$.

EX 5.54 : Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, $g(x) = x + 5$, $g \circ f = ?$ $f \circ g = ?$

Sol. $(g \circ f)(x) = g(f\,x)) = g(x^2) = x^2 + 5$,

$(f \circ g)(x) = f(g(x)) = f(x + 5) = (x + 5)^2 = x^2 + 10\,x + 25$.

§ 5.6 Function Composition and Inverse Functions

Note : ① $f \circ g$ 不一定 $= g \circ f$, i.e. the composition of function is not a commutative operation, in general.

② If $g \circ f\,\exists$, then "the range of f" $\subseteq$ "the domain of g".

③ $f : A \to B$, $f \circ 1_A = f = 1_B \circ f$.

Thm 5.15 : $f : A \to B$, $g : B \to C$

(a) If f, g are 1 - 1 , then $g \circ f$ is 1 - 1,

(b) If f, g are onto, then $g \circ f$ is onto.

Proof.(1/2)

(a) Let $a_1, a_2 \in A$, $(g \circ f)\,(a_1) = (g \circ f)\,(a_2)$

$\Rightarrow g(f(a_1)) = g(f(a_2))$

$\because$ g is 1 - 1, $\therefore f(a_1) = f(a_2)$

$\because$ f is 1 - 1, $\therefore$ $a_1 = a_2 \Rightarrow g \circ f$ is 1 - 1

(b) If f, g are onto, then $g \circ f$ is onto.

Proof.(2/2)

(b) $g \circ f : A \to C$, let $z \in C$

$\because$ g is onto, $\therefore \exists\, y \in B$ s.t. $g(y) = z$,

$\because$ f is onto, $\therefore \exists\, x \in A$ s.t. $f(x) = y$,

$\Rightarrow \forall\, z \in C, \exists\, x \in A$ s.t. $(g \circ f)(x) = g(f(x)) = g(y) = z$.

$\therefore$ $g \circ f$ is onto.

EX 5.55 : Let $f, g, h : \mathbb{R} \to \mathbb{R}$, where $f(x) = x^2$, $g(x) = x + 5$, $h(x) = \sqrt{x^2+2}$ $\quad (h \circ g) \circ f = h \circ (g \circ f)$?

Sol. $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = (h \circ g)(x^2) = h(g(x^2)) = h(x^2 + 5)$

$= \sqrt{(x^2+5)^2+2} = \sqrt{x^4+10x^2+27}$

$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x))) = h(g(x^2)) = h(x^2 + 5)$

$= \sqrt{(x^2+5)^2+2} = \sqrt{x^4+10x^2+27}$

$\therefore$ $((h \circ g) \circ f)(x) = (h \circ (g \circ f))(x)$, $\forall\, x \in \mathbb{R}$ with the same domain and codomain. $\quad \therefore (h \circ g) \circ f = h \circ (g \circ f)$.

§ 5.6 Function Composition and Inverse Functions

Thm 5.6 : If $f : A \to B$, $g : B \to C$, and $h = C \to D$,
then $(h \circ g) \circ f = h \circ (g \circ f)$

Proof.

① $(h \circ g) \circ f$, $h \circ (g \circ f)$ have the same domain A, and codomain D

② $\forall\, x \in A$, $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$
$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$

∴ the composition of function is an associative operation.

Note : $h \circ g \circ f = (h \circ g) \circ f = h \circ (g \circ f)$

Def 5.19 : If $f : A \to A$, we define $f^1 = f$,
and $\forall\, n \in \mathbb{Z}^+$, $f^{n+1} = f \circ (f^n)$.

§ 5.6 Function Composition and Inverse Functions

EX 5.56 : $A = \{1, 2, 3, 4\}$, $f : A \to A$ defined by $f = \{(1, 2), (2, 2), (3, 1), (4, 3)\}$.

$f^2 = f \circ f = \{(1, 2), (2, 2), (3, 2), (4, 1)\}$,

$f^3 = f \circ f^2 = f \circ f \circ f = \{(1, 2), (2, 2), (3, 2), (4, 2)\}$,

$f^4 = ?$ $f^5 = ?$

Def 5.20 : $\forall\ A, B \subseteq \mathcal{U}$. $\mathcal{R}$ is a relation from A to B, that the *converse* of $\mathcal{R}$, denoted $\mathcal{R}^c \equiv$ the relation from B to $A = \{(b, a) \mid (a, b) \in \mathcal{R}\}$.

EX 5.57 : ① $A = \{1, 2, 3\}$, $B = \{w, x, y\}$, $f : A \to B$ be given by $f = \{(1, w), (2, x), (3, y)\}$ then

$f^c = \{(w, 1), (x, 2), (y, 3)\}$ is a function from B to A, and $f^c \circ f = 1_A$ and $f \circ f^c = 1_B$

§ 5.6 Function Composition and Inverse Functions

EX 5.57 : ② $A = \{1, 2, 3, 4\}$, $B = \{w, x, y\}$, $f : A \to B$ where

$f = \{(1, w), (2, x), (3, y), (4, x)\}$

$f^c = \{(w, 1), (x, 2), (y, 3), (x, 4)\}$ is not a function.

Def 5.21 : If $f : A \to B$, then f is said to be ***invertible*** if $\exists\, g: B \to A$ is a function s.t. $g \circ f = 1_A$ and $f \circ g = 1_B$.

EX 5.58 : Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 5$,

$g(x) = ½(x - 5)$

$(g \circ f)(x) = g(f(x)) = g(2x + 5) = ½\, [(2x + 5) - 5] = x$

$(f \circ g)(x) = f(g(x)) = f(\, ½\, (x - 5)) = 2\, [\, ½\, (x - 5)] + 5 = x$

$\therefore g \circ f = 1_{\mathrm{R}}, f \circ g = 1_{\mathrm{R}}$

$\therefore$ f and g are invertible functions.

§ 5.6 Function Composition and Inverse Functions

Thm 5.7 : If a function $f : A \to B$ is invertible and a function $g : B \to A$ satisfy $g \circ f = 1_A$ and $f \circ g = 1_B$, then this function g is unique.

Proof.

If g is not unique, then let $h : B \to A$ with $h \circ f = 1_A$ and $f \circ h = 1_B$

$\because h = h \circ 1_B = h \circ (f \circ g) = (h \circ f) \circ g = 1_A \circ g = g$

$\therefore g$ is unique.

Def : In Def 5.21, g is called *the inverse* of f, and $g = f^{-1} = f^{c}$.

Note : If f is invertible, then f^{-1} is invertible and $(f^{-1})^{-1} = f$.

§ 5.6 Function Composition and Inverse Functions

<u>Thm 5.8</u> : A function $f : A \to B$ is invertible $\Leftrightarrow$ f is 1 - 1 and onto.

Proof. (1/2)

($\Rightarrow$) Assume $f : A \to B$ is invertible,

$\exists\, !\, g : B \to A$ s.t. $g \circ f = 1_A$ and $f \circ g = 1_B$

① $\forall\, a_1, a_2 \in A$ with $f(a_1) = f(a_2)$

$\Rightarrow g(f(a_1)) = g(f(a_2))$

$\Rightarrow (g \circ f)(a_1) = (g \circ f)(a_2)$

$\Rightarrow 1_A(a_1) = 1_A(a_2)$

$\Rightarrow a_1 = a_2$ $\quad\quad \therefore f$ is $1 - 1$.

② $\forall\, b \in B$, take $g(b) \in A$:

$f(g(b)) = (f \circ g)(b) = 1_B(b) = b.$

$\therefore f$ is onto.

§ 5.6 Function Composition and Inverse Functions

A function $f : A \to B$ is invertible $\Leftrightarrow$ f is 1 - 1 and onto.

Proof. (2/2)

($\Leftarrow$) Suppose $f : A \to B$ is bijective,

$\because$ f is onto : $\forall\, b \in B, \exists\, a \in A$ with $f(a) = b$.

$\therefore$ Define the function $g : B \to A$ by $g(b) = a$ where $f(a) = b$.

$\because$ f is one - to - one : $\forall\, b \in B, \exists\, !\, a \in A$, with $f(a) = b$

$\therefore$ $\forall\, b \in B, \exists\, !\, a \in A, g(b) = a$

i.e. g is a unique function.

And $g \circ f = 1_A$ and $f \circ g = 1_B$. $\therefore$ f is invertible with $g = f^{-1}$.

<u>EX 5.59</u> : ① $f_1 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = x^2$ is <u>not</u> invertible

$\because$ neither 1 - 1 nor onto.

② $f_2 : [0, +\infty) \to [0, +\infty)$ defined by $f_2(x) = x^2$ is invertible with $f_2^{-1}(x) = \sqrt{x}$

§ 5.6 Function Composition and Inverse Functions

Thm 5.9 : If $f : A \to B$, $g : B \to C$ are both invertible functions, then $g \circ f : A \to C$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

EX 5.60 : $\forall\, m, b \in \mathbb{R}$, $m \neq 0$, $f : \mathbb{R} \to \mathbb{R}$ defined by $f = \{(x, y) \mid y = mx + b\}$ is an invertible function ($\because$ it is 1 - 1 and onto). $f^{-1} = ?$

Sol. $f^{-1} = \{(x, y) \mid y = mx + b\}^c = \{(y, x) \mid y = mx + b\}$
$= \{(x, y) \mid x = my + b\} = \{(x, y) \mid y = (1 / m)(x - b)\}$
$\therefore f(x) = mx + b;\ f^{-1}(x) = (x - b) / m.$

EX 5.61 : Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x$ (f : 1 - 1 and onto). $f^{-1} = ?$

Sol. $f^{-1} = \{(x, y) \mid y = e^x\}^c = \{(x, y) \mid x = e^y\} = \{(x, y) \mid y = \ln x\}.$
$\therefore f^{-1}(x) = \ln x$.

Note : The graphs of f and f^{-1} are symmetric about $y = x$. (see Fig. 5.10)

§ 5.6 Function Composition and Inverse Functions

Def 5.22 : If $f : A \to B$ and $B_1 \subseteq B$, then $f^{-1}(B_1) = \{x \mid f(x) \in B_1\}$
The set $f^{-1}(B_1)$ is called the *preimage of B_1 under f*.

Note : We cannot assume the existence of an inverse for a function f just because we find the symbol f^{-1} being used.

EX 5.62 : Let $A, B \subseteq \mathbb{Z}^+$ where
$A = \{1, 2, 3, 4, 5, 6\}$, $B = \{6, 7, 8, 9, 10\}$.
$f = A \to B$ with $f = \{(1, 7), (2, 7), (3, 8), (4, 6), (5, 9), (6, 9)\}$
(a) $B_1 = \{6, 8\} \subseteq B$, $f^{-1}(B_1) = \{3, 4\}$
(note : $|f^{-1}(B_1)| = 2 = |B_1|$)
(b) $B_2 = \{7, 8\} \subseteq B$, $f^{-1}(B_2) = \{1, 2, 3\}$
(note : $|f^{-1}(B_2)| = 3 > 2 = |B_2|$)

§ 5.6 Function Composition and Inverse Functions

$f = \{(1, 7), (2, 7), (3, 8), (4, 6), (5, 9), (6, 9)\}$

(c) $B_3 = \{8, 9\} \subseteq B$. $f^{-1}(B_3) = \{3, 5, 6\}$.

(note : $|f^{-1}(B_3)| = 3 > 2 = |B_3|$)

(d) $B_4 = \{8, 9, 10\} \subseteq B$. $f^{-1}(B_4) = \{3, 5, 6\} = f^{-1}(B_3)$.

($B_4 \supset B_3$)

($\because f^{-1}(\{10\}) = \phi$)

(e) $B_5 = \{8, 10\} \subseteq B$. $f^{-1}(B_5) = \{3\}$.

(note : $|f^{-1}(B_5)| = 1 < 2 = |B_5|$)

<u>Remark</u> : Write $f^{-1}(b)$ instead of $f^{-1}(\{b\})$

ex : 上EX中 : $f^{-1}(6) = \{4\}, f^{-1}(7) = \{1, 2\}, f^{-1}(8) = \{3\}, \ldots$

§ 5.6 Function Composition and Inverse Functions

EX 5.63 : Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 3x - 5, & x > 0 \\ -3x + 1, & x \le 0. \end{cases}$

ⓐ Determine $f(0), f(1), f(-1), f(5/3), f(-5/3)$

Sol. $f(0) = -3(0) + 1 = 1$; $f(1) = 3(1) - 5 = -2$

$f(-1) = -3(-1) + 1 = 4$; $f(5/3) = 3(5/3) - 5 = 0$

$f(-5/3) = -3(-5/3) + 1 = 6$

ⓑ Find $f^{-1}(0), f^{-1}(1), f^{-1}(-1), f^{-1}(3), f^{-1}(-3), f^{-1}(-6)$

Sol. (1/2) $f^{-1}(0) = \{x \in \mathbb{R} \mid f(x) \in \{0\}\} = \{x \in \mathbb{R} \mid f(x) = 0\}$

$= \{x \in \mathbb{R} \mid x > 0 \text{ and } 3x - 5 = 0\} \cup$

$\{x \in \mathbb{R} \mid x \le 0 \text{ and } -3x + 1 = 0\}$

$= \{5/3\} \cup \phi = \{5/3\}.$

Sol. (2/2)

f^{-1} … }

In the same

see Fig 5.11

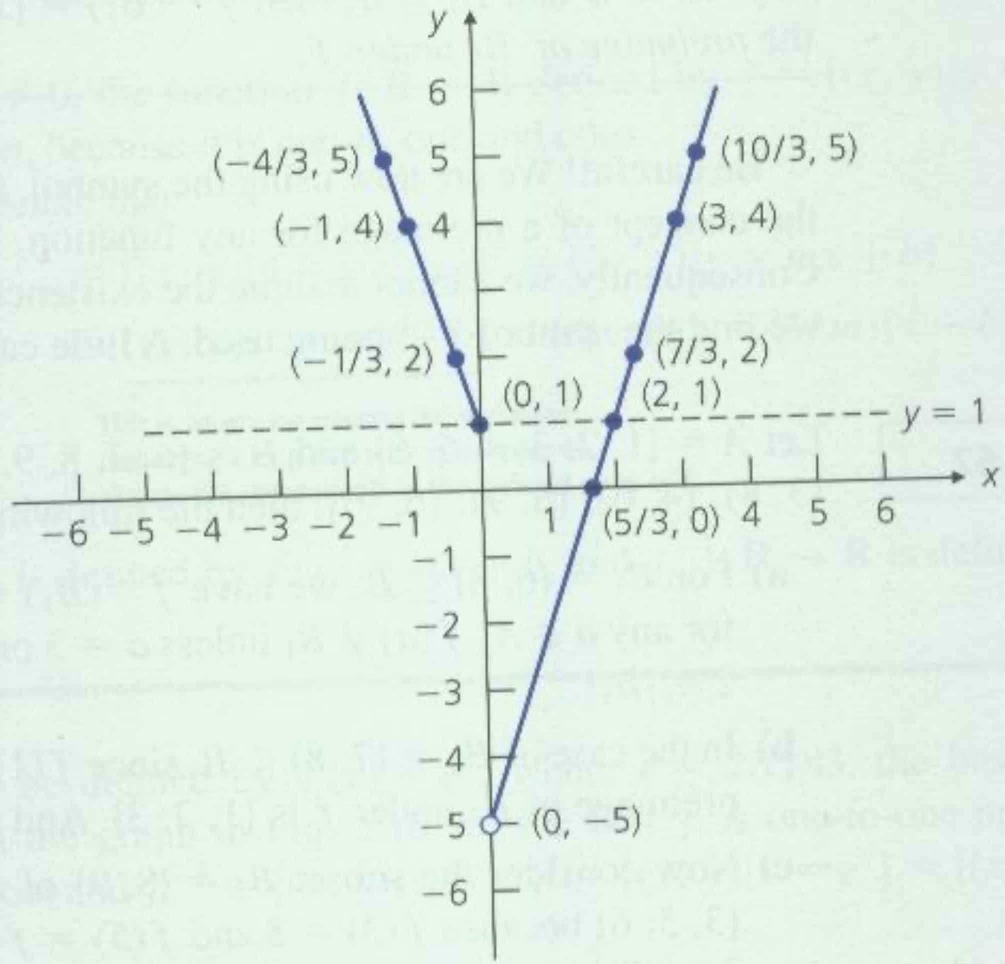

Figure 5.11

§ 5.6 Function Composition and Inverse Functions

© **What are $f^{-1}([-5, 5])$ and $f^{-1}([-6, 5])$?**

Sol.

$f^{-1}([-5, 5]) = \{x \mid f(x) \in [-5, 5]\} = \{x \mid -5 \le f(x) \le 5\}$

Case 1 $x > 0$: $-5 \le 3x - 5 \le 5 \Rightarrow 0 \le 3x \le 10 \Rightarrow 0 \le x \le 10/3$

$\Rightarrow 0 < x \le 10/3$

Case 2 $x \le 0$: $-5 \le 3x + 1 \le 5 \Rightarrow -6 \le -3x \le 4 \Rightarrow 2 \ge x \ge -4/3$

$\Rightarrow -4/3 \le x \le 0$

Hence $f^{-1}([-5, 5]) = \{x \mid -4/3 \le x \le 0 \text{ or } 0 < x \le 10/3\}$

$= [-4/3, 10/3]$

In the same way, $f^{-1}([-6, 5]) = f^{-1}([-5, 5]) = [-4/3, 10/3]$.

(see Fig 5.11)

§ 5.6 Function Composition and Inverse Functions

<u>EX 5.64</u> : (a) Let $f : \mathbb{Z} \to \mathbb{R}$ be defined by $f(x) = x^2 + 5$,
$f^{-1}(B)$ for $B \in \mathbb{R}$:

B	$f^{-1}(B)$
$\{6\}$	$\{-1, 1\}$
$[6, 7]$	$\{-1, 1\}$
$[6, 10]$	$\{-2, -1, 1, 2\}$
$[-4, 5)$	ϕ
$[-4, 5]$	$\{0\}$
$[5, +\infty)$	$\mathbb{Z}$

B	$g^{-1}(B)$
$\{6\}$	$\{-1, 1\}$
$[6, 7]$	$[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$
$[6, 10]$	$[-\sqrt{5}, -1] \cup [1, \sqrt{5}]$
$[-4, 5)$	ϕ
$[-4, 5]$	$\{0\}$
$[5, +\infty)$	$\mathbb{R}$

(b) Let $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = x^2 + 5$, $g^{-1}(B) = ?$

§ 5.6 Function Composition and Inverse Functions

Thm 5.10 : If $f : A \to B$ and $B_1, B_2 \subseteq B$, then

$$\text{(a) } f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$

$$\text{(b) } f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$$

$$\text{(c) } f^{-1}(\overline{B_1}) = \overline{f^{-1}(B_1)}$$

Proof.

$$\text{(b) } \forall\, a \in A,\ a \in f^{-1}(B_1 \cup B_2) \Leftrightarrow f(a) \in B_1 \cup B_2$$

$$\Leftrightarrow f(a) \in B_1 \text{ or } f(a) \in B_2$$

$$\Leftrightarrow a \in f^{-1}(B_1) \text{ or } a \in f^{-1}(B_2)$$

$$\Leftrightarrow a \in f^{-1}(B_1) \cup f^{-1}(B_2).$$

Note : $f : A \to B$ is $1-1 \Leftrightarrow |f^{-1}(b)| \leq 1\ \forall\, b \in B$.

§ 5.6 Function Composition and Inverse Functions

<u>Thm 5.11</u> : Let $f : A \to B$ for <u>finite</u> sets A, B where $|A| = |B|$.
TFSAE : (a) f is $1-1$ (b) f is onto (c) f is invertible.

Proof.

(c) $\Rightarrow$ (a), (c) $\Rightarrow$ (b) : <u>Thm 5.8</u>, (a) and (b) $\Rightarrow$ (c) : <u>Thm 5.8</u>, only need to prove (a) $\Leftrightarrow$ (b):

(b) $\Rightarrow$ (a) : Assume f is onto, if f is not $1-1$, then $(|A| \geq |B|)$
$\exists\, a_1, a_2 \in A_1$, with $a_1 \neq a_2$, but $f(a_1) = f(a_2)$
Then $|A| > |f(A)| = |B|$ $\rightarrow\leftarrow$ $(|A| \neq |B|)$

(b) $\Leftarrow$ (a) : Assume f is $1-1$, if f is not onto, then
$\exists\, b \in B$ with $\forall\, a \in A$, $f(a) \neq b$. $\Rightarrow |f(A)| < |B|$
$\because |A| = |B| > |f(A)|$
By the Pigeonhole Principle. f is not $1-1$
$\rightarrow\leftarrow$

§ 5.6 Function Composition and Inverse Functions

<u>Note</u> : ① If $|A| = |B| = n \in Z^+$, then there are n ! one-to-one function from A to B and $\sum^{n}_{k=0} (-1)^k \binom{n}{n-k} (n-k)^n$ onto function By <u>Thm 5.11 (a) (b)</u>,

$$\therefore n! = \sum^{n}_{k=0} (-1)^k \binom{n}{n-k} (n-k)^n$$

② $S(n, n) = 1$.

§ 5.7 & 5.8

§ 5.7 see textbook : <u>Def 5.23</u> 及下面的說明; <u>Table 5.11</u>

§ 5.8 see textbook : <u>EX 5.70</u> 及其上說明; <u>Fig 5.17</u>, <u>Table 5.12</u>

Table 5.11

Big-Oh Form	Name
$O(1)$	Constant
$O(\log_2 n)$	Logarithmic
$O(n)$	Linear
$O(n \log_2 n)$	$n \log_2 n$
$O(n^2)$	Quadratic
$O(n^3)$	Cubic
$O(n^m)$, $m = 0, 1, 2, 3, \ldots$	Polynomial
$O(c^n)$, $c > 1$	Exponential
$O(n!)$	Factorial

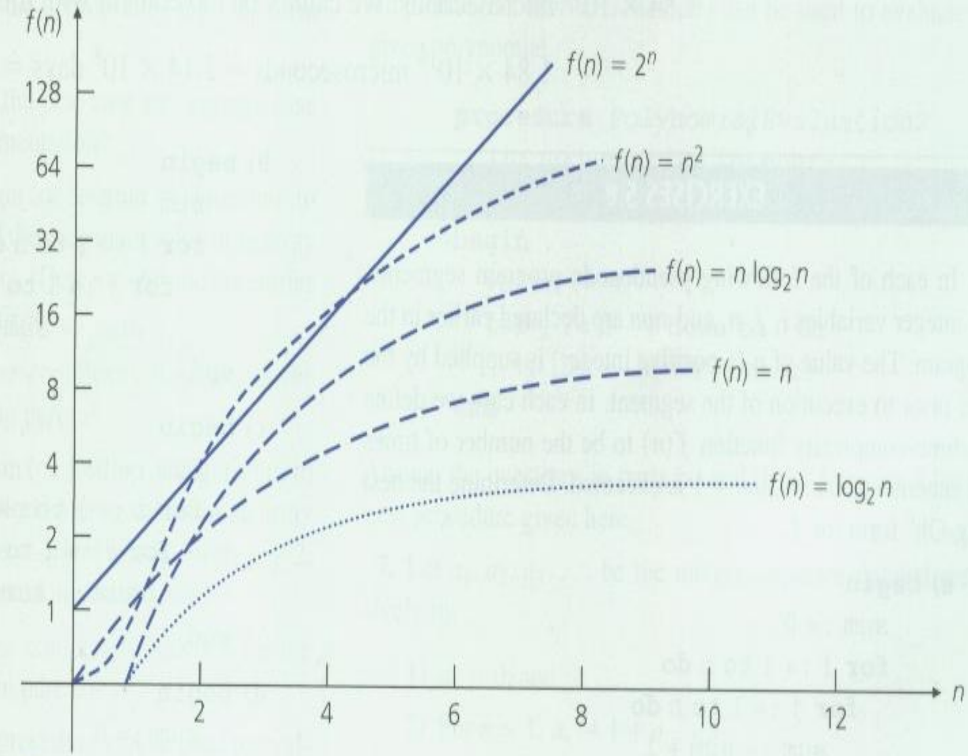

Figure 5.17