Chap 5 Relations and Functions

§ 5.1 Cartesian Products and Relations

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics (5th Edition)
by Ralph P. Grimaldi
§ 5.1 Cartesian Products and Relations

Def 5.1 : ① For sets $A, B$, the Cartesian product, or cross product, of $A$ and $B$ is denoted by $A \times B \equiv \{(a, b) | a \in A, b \in B\}$.
② The elements of $A \times B$ are ordered pairs
    For $(a, b), (c, d) \in A \times B$, $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

EX 5.1 : Let $A = \{2, 3, 4\}, B = \{4, 5\}$ Then
    a) $A \times B = \{(2, 4), (2, 5), (3, 4), (3, 5), (4, 4), (4, 5)\}$,
    b) $B \times A = \{(4, 2), (4, 3), (4, 4), (5, 2), (5, 3), (5, 4)\}$,

Note : ① $A \times B \neq B \times A$, $|A \times B| = |B \times A| = |A| \cdot |B|$.
② Here $A \subseteq \mathcal{U}_1$, $B \subseteq \mathcal{U}_2$.
③ If $A, B \subseteq \mathcal{U}$, but $A \times B \subseteq \mathcal{U}$ is not necessary!! i.e. “$\times$” is not necessarily closed.
§ 5.1 Cartesian Products and Relations

Def: ① For sets $A_1, A_2, ..., A_n$. ($n \in \mathbb{Z}^+, n \geq 3$), the \textit{(n-fold) product} of $A_1, A_2, ..., A_n$ is denoted by $A_1 \times A_2 \times ... \times A_n$  

$\equiv \{ (a_1, a_2, ..., a_n) \mid a_i \in A_i, 1 \leq i \leq n \}$

② The elements of $A_1 \times A_2 \times ... \times A_n$ are called \textit{ordered n-tuples}, (3-tuple $\equiv$ \textit{triple}).

For $(a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n) \in A_1 \times ... \times A_n$,

$(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n)$ iff $a_i = b_i, \ \forall \ 1 \leq i \leq n$.

EX 5.1: Let $A = \{2, 3, 4\}, B = \{4, 5\}$ Then

\begin{align*}
\text{c) } B^2 &= B \times B = \{(4, 4), (4, 5), (5, 4), (5, 5)\}, \\
\text{d) } B^3 &= B \times B \times B = \{(4, 4, 4), (4, 4, 5), (4, 5, 4), (4, 5, 5), (5, 4, 4), (5, 4, 5), (5, 5, 4), (5, 5, 5)\} \\
&= \{(a, b, c) \mid a, b, c \in B\}.
\end{align*}
EX 5.2: \( \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\} \) : 二維實數坐標平面.
\( \mathbb{R}^+ \times \mathbb{R}^+ \): 第一象限.
\( \mathbb{R}^3 \): Euclidean three - space .

Note : \( A_1 \times A_2 \times A_3 \neq (A_1 \times A_2) \times A_3 \neq A_1 \times (A_2 \times A_3) \)

\[\begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow \\
\end{array}\]

\[\therefore (a_1, a_2, a_3) \neq ((a_1, a_2), a_3) \neq (a_1, (a_2, a_3)) \]

但不考慮後兩種 (雖然也很重要...)，
本書皆只考慮第一種。
§ 5.1 Cartesian Products and Relations

EX 5.3: Let $A = \{2, 3, 4\}$, $B = \{4, 5\}$, $C = \{x, y\}$:

$A \times B$, $B \times A$, $A \times B \times C$: use tree diagram:

- $A \times B$:
  - 2: (2, 4), (2, 5)
  - 3: (3, 4), (3, 5)
  - 4: (4, 4), (4, 5)

- $B \times A$:
  - 4: (4, 2), (4, 3), (4, 4)
  - 5: (5, 2), (5, 3), (5, 4)

- $A \times B \times C$:
  - (2, 4): (2, 4, x), (2, 4, y)
  - (2, 5): (2, 5, x), (2, 5, y)
  - (3, 4): (3, 4, x), (3, 4, y)
  - (3, 5): (3, 5, x), (3, 5, y)
  - (4, 4): (4, 4, x), (4, 4, y)
  - (4, 5): (4, 5, x), (4, 5, y)

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§ 5.1 Cartesian Products and Relations

EX 5.4 : At the Wimbledon Tennis Championships : The winner is the First to win two sets. Let $N, E$ denote the two players in a match :

![Diagram showing the tennis match structure](image)

**Figure 5.2**

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Def 5.2 : For sets $A, B$,

1. Any subset of $A \times B$ is called a relation from $A$ to $B$.
2. Any subset of $A \times A$ is called a binary relation on $A$.

EX 5.5 : $A, B$ as EX 5.1, $A = \{2, 3, 4\}, B = \{4, 5\}$.

\[ |A \times B| = 6, \exists 2^6 \text{ possible relations from } A \text{ to } B. \]

Such as : a) $\emptyset$ b) $\{(2, 4)\}$ c) $\{(2, 4), (2, 5)\}$ d) $\{(2, 4), (3, 4), (4, 4)\}$ e) $\{(2, 4), (3, 4), (4, 5)\}$ f) $A \times B$.

Note : ① For $A, B$ : finite sets with $|A| = m, |B| = n$;

\[ \exists 2^{mn} \text{ relation from } A \text{ to } B, \text{ including } \emptyset \text{ and } A \times B. \]

② $\exists 2^{mn} \text{ relation from } B \text{ to } A :$

\[ \mathcal{R}_1 \subseteq A \times B \text{ is a relation } \iff \mathcal{R}_2 \subseteq B \times A \text{ is a relation.} \]

where $\mathcal{R}_2 = \{(b, a) \mid (a, b) \in \mathcal{R}_1\}$. 

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**EX 5.6**: Let \( B = \{1, 2\} \), \( A = \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \).

Let \( R \) (a binary relation on \( A \)) = \{(\emptyset, \emptyset), (\emptyset, \{1\}), (\emptyset, \{2\}), (\emptyset, \{1, 2\}), (\{1\}, \{1\}), (\{1\}, \{1, 2\}), (\{2\}, \{2\}), (\{2\}, \{1, 2\}), (\{1, 2\}, \{1, 2\})\}

\( R \) is the subset relation.

**Def**: ① A binary relation \( R \) on \( \mathcal{P}(B) \) is the *subset relation* 
\((C, D) \in R\) iff \( C, D \subseteq B \) and \( C \subseteq D \).

② *infix notation* for a relation \( R \) : \( a \ R b \equiv (a, b) \in R \),
\( c \ R d \equiv (c, d) \notin R \).
EX 5.7: $A = \mathbb{Z}^+$. $R$ is a binary relation on $A = \{(x, y) \mid x \leq y\}$, "less than or equal to" relation. 
$(7, 7), (7, 11) \in R$; $(8, 2) \notin R$, i.e. $7 \mathrel{R} 11$; $8 \mathrel{R} 2$.

EX 5.8: Let $R \subseteq \mathbb{N} \times \mathbb{N}$, $R = \{(m, n) \mid n = 7m\}$. $R$ can be defined recursively as:
1) $(0, 0) \in R$; and
2) If $(s, t) \in R$, then $(s + 1, t + 7) \in R$.
check $(3, 21) \in R$: (i) $(0, 0) \in R \implies (0 + 1, 0 + 7) = (1, 7) \in R$, 
(ii) $(1, 7) \in R \implies (1 + 1, 7 + 7) = (2, 14) \in R$, 
(iii) $(2, 14) \in R \implies (2 + 1, 14 + 7) = (3, 21) \in R$. 

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§ 5.1 Cartesian Products and Relations

Remark : \( \forall \) sets \( A, A \times \phi = \phi \times A = \phi \)

If \( A \times \phi \neq \phi \), then let \( (a, b) \in A \times \phi \)

Then \( a \in A \) and \( b \in \phi \)

Thm 5.1 : For any sets \( A, B, C \subseteq \mathcal{U} : \)

(a) \( A \times (B \cap C) = (A \times B) \cap (A \times C) \)

(b) \( A \times (B \cup C) = (A \times B) \cup (A \times C) \)

(c) \( (A \cap B) \times C = (A \times C) \cap (B \times C) \)

(d) \( (A \cup B) \times C = (A \times C) \cup (B \times C) \)

Proof. (a) \( \forall a, b \in \mathcal{U}, (a, b) \in A \times (B \cap C) \)

\[ \iff a \in A \land b \in (B \cap C) \]

\[ \iff (a \in A \land a \in A) \land (b \in B \land b \in C) \]

\[ \iff a \in A \land b \in B \land a \in A \land b \in C \]

\[ \iff (a, b) \in A \times B \land (a, b) \in A \times C \]

\[ \iff (a, b) \in (A \times B) \cap (A \times C) . \]
Chap 5 Relations and Functions

§ 5.2 Functions : Plain and One-to-One

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§ 5.2 Functions : Plain and One-to-One

Def 5.3 : For nonempty sets $A, B$, a function, or mapping, $f$ from $A$ to $B$, denoted by $f : A \rightarrow B \equiv$ a relation $R$ from $A$ to $B$ in which $\forall a \in A, \exists! (x, y) \in R$ s.t. $a = x$.

Note : ① We often write $f(a) = b$ when $(a, b)$ is an ordered pair in function $f$.
② for $(a, b) \in f$, $b$ is called the image of $a$ under $f$, $a$ is called the preimage of $b$ under $f$.
③ $f$ is a function $\iff \forall a \in A, \exists! b \in B$ s.t. $f(a) = b$.
④ $f$ is a function $\implies “(a, b), (a, c) \in f \implies b = c”$. 
§ 5.2 Functions : Plain and One-to-One

EX 5.9 : \( A = \{1, 2, 3\}, B = \{w, x, y, z\} \)

① \( f = \{(1, w), (2, x), (3, w)\} \) is a function.
② \( R_1 = \{(1, w), (2, x)\}, R_2 = \{(1, w), (2, w), (2, x), (3, z)\} \) are relation, not function!!

Def 5.4 : For the function \( f : A \rightarrow B \),

① \( A \) is called the **domain** of \( f \),
② \( B \) is called the **codomain** of \( f \),
③ \( f(A) = \{b \mid (a, b) \in f, \text{for some } a \in A\} \equiv \text{the range of } f. \)
④ \( a : \text{input}, f(a) : \text{output}, f : \text{transformed} \)

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\section*{5.2 Functions : Plain and One-to-One}

**EX 5.10** : (a) The \textit{greatest integer function} or \textit{floor function} :  
\[ f : \mathbb{R} \to \mathbb{Z}, \]
\[ f(x) = \lfloor x \rfloor = \text{the greatest integer less than or equal to } x \]
\[ = \max \{ a \mid a \leq x, a \in \mathbb{Z} \}. \]
\[ \text{ex : }  \]
\[ 1 \begin{array}{l} \lfloor 3.8 \rfloor = 3, \lfloor 3 \rfloor = 3, \lfloor -3.8 \rfloor = -4, \lfloor -3 \rfloor = -3. \\ \end{array} \]
\[ 2 \begin{array}{l} \lfloor 7.1 + 8.2 \rfloor = \lfloor 15.3 \rfloor = 15 = 7 + 8 = \lfloor 7.1 \rfloor + \lfloor 8.2 \rfloor . \\ \end{array} \]
\[ 3 \begin{array}{l} \lfloor 7.7 + 8.4 \rfloor = \lfloor 16.1 \rfloor = 16 \neq 7 + 8 = \lfloor 7.7 \rfloor + \lfloor 8.4 \rfloor . \\ \end{array} \]

(b) The \textit{ceiling function} : \[ g : \mathbb{R} \to \mathbb{Z}, \]  
\[ g(x) = \lceil x \rceil = \text{the least integer greater than or equal to } x \]
\[ = \min \{ a \mid a \geq x, a \in \mathbb{Z} \}. \]
\[ \text{ex : } \]
\[ 1 \begin{array}{l} \lceil 3 \rceil = 3, \lceil 3.01 \rceil = \lceil 3.7 \rceil = 4 = \lceil 4 \rceil , \lceil -3 \rceil = -3, \\ \lceil -3.01 \rceil = \lceil -3.7 \rceil = -3. \\ \end{array} \]
\[ 2 \begin{array}{l} \lceil 3.6 + 4.5 \rceil = \lceil 8.1 \rceil = 9 = 4 + 5 = \lceil 3.6 \rceil + \lceil 4.5 \rceil . \\ \end{array} \]
\[ 3 \begin{array}{l} \lceil 3.3 + 4.2 \rceil = \lceil 7.5 \rceil = 8 \neq 9 = 4 + 5 = \lceil 3.3 \rceil + \lceil 4.2 \rceil . \\ \end{array} \]
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(c) \textit{trunc (truncation)} :
\[ \mathbb{R} \to \mathbb{Z} \equiv \text{deletes the fractional part of a real number} \]
\begin{itemize}
  \item \textbf{1)} \( \text{trunc} (3.78) = 3, \text{trunc} (5) = 5, \text{trunc} (-7.22) = -7 \)
  \item \textbf{2)} \( \text{trunc} (3.78) = \lfloor 3.78 \rfloor = 3; \quad \textbf{3)} \text{trunc} (-3.78) = \lceil -3.78 \rceil = -3 \)
\end{itemize}
\[ \text{trunc} (x) = \begin{cases} 
  \lfloor x \rfloor, & \text{if } x \geq 0; \\
  \lceil x \rceil, & \text{if } x < 0.
\end{cases} \]

(d) Storing a matrix in a one-dim. array as the \textit{access function} \( f \) from the entries \( a_{ij} \) of \( A_{m \times n} \) to the positions, 1, 2, \ldots, \( mn \) :
\[
\begin{array}{cccccccccccc}
  & a_{11} & a_{12} & \cdots & a_{1n} & a_{21} & a_{22} & \cdots & a_{2n} & a_{31} & \cdots & \cdots & \cdots & a_{ij} & \cdots & a_{mn} \\
1 & 2 & \cdots & n & n + 1 & n + 2 & \cdots & 2n & 2n + 1 & \cdots & (i - 1)n + j & \cdots & (m - 1)n + n &= mn \\
\end{array}
\]
\begin{itemize}
  \item \textbf{ex :} \( a_{21} \) is found in position \( n + 1 \),
  \item \( a_{34} \) is found in position \( 2n + 4 \).
\end{itemize}
§ 5.2 Functions : Plain and One-to-One

**EX 5.11 :**

(a) \( \forall a, b \in \mathbb{Z}, b > 0, \exists ! q, r \in \mathbb{Z} \) s.t. \( a = q b + r, 0 \leq r < b \):

\[
q = \lfloor a / b \rfloor \text{ and } r = a - \lfloor a / b \rfloor \cdot b.
\]

(b) \( n \in \mathbb{Z}^+, n > 1, n = p_1^{e(1)} p_2^{e(2)} \ldots p_k^{e(k)} \) where \( k \in \mathbb{Z}^+, p_i \) is prime \( \forall 1 \leq i \leq k, p_i \neq p_j \) \( \forall 1 \leq i < j \leq k, e(i) \in \mathbb{Z}^+, \forall 1 \leq i \leq k. \)

Then if \( r \in \mathbb{Z}^+ \), the number of positive divisors of \( n \) that are perfect \( r \)th powers is \( \prod_{i=1}^{k} \left\lfloor (e(i) + 1) / r \right\rfloor = \prod_{i=1}^{k} \left( \left\lfloor e(i) / r \right\rfloor + 1 \right) \)

Let \( \left\lfloor (e(i)+1) / r \right\rfloor = h \Leftrightarrow h - 1 < (e(i)+1) / r \leq h \)

\[
\Leftrightarrow r h - r < e(i)+1 \leq r h
\]

\[
\Leftrightarrow r h - r - 1 < e(i) \leq r h - 1
\]

\[
\Leftrightarrow r h - r \leq e(i) < r h
\]

\[
\Leftrightarrow h - 1 \leq e(i) / r < h
\]

\[
\Leftrightarrow \left\lfloor e(i) / r \right\rfloor = h - 1.
\]
§ 5.2 Functions : Plain and One-to-One

EX 5.12 : ① A sequence of real numbers is a function

\[ f : \mathbb{Z}^+ \rightarrow \mathbb{R} \text{ where } f(n) = r_n. \]

② A sequence of integer numbers is a function

\[ g : \mathbb{N} \rightarrow \mathbb{Z} \text{ where } g(n) = a_n \]

Note : There are \(|B|^{|A|}\) functions from \(A\) to \(B \neq |A|^{|B|}\) functions from \(B\) to \(A\). (see textbook)

Def 5.5 : A functions \(f : A \rightarrow B\) is called one-to-one, or injective. 

( 1 – 1 ) \[ \equiv \forall b \in B, \text{ } b \text{ appears at most once as the image of an element of } A. \]

Note : ① If \(f : A \rightarrow B\) is 1 – 1, with \(A, B\) finite, then \(|A| \leq |B|\).
② \(f : A \rightarrow B\) is 1 – 1 \[ \iff \forall a_1, a_2 \in A, f(a_1) = f(a_2) \Rightarrow a_1 = a_2. \]
EX 5.13: ① \( f : \mathbb{R} \to \mathbb{R} \) where \( f(x) = 3x + 7 \) for all \( x \in \mathbb{R} \).

\[
\forall x_1, x_2 \in \mathbb{R} \quad f(x_1) = f(x_2) \Rightarrow 3x_1 + 7 = 3x_2 + 7 \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2.
\]

\( \therefore f \) is 1–1.

② \( g : \mathbb{R} \to \mathbb{R} \) where \( g(x) = x^4 - x \) for all \( x \in \mathbb{R} \).

\[
g(0) = 0^4 - 0 = 0 \quad \text{and} \quad g(1) = 1^4 - 1 = 1 - 1 = 0
\]

\( \therefore g \) is not 1–1. since \( g(0) = g(1) \) but \( 0 \neq 1 \).

EX 5.14: Let \( A = \{1, 2, 3\} \) and \( B = \{1, 2, 3, 4, 5\} \)

① \( f = \{(1, 1), (2, 3), (3, 4)\} \)

② \( g = \{(1, 1), (2, 3), (3, 3)\} \)

\( f \) is 1–1; \( g \) is not 1–1 because \( g(2) = g(3) \) but \( 2 \neq 3 \).
§ 5.2 Functions: Plain and One-to-One

Note: $|A| = m, |B| = n$, the number of $1 - 1$ function from $A$ to $B$ is $n (n - 1) (n - 2) \ldots (n - m + 1) = n! / (n - m)! = P^n_m = P(|B|, |A|)$ (see textbook)

Def 5.6: If $f : A \rightarrow B$ and $A_1 \subseteq A$, then $f(A_1) = \{b \in B \mid b = f(a), \text{ for some } a \in A_1\}$, is called the image of $A_1$ under $f$.

EX 5.15: $A = \{1, 2, 3, 4, 5\}, B = \{w, x, y, z\}$, let $f : A \rightarrow B$ be given by $f = \{(1, w), (2, x), (3, x), (4, y), (5, y)\}$ then for $A_1 = \{1\}, A_2 = \{1, 2\}, A_3 = \{1, 2, 3\}, A_4 = \{2, 3\}, A_5 = \{2, 3, 4, 5\}$:

$f(A_1) = \{f(a) \mid a \in A_1\} = \{f(a) \mid a \in \{1\}\} = \{f(1)\} = \{w\};$

$f(A_2) = \{f(a) \mid a \in \{1, 2\}\} = \{f(1), f(2)\} = \{w, x\};$

$f(A_3) = \{f(1), f(2), f(3)\} = \{w, x\} = f(A_2);$

$f(A_4) = \{x\}; \quad f(A_5) = \{x, y\}$

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**§ 5.2 Functions : Plain and One-to-One**

**EX 5.16 :** (a) Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^2$.

- $g(\mathbb{R}) =$ the range of $g = [0, +\infty)$;
- $g(\mathbb{Z}) =$ the image of $\mathbb{Z}$ under $g = \{0, 1, 4, 9, 16, \ldots\}$;
- For $A_1 = [-2, 1]$, $g(A_1) = [0, 4]$.

(b) Let $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $h(x, y) = 2x + 3y$.

- The domain of $h$ is $\mathbb{Z} \times \mathbb{Z}$;
- The codomain is $\mathbb{Z}$.

$$h(0, 0) = 2(0) + 3(0) = 0$$
$$h(-3, 7) = 2(-3) + 3(7) = 15$$
$$h(2, -1) = 2(2) + 3(-1) = 1$$
$$\forall \ n \in \mathbb{Z}, \ h(2n, -n) = 2(2n) + 3(-n) = 4n - 3n = n$$

- $h(\mathbb{Z} \times \mathbb{Z}) =$ the range of $h = \mathbb{Z}$.
- For $A_1 = \{(0, n) \mid n \in \mathbb{Z}^+\} = \{0\} \times \mathbb{Z}^+ \subseteq \mathbb{Z} \times \mathbb{Z}$,
  $$h(A_1) =$ the image of $A_1$ under $h = \{3, 6, 9, \ldots\} = \{3n \mid n \in \mathbb{Z}^+\}$$

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§ 5.2 Functions : Plain and One-to-One

Thm 5.2 : Let \( f : A \rightarrow B \), with \( A_1, A_2 \subseteq A \). Then

(a) \( f(A_1 \cup A_2) = f(A_1) \cup f(A_2) \);
(b) \( f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2) \);
(c) \( f(A_1 \cap A_2) = f(A_1) \cap f(A_2) \) when \( f \) is injective.

Proof.

(b) \( b \in B, b \in f(A_1 \cap A_2) \Rightarrow \exists a \in A_1 \cap A_2 \) s.t. \( b = f(a) \)
\( \Rightarrow [\exists a \in A_1 \) s.t. \( b = f(a)] \) and \( [\exists a \in A_2 \) s.t. \( b = f(a)] \)
\( \Rightarrow [b \in f(A_1)] \) and \( [b \in f(A_2)] \)
\( \Rightarrow b \in f(A_1) \cap f(A_2) \)
\( \therefore f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2) \).

Def 5.7 : If \( f : A \rightarrow B \) and \( A_1 \subseteq A \), then \( f|_{A_1} : A_1 \rightarrow B \) is called the \underline{restriction} of \( f \) to \( A_1 \equiv f|_{A_1}(a) = f(a), \forall a \in A_1. \)
Def 5.8: Let $A_1 \subseteq A$ and $f : A_1 \to B$. If $g : A \to B$ and $g(a) = f(a) \ \forall \ a \in A_1$, then we call $g$ an extension of $f$ to $A$.

EX 5.17: For $A = \{1, 2, 3, 4, 5\}$, let $f : A \to \mathbb{R}$ be defined by $f = \{(1, 10), (2, 13), (3, 16), (4, 19), (5, 22)\}$.
Let $g : \mathbb{Q} \to \mathbb{R}$ where $g(q) = 3q + 7. \ \forall \ g \in \mathbb{Q}$
Let $h : \mathbb{R} \to \mathbb{R}$ where $h(r) = 3r + 7. \ \forall \ r \in \mathbb{R}$. then
i) $g$ is an extension of $f$ (from $A$) to $\mathbb{Q}$,
ii) $f$ is the restriction of $g$ (from $\mathbb{Q}$) to $A$,
iii) $h$ is an extension of $f$ (from $A$) to $\mathbb{R}$,
iv) $f$ is the restriction of $h$ (from $\mathbb{R}$) to $A$,
v) $h$ is an extension of $g$ (from $\mathbb{Q}$) to $\mathbb{R}$,
vi) $g$ is the restriction of $h$ (from $\mathbb{R}$) to $\mathbb{Q}$.
§ 5.2 Functions: Plain and One-to-One

EX 5.18: Let $A = \{w, x, y, z\}$, $B = \{1, 2, 3, 4, 5\}$, $A_1 = \{w, y, z\}$

Let $f: A \rightarrow B$, $g: A_1 \rightarrow B$ by represented by the diagrams:

① $f$ is an extension of $g$ from $A_1$ to $A$ (有五種)
② $g = f|_{A_1}$

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Chap 5 Relations and Functions

§ 5.3 Onto Functions:

Stirling Numbers of the Second Kind

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§ 5.3 Onto Functions : Stirling Numbers of the Second Kind

Def 5.9: A function $f : A \rightarrow B$ is called onto, (or surjection) $\equiv f(A) = B$; i.e $\forall b \in B$, $\exists a \in A$ s.t. $f(a) = b$.

EX 5.19: ① $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^3$ is an onto function:

- $\forall r \in \mathbb{R}$ (codomain of $f$), $\exists \sqrt[3]{r} \in \mathbb{R}$ (domain of $f$)
- s.t. $f(\sqrt[3]{r}) = (\sqrt[3]{r})^3 = r$.

∴ the codomain of $f = \mathbb{R} =$ the range of $f$.

② $g : \mathbb{R} \rightarrow \mathbb{R}$, where $g(x) = x^2$ is not an onto function:

- $\exists -9 \in \mathbb{R}$, but $\nexists$ a real number $r$ s.t. $g(r) = -9$.

∴ the range of $g = g(\mathbb{R}) = [0, +\infty) \subset \mathbb{R}$.

(h : $\mathbb{R} \rightarrow [0, +\infty)$ defined by $h(x) = x^2$ is an onto function).
§ 5.3 Onto Functions : Stirling Numbers of the Second Kind

EX 5.20 : ① \( f : \mathbb{Z} \rightarrow \mathbb{Z} \), where \( f(x) = 3x + 1 \) for \( x \in \mathbb{Z} \).
\[
f(\mathbb{Z}) = \{\ldots, -8, -5, -2, 1, 4, 7, \ldots\} \subset \mathbb{Z},
\]
\[
\therefore f \text{ is not an onto function.}
\]
\[
\text{ex: } 8 \in \mathbb{Z}, \text{ but if } 3x + 1 = 8 \Rightarrow x = 7/3 \notin \mathbb{Z}
\]
\[
\therefore \text{ there is no } x \text{ in the domain } \mathbb{Z} \text{ with } f(x) = 8.
\]
② \( g : \mathbb{Q} \rightarrow \mathbb{Q} \), where \( g(x) = 3x + 1 \) for \( x \in \mathbb{Q} \).
③ \( h : \mathbb{R} \rightarrow \mathbb{R} \), where \( h(x) = 3x + 1 \) for \( x \in \mathbb{R} \).
① ② ③ \( f, g, h \) are one-to-one.

EX 5.21 : \( A = \{1, 2, 3, 4\}, B = \{x, y, z\} \),
① \( f_1 = \{(1, z), (2, y), (3, x), (4, y)\}, f_2 = \{(1, x), (2, x), (3, y), (4, z)\} \) are both functions from \( A \) onto \( B \)
② \( g = \{(1, x), (2, x), (3, y), (4, y)\} \) is not onto,
\[
\therefore g(A) = \{x, y\} \subset B.
\]

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Q : The number of onto functions \( f : A \rightarrow B \)
where \( |A| = m \geq n = |B| = ? \)

Chap 8 有詳解, 現在, 先看例子:

EX 5.22 : ① \( A = \{x, y, z\}, B = \{1, 2\}, \)
the number of \( f : A \rightarrow B \) are onto?
\( f_1 = \{(x, 1), (y, 1), (z, 1)\}, \)
\( f_2 = \{(x, 2), (y, 2), (z, 2)\} \) are not onto
called the constant functions :
\(|B|^{|A|} - 2 = 2^3 - 2 = 6.\)

② \( |A| = m \geq 2, |B| = 2, \) then
there are \( 2^m - 2 \) onto functions from \( A \) to \( B. \)
§ 5.3 Onto Functions : Stirling Numbers of the Second Kind

EX 5.23 : ① \( A = \{w, x, y, z\}, \ B = \{1, 2, 3\}, \) 

\# of onto functions from \( A \) to \( B \) = ?

Sol.

\[
3^4 - 3 \cdot 2^4 + 3 \cdot 1^4 = (3^3) 3^4 - (2^3) 2^4 + (1^3) 1^4 = 36.
\]

(說明: 任意 \( A \rightarrow \{1, 2\} \) + 重覆減的 \( A \rightarrow \{1\} \)

\( A \rightarrow \{1, 3\} \) + 重覆減的 \( A \rightarrow \{2\} \)

\( A \rightarrow \{2, 3\} \) + 重覆減的 \( A \rightarrow \{3\} \))

② \( |A| = m \geq 3, \ |B| = 3, \) then there are

\[
(3^3) 3^m - (2^3) 2^m + (1^3) 1^m
\]

functions from \( A \) onto \( B \).

Formula : \( \forall \) finite set \( A, \ B \) with \( |A| = m, \ |B| = n, \) there are

\[
\binom{n}{n} n^m - \binom{n-1}{n} (n - 1)^m + \binom{n-2}{n} (n - 2)^m - \ldots
\]

\[+ (-1)^{n-2} \binom{2}{n} 2^m + (-1)^{n-1} \binom{1}{n} 1^m
\]

\[
= \sum_{k=0}^{n-1} (-1)^k \binom{n}{n-k} (n-k)^m = \sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m
\]

onto functions from \( A \) to \( B \).

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§ 5.3 Onto Functions : Stirling Numbers of the Second Kind

EX 5.24 : Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{w, x, y, z\}$ : 

$m = 7, \ n = 4 : (\binom{4}{4})^7 - (\binom{3}{4})^7 + (\binom{2}{4})^7 - (\binom{1}{4})^7 = 8400.$

i.e. $\exists$ 8400 functions from $A$ onto $B.$

**Note:** $\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} (n-k)^{m} = 0$ when $m < n.$

$\therefore$ no such onto function $f : A \rightarrow B$ for $|A| = m < n = |B|.$

Ex 5.25 : Joan : supervisor, Teresa : secretary, 

3 : Administrative assistants. 7 account and Teresa’s work includes the most expensive account?

(a) Teresa works only on the most expensive account :

$\sum_{k=0}^{3} (-1)^{k} \binom{3}{k} (3-k)^{6} = 540.$

(b) Teresa does more : $\sum_{k=0}^{4} (-1)^{k} \binom{4}{k} (4-k)^{6} = 1560.$

$\Rightarrow 540 + 1560 = 2100 \ (= (1/4)8400)$

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§ 5.3 Onto Functions: Stirling Numbers of the Second Kind

**Def**: $S(m, n)$ (*Stirling number of the second kind*)

\[
S(m, n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m. \quad \text{for } m \geq n.
\]

= #(將 $m$ 相異物放入 $n$ 個相同的箱子，使無空箱.)

**Note**: $|A| = m \geq n = |B|$, $\exists n! \cdot S(m, n)$ onto functions from $A$ to $B$.  

**Table 5.1**

<table>
<thead>
<tr>
<th></th>
<th>$S(m,n)$</th>
</tr>
</thead>
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<tr>
<td>$m$</td>
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</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
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</tr>
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<td>1</td>
</tr>
<tr>
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<td>1</td>
</tr>
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</tr>
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<td>6</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
</tr>
</tbody>
</table>
§ 5.3 Onto Functions : Stirling Numbers of the Second Kind

EX 5.27: 將 m 相異物放入 n 個相同的箱子, 但允許空箱.
ex : m = 4, n = 3, $S(4, 1) + S(4, 2) + S(4, 3) = 1 + 7 + 6 = 14$

Thm 5.3: Let $m, n \in \mathbb{N}$ such that $1 < n \leq m$, then
$S(m + 1, n) = S(m, n – 1) + n S(m, n)$

Proof.
Let $A = \{a_1, a_2, \ldots, a_m, a_{m+1}\}$.

$S(m + 1, n) = \#$ of the way in which the objects of $A$ can be distributed among $n$ identical containers, with no container left empty.

2 case : (a) $a_{m+1}$ is in a container by itself (in $n$): $S(m, n – 1)$
(b) $a_{m+1}$ is in a container with other (in $1 \sim n$): $n(S(m, n))$
\[ \therefore S(m + 1, n) = S(m, n – 1) + n S(m, n). \]

Note: $(1 / n)[n! S(m + 1, n)] = [(n–1)! S(m, n – 1) + n! S(m, n)]$
§ 5.3 Onto Functions : Stirling Numbers of the Second Kind

Note: \((1 / n)[n! S(m + 1, n)] = [(n−1)! S(m, n − 1) + n! S(m, n)]\)

Let \(A = \{a_1, a_2, \ldots, a_m, a_{m+1}\}, B = \{b_1, b_2, \ldots, b_{n−1}, b_n\}\). with \(m \geq n−1\).

\[
(1 / n) \text{[# of functions from } A \text{ onto } B] \\
= \text{[# of functions from } A - \{a_{m+1}\} \text{ onto } B - \{b_i\}] + \\
\text{[# of functions from } A - \{a_{m+1}\} \text{ onto } B]
\]

\[
(1 / n) \left| \{ h \mid h : A \rightarrow B, h \text{ is onto}\} \right|
= \left| \{ f \mid f : (A - \{a_{m+1}\}) \rightarrow (B - \{b_i\}), f \text{ is onto and } f(a_{m+1}) = b_i \} \right| + \\
\left| \{ g \mid g : (A - \{a_{m+1}\}) \rightarrow B, g \text{ is onto and } f(a_{m+1}) = b_i \} \right|
\]

see EX 5.25

**EX 5.28** : \(30030 = 2 \times 3 \times 5 \times 7 \times 11 \times 13\)

(i) \(30 \times 1001 = (2 \times 3 \times 5) (7 \times 11 \times 13)\)

(ii) \(110 \times 273 = (2 \times 5 \times 11) (3 \times 7 \times 13)\)

(iii) \(2310 \times 13 = (2 \times 3 \times 5 \times 7 \times 11) (13)\)

\[S(6, 2) = 31\text{個其中之三}\]
§ 5.3 Onto Functions: Stirling Numbers of the Second Kind

EX 5.28: \(30030 = 2 \times 3 \times 5 \times 7 \times 11 \times 13\)

(iv) \(14 \times 33 \times 65 = (2 \times 7) (3 \times 11) (5 \times 13)\) \[S(6, 3) = 90\] 个

(v) \(22 \times 35 \times 39 = (2 \times 11) (5 \times 7) (3 \times 13)\)

i.e. \(\exists S(6, 2)\) ways to factor 30030 as \(m \cdot n\),
where \(m, n \in \mathbb{Z}^+\) for \(1 < m, n < 30030\). (不管顺序)

\(\exists S(6, 3)\) ways to factor 30030 as \(m \cdot n \cdot l\),
where \(m, n, l \in \mathbb{Z}^+\) for \(1 < m, n, l < 30030\). (不管顺序)

\(Q_1:\) At least two factors (> 1) in each of these unordered factorizations = ?
\(A : \sum_{i=2}^{6} S(6, i) = 202.\)

\(Q_2:\) Include the one-factor factorization 30030.
\(A : 202 + 1 = 203.\)