Computer Science and Information Engineering
National Chi Nan University

Discrete Mathematics

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Chap 5 Relations and Functions

§ 5.1 Cartesian Products and Relations

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

§ 5.1 Cartesian Products and Relations

Def 5.1 : ① For sets A, B, the ***Cartesian product***, or ***cross product***, of A and B is denoted by $A \times B \equiv \{(a, b) \mid a \in A, b \in B\}$.

② The elements of $A \times B$ are ***ordered pairs*** ≡

For $(a, b), (c, d) \in A \times B$, $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

EX 5.1 : Let $A = \{2, 3, 4\}$, $B = \{4, 5\}$ Then

a) $A \times B = \{(2, 4), (2, 5), (3, 4), (3, 5), (4, 4), (4, 5)\}$,

b) $B \times A = \{(4, 2), (4, 3), (4, 4), (5, 2), (5, 3), (5, 4)\}$,

Note : ① $A \times B \neq B \times A$, $|A \times B| = |B \times A| = |A| \cdot |B|$.

② Here $A \subseteq \mathcal{U}_1$, $B \subseteq \mathcal{U}_2$.

③ If $A, B \subseteq \mathcal{U}$, but $A \times B \subseteq \mathcal{U}$ is not necessary!! i.e. "×" is not necessarily closed.

§ 5.1 Cartesian Products and Relations

Def : ① For sets $A_1, A_2, \ldots, A_n$. ($n \in \mathbb{Z}^+$, $n \geq 3$), the (*n - fold*) *product* of $A_1, A_2, \ldots, A_n$ is denoted by $A_1 \times A_2 \times \ldots \times A_n \equiv \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, 1 \leq i \leq n\}$

② The elements of $A_1 \times A_2 \times \ldots \times A_n$ are called *ordered n-tuples*, (3-tuple ≡ *triple*).

For $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in A_1 \times \ldots \times A_n$,
$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ iff $a_i = b_i$, $\forall\ 1 \leq i \leq n$.

EX 5.1 : Let $A = \{2, 3, 4\}$, $B = \{4, 5\}$ Then

c) $B^2 = B \times B = \{(4, 4), (4, 5), (5, 4), (5, 5)\}$,

d) $B^3 = B \times B \times B = \{(4, 4, 4), (4, 4, 5), (4, 5, 4), (4, 5, 5), (5, 4, 4), (5, 4, 5), (5, 5, 4), (5, 5, 5)\}$
$= \{(a, b, c) \mid a, b, c \in B\}$.

§ 5.1 Cartesian Products and Relations

EX 5.2 : $\mathbf{R} \times \mathbf{R} = \{(x, y) \mid x, y \in \mathbf{R}\}$: 二維實數坐標平面.

$\mathbf{R}^+ \times \mathbf{R}^+$: 第一象限.

$\mathbf{R}^3$: Euclidean three - space .

Note : $A_1 \times A_2 \times A_3 \neq (A_1 \times A_2) \times A_3 \neq A_1 \times (A_2 \times A_3)$

$\uparrow \qquad \uparrow \qquad \uparrow$

$\because (a_1, a_2, a_3) \neq ((a_1, a_2), a_3) \neq (a_1, (a_2, a_3))$

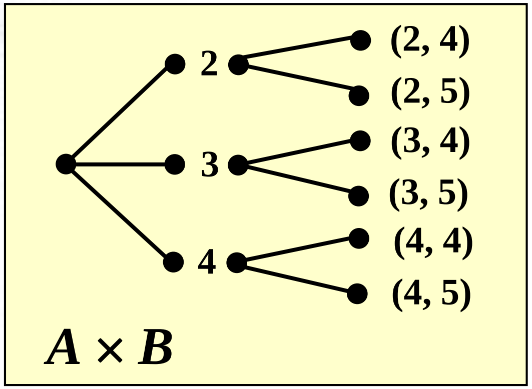
但不考慮後兩種 (雖然也很重要...)，

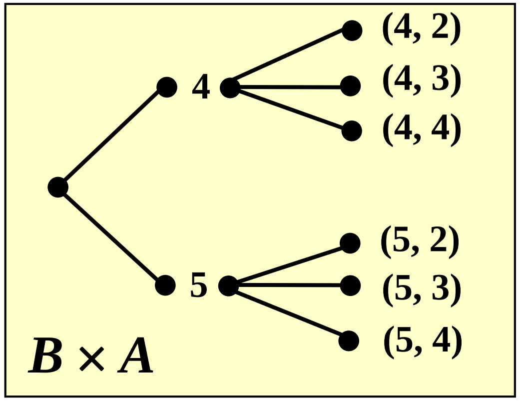
本書皆只考慮第一種。

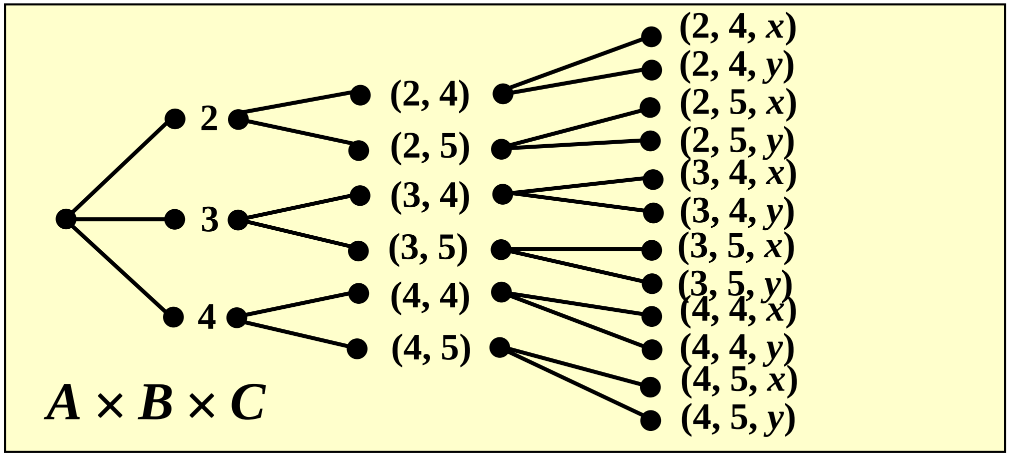
§ 5.1 Cartesian Products and Relations

EX 5.3 : Let $A = \{2, 3, 4\}$, $B = \{4, 5\}$, $C = \{x, y\}$:

$A \times B$, $B \times A$, $A \times B \times C$: use ***tree diagram*** :







EX 5.4 : At the Wimbledon Tennis Championships : The winner is the First to win two sets. Let *N*, *E* denote the two players in a match :

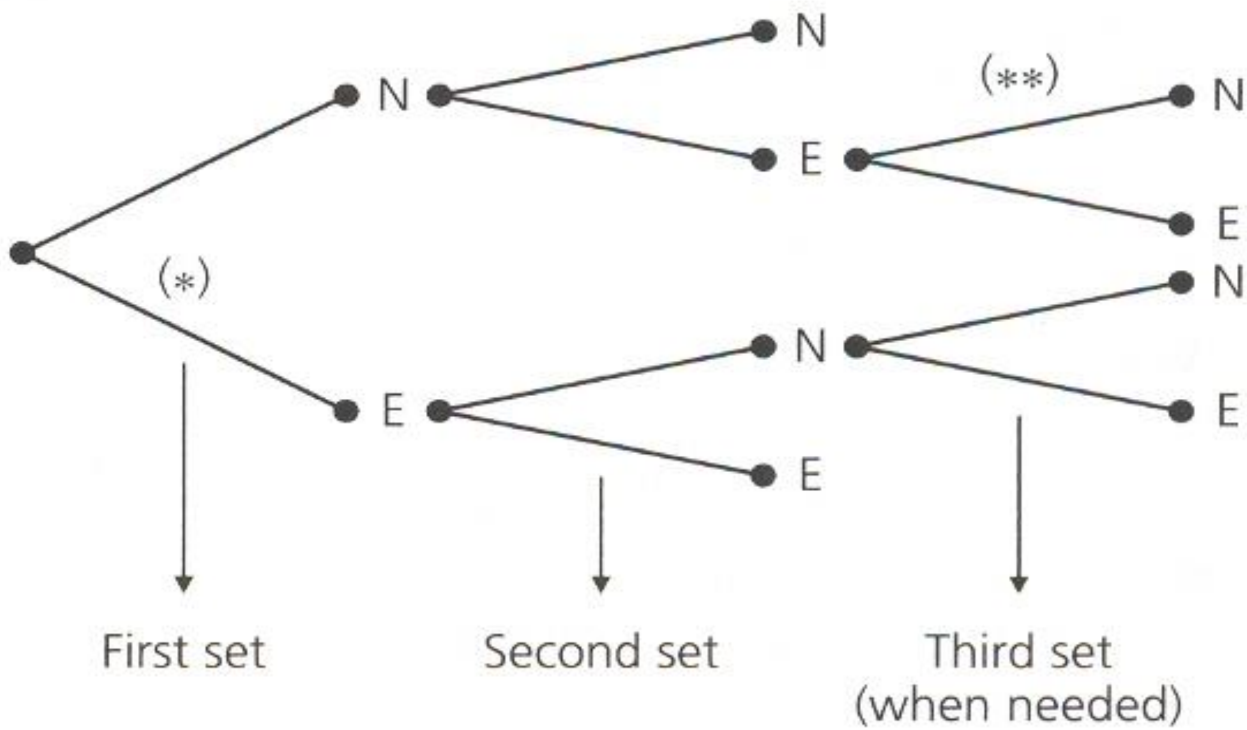


Figure 5.2

§ 5.1 Cartesian Products and Relations

Def 5.2 : For sets A, B,
① Any subset of $A \times B$ is called a ***relation*** from A to B.
② Any subset of $A \times A$ is called a ***binary relation*** on A.

EX 5.5 : A, B as **EX 5.1**, $A = \{2, 3, 4\}$, $B = \{4, 5\}$.
$\because$ $|A \times B| = 6$, $\exists\ 2^6$ possible relations from A to B.
Such as : a) ϕ b) $\{(2, 4)\}$ c) $\{(2, 4), (2, 5)\}$
d) $\{(2, 4), (3, 4), (4, 4)\}$ e) $\{(2, 4), (3, 4), (4, 5)\}$ f) $A \times B$.

Note : ① For A, B : finite sets with $|A| = m$, $|B| = n$;
$\exists\ 2^{mn}$ relation from A to B, including ϕ and $A \times B$.
② $\exists\ 2^{mn}$ relation from B to A :
$\mathcal{R}_1 \subseteq A \times B$ is a relation $\Leftrightarrow$ $\mathcal{R}_2 \subseteq B \times A$ is a relation.
where $\mathcal{R}_2 = \{(b, a) \mid (a, b) \in \mathcal{R}_1\}$.

§ 5.1 Cartesian Products and Relations

EX 5.6 : Let $B = \{1, 2\}$, $A = \mathcal{P}(B) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$.

Let $\mathcal{R}$ (a binary relation on A) $= \{(\phi, \phi), (\phi, \{1\}), (\phi, \{2\}), (\phi, \{1, 2\}), (\{1\}, \{1\}), (\{1\}, \{1, 2\}), (\{2\}, \{2\}), (\{2\}, \{1, 2\}), (\{1, 2\}, \{1, 2\})\}$

$\mathcal{R}$ is the **subset relation.**

Def : ① A binary relation $\mathcal{R}$ on $\mathcal{P}(B)$ is the ***subset relation*** $(C, D) \in \mathcal{R}$ iff $C, D \subseteq B$ and $C \subseteq D$.

② ***infix*** notation for a relation $\mathcal{R}$: $a \mathcal{R} b \equiv (a, b) \in \mathcal{R}$, $c \not\mathcal{R} d \equiv (c, d) \notin \mathcal{R}$.

§ 5.1 Cartesian Products and Relations

<u>EX 5.7</u> : $A = \mathbb{Z}^+$. $\mathcal{R}$ is a binary relation on $A = \{(x, y) \mid x \le y\}$, "less than or equal to" relation.

$(7, 7), (7, 11) \in \mathcal{R}$; $(8, 2) \notin \mathcal{R}$,
i.e. $7\ \mathcal{R}\ 11$; $8\ \not\mathcal{R}\ 2$.

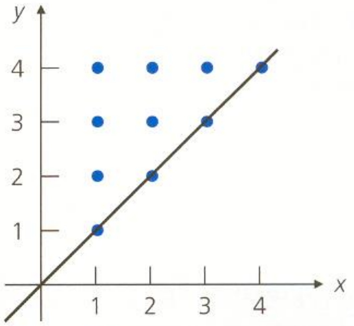


Figure 5.3

<u>EX 5.8</u> : Let $\mathcal{R} \subseteq \mathbb{N} \times \mathbb{N}$, $\mathcal{R} = \{(m, n) \mid n = 7m\}$.

$\mathcal{R}$ can be defined recursively as:

1) $(0, 0) \in \mathcal{R}$; and

2) If $(s, t) \in \mathcal{R}$, then $(s + 1, t + 7) \in \mathcal{R}$.

check $(3, 21) \in \mathcal{R}$: (i) $(0, 0) \in \mathcal{R} \Rightarrow (0 + 1, 0 + 7) = (1, 7) \in \mathcal{R}$,
(ii) $(1, 7) \in \mathcal{R} \Rightarrow (1 + 1, 7 + 7) = (2, 14) \in \mathcal{R}$,
(iii) $(2, 14) \in \mathcal{R} \Rightarrow (2 + 1, 14 + 7) = (3, 21) \in \mathcal{R}$.

§ 5.1 Cartesian Products and Relations

Remark : $\forall$ sets A, $A \times \phi = \phi \times A = \phi$

If $A \times \phi \neq \phi$, then let $(a, b) \in A \times \phi$

Then $a \in A$ and $b \in \phi \rightarrow\leftarrow$

Thm 5.1 : For any sets $A, B, C \subseteq \mathcal{U}$:

(a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(d) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Proof. (a) $\forall\, a, b \in \mathcal{U}$, $(a, b) \in A \times (B \cap C)$

$\Leftrightarrow a \in A \wedge b \in (B \cap C)$

$\Leftrightarrow (a \in A \wedge a \in A) \wedge (b \in B \wedge b \in C)$

$\Leftrightarrow a \in A \wedge b \in B \wedge a \in A \wedge b \in C$

$\Leftrightarrow (a, b) \in A \times B \wedge (a, b) \in A \times C$

$\Leftrightarrow (a, b) \in (A \times B) \cap (A \times C)$.

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§ 5.2 Functions : Plain and One-to-One

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§ 5.2 Functions : Plain and One-to-One

Def 5.3 : For nonempty sets A, B, a ***function***, or ***mapping***, f **from A to B**, denoted by $f : A \to B$ $\equiv$ a relation $\mathcal{R}$ from A to B in which $\forall a \in A$, $\exists!\ (x, y) \in \mathcal{R}$ s.t. $a = x$.

Note : ① We often write $f(a) = b$ when (a, b) is an ordered pair in function f.

② for $(a, b) \in f$, $\begin{cases} b \text{ is called the } \textit{image} \text{ of } a \text{ under } f, \\ a \text{ is called the } \textit{preimage} \text{ of } b \text{ under } f. \end{cases}$

③ f is a function $\Leftrightarrow \forall a \in A, \exists!\ b \in B$ s.t. $f(a) = b$.

④ f is a function $\Rightarrow$ "$(a, b), (a, c) \in f \Rightarrow b = c$".

§ 5.2 Functions : Plain and One-to-One

EX 5.9 : $A = \{1, 2, 3\}$, $B = \{w, x, y, z\}$

① $f = \{(1, w), (2, x), (3, w)\}$ is a function.

② $\mathcal{R}_1 = \{(1, w), (2, x)\}$, $\mathcal{R}_2 = \{(1, w), (2, w), (2, x), (3, z)\}$
are relation, not function!!

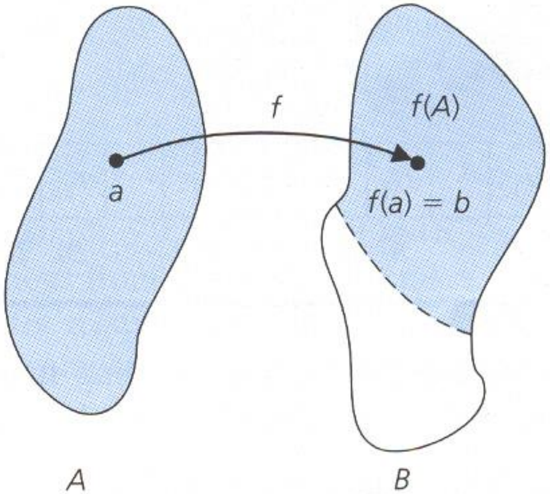

Figure 5.4

Def 5.4 : For the function $f : A \to B$,

① A is called the *domain* of f,

② B is called the *codomain* of f,

③ $f(A) = \{b \mid (a, b) \in f$, for some $a \in A\}$
$\equiv$ the *range* of f.

④ a : *input*, $f(a)$: *output*, f : *transformed*

§ 5.2 Functions : Plain and One-to-One

EX 5.10 : (a) The *greatest integer function* or *floor function* :

$f : \mathbb{R} \to \mathbb{Z}$,

$f(x) = \lfloor x \rfloor$ = the greatest integer less than or equal to x

$= \max \{a \mid a \le x, a \in \mathbb{Z}\}$.

ex : ① $\lfloor 3.8 \rfloor = 3, \lfloor 3 \rfloor = 3, \lfloor -3.8 \rfloor = -4, \lfloor -3 \rfloor = -3$.

② $\lfloor 7.1 + 8.2 \rfloor = \lfloor 15.3 \rfloor = 15 = 7 + 8 = \lfloor 7.1 \rfloor + \lfloor 8.2 \rfloor$.

③ $\lfloor 7.7 + 8.4 \rfloor = \lfloor 16.1 \rfloor = 16 \neq 7 + 8 = \lfloor 7.7 \rfloor + \lfloor 8.4 \rfloor$.

(b) The *ceiling function* : $g : \mathbb{R} \to \mathbb{Z}$, is defined by

$g(x) = \lceil x \rceil$ = the least integer greater than or equal to x

$= \min \{a \mid a \ge x, a \in \mathbb{Z}\}$.

ex : ① $\lceil 3 \rceil = 3, \lceil 3.01 \rceil = \lceil 3.7 \rceil = 4 = \lceil 4 \rceil, \lceil -3 \rceil = -3$,

$\lceil -3.01 \rceil = \lceil -3.7 \rceil = -3$.

② $\lceil 3.6 + 4.5 \rceil = \lceil 8.1 \rceil = 9 = 4 + 5 = \lceil 3.6 \rceil + \lceil 4.5 \rceil$.

③ $\lceil 3.3 + 4.2 \rceil = \lceil 7.5 \rceil = 8 \neq 9 = 4 + 5 = \lceil 3.3 \rceil + \lceil 4.2 \rceil$.

§ 5.2 Functions : Plain and One-to-One

(c) ***trunc*** **(*****truncation*****) :**

$\mathbb{R} \rightarrow \mathbb{Z} \equiv$ deletes the fractional part of a real number

ex : ① trunc (3.78) = 3, trunc (5) = 5, trunc (–7. 22) = – 7

② trunc (3.78) = $\lfloor 3.78 \rfloor$ = 3; ③ trunc (– 3.78) = $\lceil -3.78 \rceil$ = – 3

$$\text{trunc}(x) = \begin{cases} \lfloor x \rfloor, & \text{if } x \geq 0; \\ \lceil x \rceil, & \text{if } x < 0. \end{cases}$$

(d) Storing a matrix in a one-dim. array as the *access function f* from the entries a_{ij} of $A_{m\times n}$ to the positions, 1, 2, …, *mn* :

a_{11}	a_{12}	$\cdots$	a_{1n}	a_{21}	a_{22}	$\cdots$	a_{2n}	a_{31}	$\cdots$	a_{ij}	$\cdots$	a_{mn}
1	2	$\cdots$	n	$n+1$	$n+2$	$\cdots$	$2n$	$2n+1$	$\cdots$	$(i-1)n+j$	$\cdots$	$(m-1)n+n\ (=mn)$

ex : a_{21} is found in position n + 1,

a_{34} is found in position 2n + 4.

§ 5.2 Functions : Plain and One-to-One

EX 5.11 :

(a) $\forall\ a, b \in \mathbb{Z}, b > 0, \exists\ !\ q, r \in \mathbb{Z}$ s.t. $a = q\,b + r, 0 \le r < b$:

$q = \lfloor a / b \rfloor$ and $r = a - \lfloor a / b \rfloor \cdot b.$

(b) $n \in \mathbb{Z}^+, n > 1, n = p_1^{e(1)}\, p_2^{e(2)} \ldots p_k^{e(k)}$ where $k \in \mathbb{Z}^+$, p_i is prime $\forall\ 1 \le i \le k$, $p_i \ne p_j\ \forall\ 1 \le i < j \le k$, $e(i) \in \mathbb{Z}^+, \forall\ 1 \le i \le k$.

Then if $r \in \mathbb{Z}^+$, the number of positive divisors of n that are perfect rth powers is $\Pi^k_{i=1} \lceil (e(i) + 1) / r \rceil = \Pi^k_{i=1} (\lfloor e(i) / r \rfloor + 1)$

Let $\lceil (e(i)+1) / r \rceil = h \Leftrightarrow h - 1 < (e(i)+1) / r \le h$

$\Leftrightarrow r\,h - r < e(i)+1 \le rh$

$\Leftrightarrow r\,h - r - 1 < e(i) \le r\,h - 1$

$\Leftrightarrow r\,h - r \le e(i) < r\,h$

$\Leftrightarrow h - 1 \le e(i) / r < h$

$\Leftrightarrow \lfloor e(i) / r \rfloor = h - 1.$

§ 5.2 Functions : Plain and One-to-One

EX 5.12 : ① A sequence of real numbers is a function $f : \mathbb{Z}^+ \rightarrow \mathbb{R}$ where $f(n) = r_n$.
② A sequence of integer numbers is a function $g : \mathbb{N} \rightarrow \mathbb{Z}$ where $g(n) = a_n$

Note : There are $|B|^{|A|}$ functions from A to $B \neq |A|^{|B|}$ functions from B to A. (see textbook)

Def 5.5 : A functions $f : A \rightarrow B$ is called *one-to-one*, or *injective*.
(1 – 1) $\equiv \forall\, b \in B$, b appears at most once as the image of an element of A.

Note : ① If $f : A \rightarrow B$ is $1 - 1$, with A, B finite, then $|A| \leq |B|$.
② $f : A \rightarrow B$ is $1 - 1 \Leftrightarrow \forall\, a_1, a_2 \in A$, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

§ 5.2 Functions : Plain and One-to-One

EX 5.13 : ① $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 3x + 7$ for all $x \in \mathbb{R}$.

$\forall\, x_1, x_2 \in \mathbb{R}$

$f(x_1) = f(x_2) \Rightarrow 3x_1 + 7 = 3x_2 + 7 \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2$.

$\therefore$ f is 1 – 1.

② $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = x^4 - x$ for all $x \in \mathbb{R}$.

$g(0) = 0^4 - 0 = 0$ and $g(1) = 1^4 - 1 = 1 - 1 = 0$

$\therefore$ g is not 1 – 1. since $g(0) = g(1)$ but $0 \neq 1$.

EX 5.14 : Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$

① $f = \{(1, 1), (2, 3), (3, 4)\}$

② $g = \{(1, 1), (2, 3), (3, 3)\}$

f is 1 – 1; g is not 1 – 1 because $g(2) = g(3)$ but $2 \neq 3$.

§ 5.2 Functions : Plain and One-to-One

Note : $|A| = m$, $|B| = n$, the number of 1 – 1 function from A to B is $n(n-1)(n-2)\ldots(n-m+1) = n!/(n-m)! = P^n{}_m = P(|B|, |A|)$ (see textbook)

Def 5.6 : If $f : A \to B$ and $A_1 \subseteq A$, then $f(A_1) = \{b \in B \mid b = f(a)$, for some $a \in A_1\}$, is called ***the image of A_1 under f***.

EX 5.15 : $A = \{1, 2, 3, 4, 5\}$, $B = \{w, x, y, z\}$, let $f : A \to B$ be given by $f = \{(1, w), (2, x), (3, x), (4, y), (5, y)\}$ then for

$A_1 = \{1\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2, 3\}$, $A_4 = \{2, 3\}$, $A_5 = \{2, 3, 4, 5\}$:

$f(A_1) = \{f(a) \mid a \in A_1\} = \{f(a) \mid a \in \{1\}\} = \{f(1)\} = \{w\}$;

$f(A_2) = \{f(a) \mid a \in \{1, 2\}\} = \{f(1), f(2)\} = \{w, x\}$;

$f(A_3) = \{f(1), f(2), f(3)\} = \{w, x\} = f(A_2)$;

$f(A_4) = \{x\}$; $f(A_5) = \{x, y\}$

§ 5.2 Functions : Plain and One-to-One

EX 5.16 : (a) Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^2$.

$g(\mathbb{R})$ = the range of g = $[0, +\infty)$;

$g(\mathbb{Z})$ = the image of $\mathbb{Z}$ under g = $\{0, 1, 4, 9, 16, \ldots\}$;

For $A_1 = [-2, 1]$, $g(A_1) = [0, 4]$.

(b) Let $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $h(x, y) = 2x + 3y$.

The domain of h is $\mathbb{Z} \times \mathbb{Z}$.;

The codomain is $\mathbb{Z}$.

$h(0, 0) = 2(0) + 3(0) = 0$

$h(-3, 7) = 2(-3) + 3(7) = 15$

$h(2, -1) = 2(2) + 3(-1) = 1$

$\forall\, n \in \mathbb{Z}$, $h(2n, -n) = 2(2n) + 3(-n) = 4n - 3n = n$

$h(\mathbb{Z} \times \mathbb{Z})$ = the range of h = $\mathbb{Z}$.

For $A_1 = \{(0, n) \mid n \in \mathbb{Z}^+\} = \{0\} \times \mathbb{Z}^+ \subseteq \mathbb{Z} \times \mathbb{Z}$,

$h(A_1)$ = the image of A_1 under h = $\{3, 6, 9, \ldots\} = \{3n \mid n \in \mathbb{Z}^+\}$

§ 5.2 Functions : Plain and One-to-One

Thm 5.2 : Let $f : A \to B$, with $A_1, A_2 \subseteq A$. Then

(a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$;

(b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$;

(c) $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ when f is injective.

Proof.

(b) $b \in B$, $b \in f(A_1 \cap A_2) \Rightarrow \exists\, a \in A_1 \cap A_2$ s.t. $b = f(a)$

$\Rightarrow [\exists\, a \in A_1$ s.t. $b = f(a)]$ and $[\exists\, a \in A_2$ s.t. $b = f(a)]$

$\Rightarrow [b \in f(A_1)]$ and $[b \in f(A_2)]$

$\Rightarrow b \in f(A_1) \cap f(A_2)$

$\therefore f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

Def 5.7 : If $f : A \to B$ and $A_1 \subseteq A$, then $f|_{A_1} : A_1 \to B$ is called the ***restriction of f to A_1*** $\equiv f|_{A_1}(a) = f(a), \forall\, a \in A_1$.

§ 5.2 Functions : Plain and One-to-One

Def 5.8 : Let $A_1 \subseteq A$ and $f : A_1 \to B$. If $g : A \to B$ and $g(a) = f(a)\ \forall\ a \in A_1$, then we call g an *extension* of f to A.

EX 5.17 : For $A = \{1, 2, 3, 4, 5\}$, let $f : A \to \mathbb{R}$ be defined by

$f = \{(1, 10), (2, 13), (3, 16), (4, 19), (5, 22)\}$.

Let $g : \mathbb{Q} \to \mathbb{R}$ where $g(q) = 3q + 7.\ \forall\ g \in \mathbb{Q}$

Let $h : \mathbb{R} \to \mathbb{R}$ where $h(r) = 3r + 7.\ \forall\ r \in \mathbb{R}$. then

i) g is an extension of f (from A) to $\mathbb{Q}$,
ii) f is the restriction of g (from Q) to A,
iii) h is an extension of f (from A) to $\mathbb{R}$,
iv) f is the restriction of h (from R) to A,
v) h is an extension of g (from Q) to $\mathbb{R}$,
vi) g is the restriction of h (from R) to $\mathbb{Q}$.

§ 5.2 Functions : Plain and One-to-One

<u>EX 5.18</u> : Let $A = \{w, x, y, z\}$, $B = \{1, 2, 3, 4, 5\}$, $A_1 = \{w, y, z\}$

Let $f : A \rightarrow B$, $g : A_1 \rightarrow B$ by represented by the diagrams :

① f is <u>an</u> extension of g from A_1 to A (有五種)

② $g = f|_{A_1}$

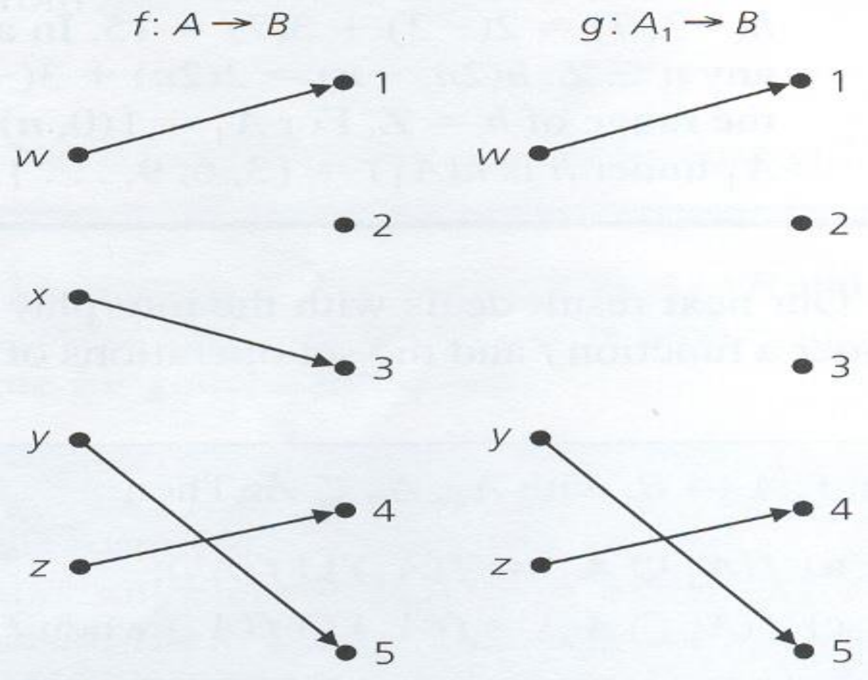

Figure 5.5