Computer Science and Information Engineering
National Chi Nan University

Discrete Mathematics

Dr. Justie Su-Tzu Juan

Chap 3 Set Theory
§ 3.4 A First Word on Probability

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

§ 3.4 A First Word on Probability

<u>Def</u> □① ***experiment*** $\mathscr{E}$ 、 ***sample space*** $\mathscr{S}$ 、 ***event*** A ($\subseteq \mathscr{S}$) 、 ***elementary event*** a ($\in A$). **Let** $|\mathscr{S}| = n$.

② $Pr(a)$ = ***The probability that a occurs*** $= \frac{1}{n} = \frac{|\{a\}|}{|S|}$

$Pr(A)$ = ***The probability that A occurs*** $= \frac{|A|}{n} = \frac{|A|}{|S|}$

<u>Ex3.28</u> ~ <u>Ex3.36</u>□see book.

§ 3.4 A First Word on Probability

<u>Ex3.31</u> □5 cards from a standard deck of 52 cards. $\binom{52}{5} = 2598960$
What is the probability:
(a) Three aces and two jacks; (b) three aces and a pair;
(c) a full house?
Sol.

(a) $\binom{4}{3} = 4$ for aces, $\binom{4}{2} = 6$ for jacks.
Let A = the event where Tanya draws three aces and two jacks.
$\therefore |A| = \binom{4}{3}\binom{4}{2} = 4\cdot 6$; $Pr(A) = 24 / 2598960 \approx 0.000009234$.

(b) $\binom{4}{3} = 4$ for aces, $\binom{12}{1}\binom{4}{2} = 12\cdot 6 = 72$ for a pair.
Let B = the event where Tanya draws three aces and a pair.
$\therefore |B| = \binom{4}{3}\binom{12}{1}\binom{4}{2} = 4\cdot 72$; $Pr(B) = 288 / 2598960 \approx 0.000110814$.

(c) $\binom{13}{1}\binom{4}{3} = 13\cdot 4$ for three something, $\binom{12}{1}\binom{4}{2} = 12\cdot 6 = 72$ for a pair
Let C = the event where Tanya draws a full house.
$\therefore |C| = \binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 13\cdot 288 = 3744$;
$Pr(C) = 3744 / 2598960 \approx 0.001440576$.

§ 3.4 A First Word on Probability

Def □③ ***Cartesian product***, or ***cross product***, of A and B = $A \times B$ $= \{(a, b) \mid a \in A, b \in B\}$.

④ ***ordered pairs*** : the element of $A \times B$. (form : (a, b))

⑤ $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Ex3.32 □$A = \{1, 2, 3\}$ and $B = \{x, y\}$, then

$A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$

$B \times A = \{(x, 1), (y, 1), (x, 2), (y, 2), (x, 3), (y, 3)\}$

$(1, x) \in A \times B$, $(1, x) \notin B \times A$

$|A \times B| = 3 \cdot 2 = 6 = |A|\,|B| = |B|\,|A| = |B \times A|$.

§ 3.4 A First Word on Probability

Ex3.37 □**120 passengers on airline:**

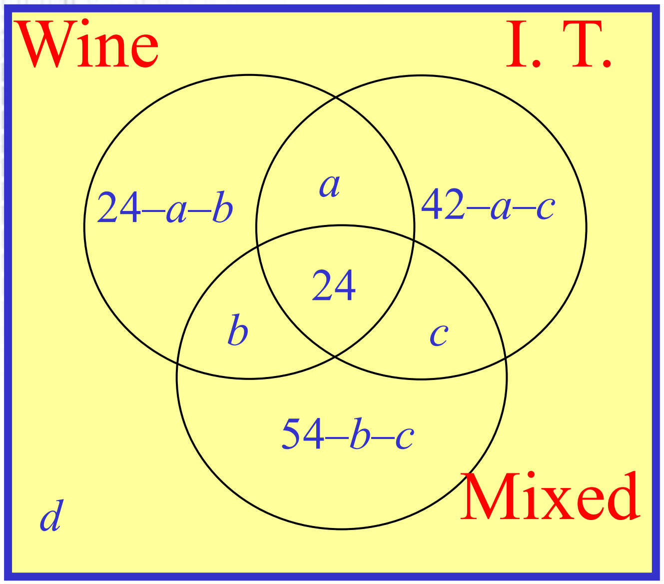
48: wine; 78: mixed drink; 66: iced tea;

36: 2 beverages; 24: 3 beverages.

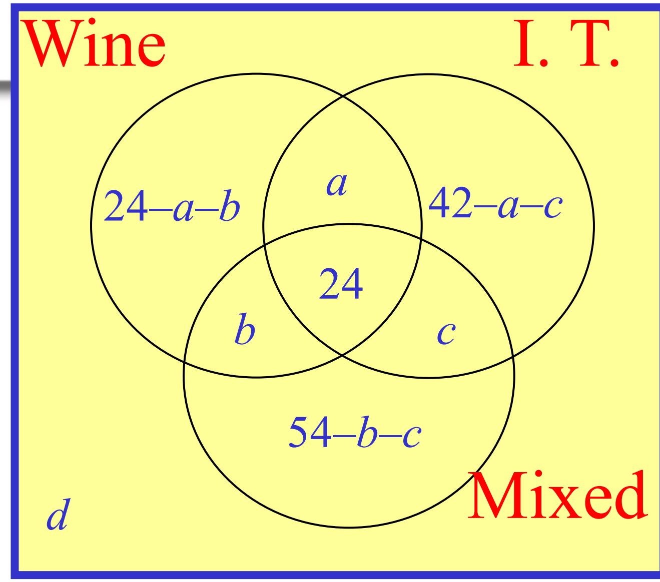
Choose two from 120 passengers: what is the probability that:

a) Event *A* : they both want only iced tea?

b) Event *B* : they both enjoy exactly two of the three beverage offerings?



§ 3.4 A First Word on probability



Sol. (1/2)

$$a + b + c = 36$$
$$24 - a - b = 24 + c - 36 = c - 12 \geq 0$$
$$42 - a - c = 42 + b - 36 = b + 6 \geq 0$$
$$54 - b - c = 54 + a - 36 = a + 18 \geq 0$$

and $120 = (c - 12) + (b + 6) + (a + 18) + a + b + c + 24 + d$
$= 36 \cdot 2 + 12 + 24 + d = 108 + d$

$\therefore d = 12$

(8 unknowns 6 equations $\therefore$ infinite selected)

ex:

let $a = b = 12$, then $c = 12$, $42 - a - c = b + 6 = 18$.

let $a = b = 10$, then $c = 16$, $42 - a - c = b + 6 = 16$.

Sol. (2/2)

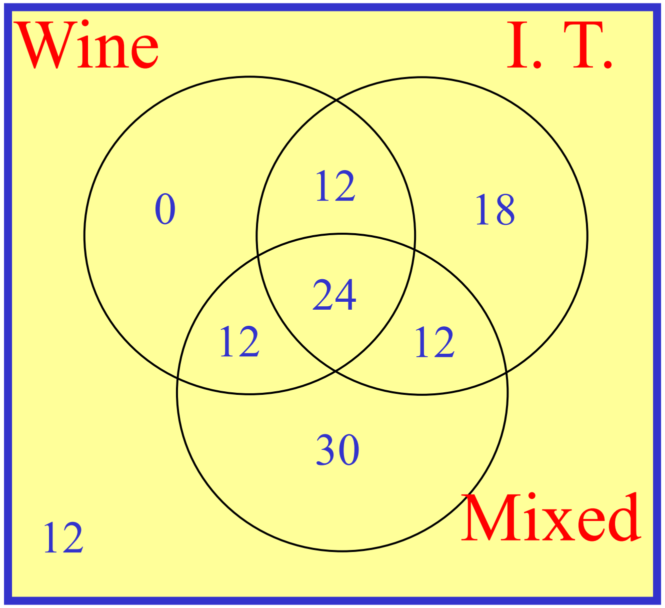
In Book:

$|\mathcal{S}| = \binom{120}{2} = 7140$

$|A| = \binom{18}{2} = 153$

$|B| = \binom{36}{2} = 630$

$\therefore Pr(A) = \frac{51}{2380}, Pr(B) = \frac{3}{34}.$



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Discrete Mathematics

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Chap 4 Properties the Integers: Mathematical Induction

§ 4.1 The Well-Ordering Principle: Mathematical Induction

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§ 4.1 The Well-Ordering Principle: Mathematical Induction

Def □*The Well-Ordering Principle* **: Every nonempty subset of Z^+ contains a smallest element. (Z^+ is *well ordered*)**

Thm 4.1 □***Finite Induction Principle*** **(or *The Principle of Mathematical Induction*):**

Let $S(n)$ denote an open mathematical statement that involves variable $n \in Z^+$.

a) If $S(1)$ is true; and

b) If whenever $S(k)$ is true, then $S(k + 1)$ is true; $k \in Z^+$.

then $S(n)$ is true for all $n \in Z^+$.

§ 4.1 The Well-Ordering Principle: Mathematical Induction

Proof. Let $F = \{t \in Z^+ \mid S(t) \text{ is false}\}$.

If $F \neq \phi$, then by the Well-Ordering Principle,

$\exists\, s \in F$ such that s is the least element of F.

$\because S(1)$ is true, $\therefore 1 \notin F$, $s \neq 1$,

$\Rightarrow s > 1$, $s - 1 \in Z^+$.

$\because s$ is the least element of F, $\therefore s - 1 \notin F$. i.e. $S(s - 1)$ is true.

$\because S(s - 1)$ is true $\Rightarrow S(s)$ is true (by (b))

$\Rightarrow s \notin F$. $\rightarrow\leftarrow$

$\therefore F = \phi$.

<u>Def</u> □(a) "$S(1)$ is true": basis step

(b) "$S(k)$ is true $\Rightarrow S(k + 1)$ is true" : inductive step

"$S(k)$ is true": induction hypothesis (I. H.)

§ 4.1 The Well-Ordering Principle: Mathematical Induction

Remark □① $1 \to n_0 \in \mathbf{Z}$. sub.

② $[S(n_0) \wedge [\forall k \geq n_0\, [S(k) \Rightarrow S(k+1)]]] \Rightarrow \forall n \geq n_0\, S(n)$

Think: Pushing dominoes:

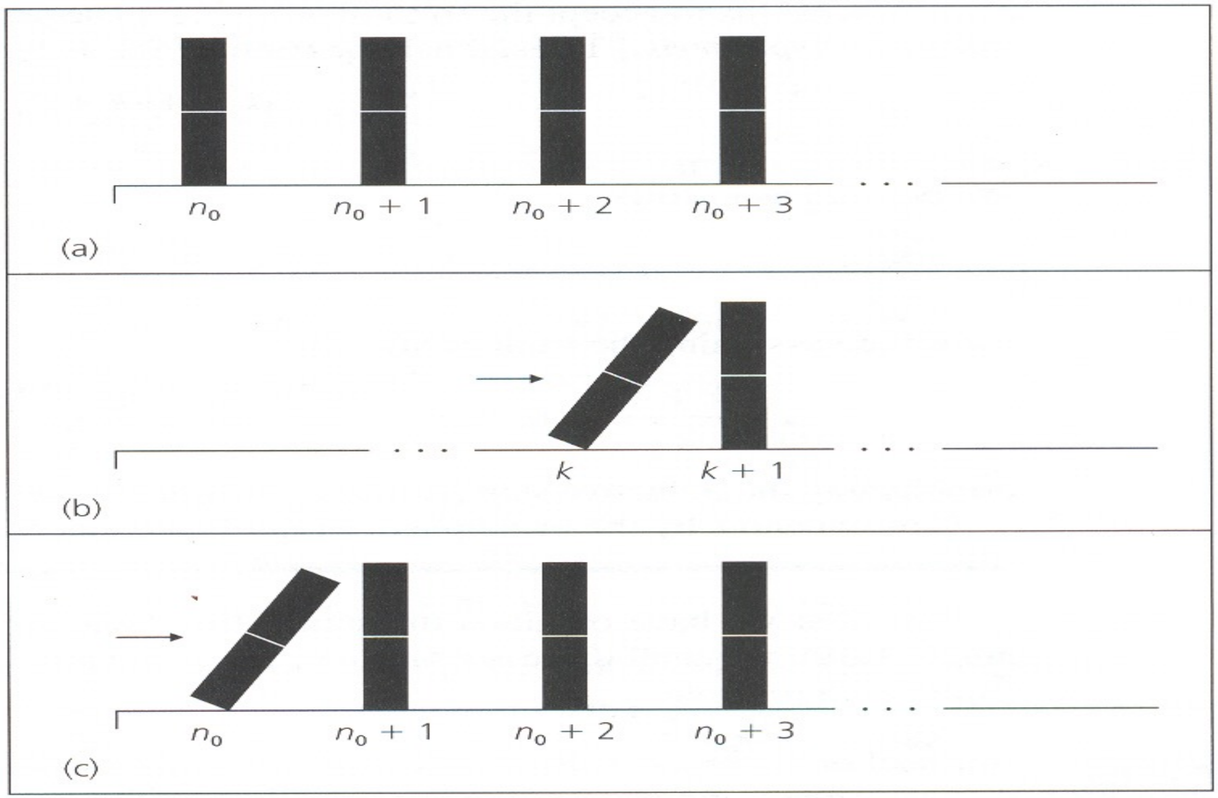

Figure 4.1

§ 4.1 The Well-Ordering Principle: Mathematical Induction

EX 4.1 □$\forall\ n \in Z^+, \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = n(n+1)/2$.

Proof. Let $S(n)$ is $\sum_{i=1}^{n} i = n(n+1)/2$.

① $n = 1$: $S(1)$: $\sum_{i=1}^{1} 1 = 1 = 1 \cdot (1+1)/2$. $\therefore S(1)$ is true.

② Assume $n = k$, $S(n)$ is true for some $k \in Z^+$,

i.e. $S(k)$: $\sum_{i=1}^{k} i = k(k+1)/2$, is true.

Then, when $n = k + 1$,

$$
\begin{aligned}
S(k+1): \sum_{i=1}^{k+1} i &= 1 + 2 + 3 + \ldots + k + (k+1) \\
&= \left(\sum_{i=1}^{k} i\right) + (k+1), \\
\text{(by I. H.)} &= (k(k+1)/2) + (k+1) \\
&= [k(k+1) + 2(k+1)]/2 \\
&= (k+1)(k+2)/2.
\end{aligned}
$$

$\therefore S(k+1)$ is true.

By the Principle of Mathematical Induction,

$S(n)$ is true for all $n \in Z^+$.

§ 4.1 The Well-Ordering Principle: Mathematical Induction

EX 4.2 □A wheel painted by 1 to 36 in a random manner. Show that ∃ 3 consecutive numbers total 55 or more.

Sol. (By contradiction) Assume $x_1, x_2, \ldots, x_{36}$ be the numbers labeled in the wheel clockwise.

For the result to be false:

$$
\left.\begin{array}{rl}
 & x_1 + x_2 + x_3 < 55 \\
 & x_2 + x_3 + x_4 < 55 \\
 & x_3 + x_4 + x_5 < 55 \\
 & \vdots \\
 & x_{34} + x_{35} + x_{36} < 55 \\
 & x_{35} + x_{36} + x_1 < 55 \\
+) & x_{36} + x_1 + x_2 < 55 \\
\hline
\end{array}\right\}
$$

$$3 \sum_{i=1}^{36} x_i < 36 \cdot 55$$

$$
\begin{aligned}
3 \sum_{i=1}^{36} x_i = 3 \sum_{i=1}^{36} i &= 3 \cdot (36 \cdot 37) / 2 \\
&= 3 \cdot 666 \\
&= 1998
\end{aligned}
$$

$$36 \cdot 55 = 1980$$

$$\Rightarrow 1998 < 1980 \quad \rightarrow\leftarrow$$

§ 4.1 The Well-Ordering Principle: Mathematical Induction

EX 4.4 □ $\forall\, n \in Z^+$, $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.

Proof.

Let $S(n) : \sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.

① Basis Step: $S(1) : \sum_{i=1}^{1} i^2 = 1 = 1(1+1)(2+1)/6$. $\therefore S(1)$ is true.

② Inductive Step:

Assume $S(k)$ is true for some $k \in Z^+$,

i.e. $\sum_{i=1}^{k} i^2 = k(k+1)(2k+1)/6$.

Then $S(k+1)$: $\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$

(By *I. H.*) $= k(k+1)(2k+1)/6 + (k+1)^2$

$= (k+1)\,[k(2k+1)/6 + (k+1)]$

$= (k+1)\,(2k^2 + 7k + 6)/6$

$= (k+1)(k+2)(2k+3)/6$, $S(k+1)$ is true

$\therefore$ By Principle of Mathematical Induction, $S(n)$ is true $\forall\, n \in Z^+$.

§ 4.1 The Well-Ordering Principle: Mathematical Induction

EX 4.6 □**Why need to establish the basis step:**
(no matter how easy it may be to verify it!)

ex: Let $S(n) : \sum_{i=1}^{n} i = (n^2 + n + 2) / 2$.

Assume $S(k)$ is true for some $k \in Z^+$,

i.e. $\sum_{i=1}^{k} i = (k^2 + k + 2) / 2$.

The $S(k + 1) : \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k + 1)$

$$\text{(By } I.H.\text{)} = (k^2 + k + 2) / 2 + (k + 1)$$
$$= [k^2 + k + 2+ 2(k +1)] / 2$$
$$= [(k + 1)^2 + (k + 1) + 2] / 2,$$ $S(k + 1)$ is true!

If we can find $S(n_0)$ is true for some $n_0 \in Z^+$,

Then $S(n)$ is true for all $n \geq n_0 \in Z^+$.

But, By Ex4.1, $\sum_{i=1}^{n} i = n(n + 1) / 2$.

$\Rightarrow$ $n(n + 1) / 2 = \sum_{i=1}^{n} i = (n^2 + n + 1) / 2$.

$\Rightarrow$ $0 = 1$ $\rightarrow\leftarrow$!!

§ 4.1 The Well-Ordering Principle: Mathematical Induction

Note □See Fig. 4.2, using ***n*** "+", ***n*** "×", v.s.
Fig. 4.3, using 2 "+", 3 "×".

```
procedure SumOfSquares1 (n: positive integer)
begin
  sum := 0
  for i := 1 to n do
    sum := sum + i²
end
```

Figure 4.2

```
procedure SumOfSquares2 (n: positive integer)
begin
  sum := n*(n + 1)*(2*n + 1)/6
end
```

Figure 4.3

§ 4.1 The Well-Ordering Principle: Mathematical Induction

Ex 4.7 □[非已知公式] Consider the sum of consecutive odd positive integers:

1) $1 = 1 = 1^2$
2) $1 + 3 = 4 = 2^2$
3) $1 + 3 + 5 = 9 = 3^2$
4) $1 + 3 + 5 + 7 = 16 = 4^2$

∴ We conjecture : $S(n)$: $\sum_{i=1}^{n} (2i - 1) = n^2$ is true.

Proof it :

① $S(1)$, $S(2)$, $S(3)$, $S(4)$ are true.

② Assume $S(k)$ is true,

i.e. $\sum_{i=1}^{k} (2i - 1) = k^2$.

Then, $S(k + 1)$: $\sum_{i=1}^{k+1} (2i - 1) = \sum_{i=1}^{k} (2i - 1) + (2k + 1)$

(By *I.H.*) $= k^2 + 2k + 1 = (k + 1)^2$.

∴ $S(k + 1)$ is true.

By Principle of Mathematical Induction,

$S(n)$ is true for all $n \in Z^+$.

§ 4.1 The Well-Ordering Principle: Mathematical Induction

Ex 4.8 □[非Σ]

Table 4.1

n	$4n$	$n^2 - 7$	n	$4n$	$n^2 - 7$
1	4	−6	5	20	18
2	8	−3	6	24	29
3	12	2	7	28	42
4	16	9	8	32	57

Conjecture $S(n)$: $\forall\ n \geq 6,\ 4n < (n^2 - 7)$.

Proof.

① $S(6)$ is true by above table.

② Assume $S(k)$ is true for some integer $k \geq 6$, i.e. $4k < k^2 - 7$.

Consider $n = k + 1$: $4(k + 1) = 4k + 4 < (k^2 - 7) + 4$ (by *I.H.*)

$\because \forall\ k \geq 6,\ 2k + 1 \geq 13 > 4$

$\therefore 4(k + 1) < (k^2 - 7) + 4 < (k^2 - 7) + 2k + 1$

$\Rightarrow 4(k + 1) < (k^2 + 2k + 1) - 7 = (k + 1)^2 - 7.$

$\therefore S(k + 1)$ is true.

By Principle of Mathematical Induction, $S(n)$ is true $\forall\ n \geq 6$.

§ 4.1 The Well-Ordering Principle: Mathematical Induction

EX 4.9 □*Harmonic number* : $H_n = 1 + 1/2 + 1/3 + \ldots + 1/n$. $\forall\, n \in Z^+$.

Property: $\forall\, n \in Z^+$, $\Sigma_{j=1}^{n} H_j = (n+1)H_n - n$.

Proof.

Let $S(n) : \Sigma_{j=1}^{n} H_j = (n+1)H_n - n$.

① $n = 1 : \Sigma_{j=1}^{1} H_1 = H_1 = 1 = 2 \cdot 1 - 1 = (1+1) \cdot H_1 - 1$.

$\therefore S(1)$ is true.

② Assume $S(k)$ is true for some $k \in Z^+$,

i.e. $\Sigma_{j=1}^{k} H_j = (k+1)H_k - k$.

Then, consider $n = k + 1$:

$\Sigma_{j=1}^{k+1} H_j = \Sigma_{j=1}^{k} H_j + H_{k+1} = [(k+1)H_k - k] + H_{k+1}$ (by *I.H.*)

$= (k+1)[H_{k+1} - 1/(k+1)] - k + H_{k+1}$

$= (k+2)H_{k+1} - 1 - k$

$= [(k+1)+1]H_{k+1} - (k+1)$ $\quad\therefore S(k+1)$ is true.

By the Principle of Mathematical Induction, $S(n)$ is true $\forall n \in Z^+$.

§ 4.1 The Well-Ordering Principle: Mathematical Induction

EX 4.10 □[Binary Search] $\forall\ n \geq 0$, let $A_n \subset \mathbb{R}$, where $|A_n| = 2^n$. And the elements of A_n are listed in ascending order. $r \in \mathbb{R}$, prove that: "determine $r \in A_n$ or not must compare $\leq n + 1$ elements in A_n".

Proof. (1/2)

Let $S(n)$="determine $r \in A_n$ or not, compare $\leq n+1$ elements in A_n".

① $n = 0$: $A_0 = \{a\}$, and only 1 comparison is needed. $1 = 0 + 1$,
$\therefore S(0)$ is true.

② Assume $S(k)$ is true. For some $k \geq 0$, consider $n = k + 1$:
$A_{k+1} \subset \mathbb{R}$ where $|A_{k+1}| = 2^{k+1}$.
Let $A_{k+1} = B_k \cup C_k$, where $|B_k| = |C_k| = 2^k$ and the element of B_k, C_k are in ascending order with
$\forall\ b \in B_k, \forall\ c \in C_k, b < c$

Now: a) First we compare r and x = the largest element x in B_k.
b) If $r \leq x$, then $r \notin C_k$.
c) If $r > x$, then $r \notin B_k$.

Proof. (2/2)

Both (*b*), (*c*) imply, ∵ $|B_k| = |C_k| = 2^k$, by *I. H.*
we can determine where $r \in B_k$ (C_k) by making $\le k + 1$ additional comparisons.
∴ at most $(k + 1) + 1$ comparisons are made, i.e. $S(k+1)$ is true.
By the P. of Math. Induction, the general result follows.

<u>EX 4.11</u> □*Program verification.* (略)

```
while n ≠ 0 do
  begin
    x := x*y
    n := n − 1
  end
answer := x
```

Figure 4.4

$S(n)$ = ∀ $x, y \in$ R, if the program reaches the top of the while loop with $n \in$ N, after the loop is bypassed (for $n = 0$) or the two loop instructions are executed n (> 0) times, then the value of the real variable *answer* is $x(y^n)$.

§ 4.1 The Well-Ordering Principle: Mathematical Induction

Sol.

$S(n) = \forall\, x, y \in \mathrm{R}$, if the program reaches the top of the while loop with $n \in \mathrm{N}$, after the loop is bypassed (for $n = 0$) or the two loop instructions are executed n (> 0) times, then the value of the real variable *answer* is $x(y^n)$.

$S(0)$ is true. Since when $n = 0$, *answer* $= x = x(1) = x\,(y^0)$.

$S(k)$ is true $\Rightarrow S(k + 1)$ is true. 利用 "x_1"!

```
while n ≠ 0 do
  begin
    x := x*y
    n := n − 1
  end
answer := x
```

Figure 4.4

已知:

1. The value of y is unchanged
2. The value of x is $x_1 = x(y^1) = xy$.
3. The value on n is $(k + 1) - 1 = k$.

By *I.H.*, after the while loop for x_1, y and $n = k$ is bypassed (for $k = 0$) or two loop instructions are executed k (> 0) times, then *answer* =

$$x_1(y^k) = (xy)(y^k) = x(y^{k+1}).$$

EX 4.13 □$S(n)$: n can be written as a sum of 3's and/or 8's.
Prove $S(n)$ is true for all $n \geq 14$.

Proof.

① $n = 14$: $14 = 3 + 3 + 8$, $S(14)$ is true.

② Assume $S(k)$ is true. For some $k \geq 14$,
i.e. $\exists\, a, b \in Z^{+} \cup \{0\}$ such that $k = a \cdot 3 + b \cdot 8$.

Consider $n = k + 1$:

By *I. H.*: $k + 1 = a \cdot 3 + b \cdot 8 + 1$ for some $a, b \in Z^{+} \cup \{0\}$.

Case 1: if $b \neq 0$: then $k + 1 = a \cdot 3 + (b - 1)8 + 9$
$= (a + 3) \cdot 3 + (b - 1) \cdot 8$.

Case 2: if $b = 0$: i.e. $k + 1 = a \cdot 3 + 1$ for some $a \in Z^{+}$.
$\because k \geq 14 \therefore a \geq 5$, i.e. $(a - 5) \in Z^{+} \cup \{0\}$.
$\therefore k + 1 = (a - 5) \cdot 3 + 2 \cdot 8$.

By Case 1 & 2, $S(k + 1)$ is true.

By the Principle of Mathematical Induction, $S(n)$ is true for all $n \geq 14$.

§ 4.1 The Well-Ordering Principle: Mathematical Induction

Thm 4.2 □*Principle of Strong Mathematical Induction* :

(*Finite Induction Principle – Alternative Form*):

Let $S(n)$ denote an open mathematical statement that involves the variable $n \in Z^+$. Let $n_0, n_1 \in Z^+$ with $n_0 \leq n_1$,

basis step a) If $S(n_0), S(n_0 + 1), \ldots, S(n_1 - 1), S(n_1)$ are true; and

inductive step b) If whenever $S(n_0), S(n_0 + 1), \ldots, S(k - 1), S(k)$ are true for some $k \in Z^+$, where $k \geq n_1$, then $S(k + 1)$ is also true.

then $S(n)$ is true $\forall\ n \geq n_0$.

Remark □As **Thm 4.1**, n_0 need not actually be a positive integer. It may be 0 or negative integer.

§ 4.1 The Well-Ordering Principle: Mathematical Induction

EX 4.14 □As **EX 4.13**, $\forall\, n \in Z^+$ where $n \geq 14$,

$S(n)$: n can be written as a sum of 3's and/or 8's.

Proof.

① $\because$ $14 = 3 + 3 + 8$; $15 = 3 + 3 + 3 + 3 + 3$; $16 = 8 + 8$.

$\therefore$ $S(14)$, $S(15)$, $S(16)$ are true. ($n_0 = 14$, $n_1 = 16$)

② Assume $S(14)$, $S(15)$, … , $S(k-1)$, $S(k)$ are true for some $k \in Z^+$ with $k \geq 16$.

Now if $n = k + 1$, then $n \geq 17$ and $k + 1 = (k - 2) + 3$.

$\because$ $n_0 = 14 \leq k - 2 \leq k$, $\therefore$ $S(k - 2)$ is true. (by I.H.)

i.e. $(k - 2)$ can be written as a sum of 3's and/or 8's;

so $k + 1 = (k - 2) + 3$ can also be written in this form.

$\therefore$ S(k+1) is true.

$\therefore$ $S(n)$ is true for all $n \geq 14$ by the Principle of Strong Mathematical Induction .

§ 4.1 The Well-Ordering Principle: Mathematical Induction

Ex 4.15 □[Using more than one prior result] Let $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ $\forall\ n \in \mathbf{Z}^+$ where $n \geq 3$.

i.e. $a_3 = a_2 + a_1 + a_0 = 3 + 2 + 1 = 6$

$a_4 = a_3 + a_2 + a_1 = 6 + 3 + 2 = 11$

$a_5 = a_4 + a_3 + a_2 = 11 + 6 + 3 = 20$

Prove: $a_n \leq 3^n$ $\forall\ n \in \mathbf{N}$.

Proof. (1/2)

Let $S'(n)$: $a_n \leq 3^n$ $\forall\ n \in \mathbf{N}$.

① i) $a_0 = 1 = 3^0 \leq 3^0$ ii) $a_1 = 2 \leq 3 = 3^1$ iii) $a_2 = 3 \leq 9 = 3^2$.

$\therefore$ $S'(0)$, $S'(1)$, $S'(2)$ are true.

② Assume $S'(0)$, $S'(1)$, $S'(2)$, … , $S'(k-1)$, $S'(k)$ are true for some $k \in \mathbf{Z}^+$ where $k \geq 2$.

§ 4.1 The Well-Ordering Principle: Mathematical Induction

Proof. (2/2)

For $n = k + 1 \geq 3$, $a_{k+1} = a_k + a_{k-1} + a_{k-2}$

$$\leq 3^k + 3^{k-1} + 3^{k-2}$$

$$\leq 3^k + 3^k + 3^k = 3(3^k) = 3^{k+1}.$$

$\therefore [S'(k-2) \wedge S'(k-1) \wedge S'(k)] \Rightarrow S'(k+1).$

By the Principle of Strong Mathematical Induction ,

$a_n \leq 3^n \ \forall \ n \in \mathrm{N}.$

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Chap 4 Properties the Integers: Mathematical Induction

§ 4.2 Recursive Definitions

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§ 4.2 Recursive Definitions

Def □① ***explicit*** **formula. ex :** $b_n = 2n\ \forall\ n \in \mathrm{N}$.

② ***recursive definition***.

ex : $\begin{cases} a_n = a_{n-1} + a_{n-2} + a_{n-3},\ \forall\ n \in \mathrm{Z}^+, n \geq 3. \\ a_0 = 1, a_1 = 2, a_2 = 3. \end{cases}$

ex : 比較 : $b_6 = 2 \cdot 6 = 12$

$a_6 = a_5 + a_4 + a_3$

$= [(a_4 + a_3 + a_2)+(a_3 + a_2 + a_1)+(a_2 + a_1 + a_0)]$

$= \ldots$

③ ***a basis for the recursion***. ex : $a_0 = 1, a_1 = 2, a_2 = 3$

the recursive process.

ex : $a_n = a_{n-1} + a_{n-2} + a_{n-3}\ \forall\ n \in \mathrm{Z}^+, n \geq 3$.

§ 4.2 Recursive Definitions

EX4.16 □Given any statements $p_1, p_2, \ldots, p_n, p_{n+1}$, we define

1) the **conjunction** of p_1, p_2 by $p_1 \wedge p_2$, and
2) the conjunction of $p_1, p_2, \ldots, p_n, p_{n+1}$ for $n \geq 2$ by

$$p_1 \wedge p_2 \wedge \ldots \wedge p_n \wedge p_{n+1} \Leftrightarrow (p_1 \wedge p_2 \wedge \ldots \wedge p_n) \wedge p_{n+1}$$

ex : Let $n \in Z^+$ where $n \geq 3$, let $r \in Z^+$ with $1 \leq r < n$. Then

$S(n)$: For any statements $p_1, p_2, \ldots, p_r, p_{r+1}, \ldots, p_n$,

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_r) \wedge (p_{r+1} \wedge \ldots \wedge p_n) \Leftrightarrow p_1 \wedge p_2 \wedge \ldots \wedge p_r \wedge p_{r+1} \wedge \ldots \wedge p_n.$$

Proof. (1/2)

① $S(3)$ is hold by the associative law of $\wedge$.

② Assume $S(k)$ is true for $k \geq 3$ and all $1 \leq r < k$,

Now, Consider $S(k + 1)$:

§ 4.2 Recursive Definitions

Proof. (2/2)

case 1. If $r = k$, then

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_k) \wedge p_{k+1} \Leftrightarrow p_1 \wedge p_2 \wedge \ldots \wedge p_k \wedge p_{k+1}$$

is true from our recursive definition.

case 2. For $1 \le r < k$, we have

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_r) \wedge (p_{r+1} \wedge \ldots \wedge p_k \wedge p_{k+1})$$

$$\Leftrightarrow (p_1 \wedge p_2 \wedge \ldots \wedge p_r) \wedge [(p_{r+1} \wedge \ldots \wedge p_k) \wedge p_{k+1}]$$

$$\Leftrightarrow [(p_1 \wedge p_2 \wedge \ldots \wedge p_r) \wedge (p_{r+1} \wedge \ldots \wedge p_k)] \wedge p_{k+1}$$

(by *I. H.*) $\Leftrightarrow (p_1 \wedge p_2 \wedge \ldots \wedge p_r \wedge p_{r+1} \wedge \ldots \wedge p_k) \wedge p_{k+1}$

$$\Leftrightarrow p_1 \wedge p_2 \wedge \ldots \wedge p_{k+1}.$$

∴ by the Principle of Mathematical Induction,
$S(n)$ is true for all $n \in Z^+$ where $n \ge 3$.

§ 4.2 Recursive Definitions

EX4.17 □[∪]

Consider $A_1, A_2, \ldots, A_{n+1}$, where $A_i \subseteq \mathcal{U}\ \forall\ 1 \le i \le n+1$, we define their **union** **recursively:**

1) The union of A_1, A_2 is $A_1 \cup A_2$.

2) The union of $A_1, A_2, \ldots A_n, A_{n+1}$, for $n \ge 2$ is

$$A_1 \cup A_2 \cup \ldots \cup A_n \cup A_{n+1} = (A_1 \cup A_2 \cup \ldots \cup A_n) \cup A_{n+1}.$$

ex □"Generalized Associative Law for ∪":

If $n, r \in Z^+$, with $n \ge 3$ and $1 \le r < n$, then

$$S(n) = (A_1 \cup A_2 \cup \ldots \cup A_r) \cup (A_{r+1} \cup \ldots \cup A_n)$$
$$= A_1 \cup \ldots \cup A_r \cup A_{r+1} \cup \ldots \cup A_n.$$

Where $A_i \subseteq \mathcal{U}$ for all $1 \le i \le n$.

Proof.

① S(3) is true from the associative law of ∪.

② Assuming the truth of $S(k)$ for some $k \in Z^+$, where $k \geq 3$ and $1 \leq r < k$.

Now consider $n = k + 1$:

<u>case 1</u>. $r = k$:

$$(A_1 \cup A_2 \cup \ldots \cup A_k) \cup A_{k+1} = A_1 \cup A_2 \cup \ldots \cup A_k \cup A_{k+1}$$

∵ The given recursive definition.

<u>case 2</u>. $1 \leq r < k$:

$$(A_1 \cup A_2 \cup \ldots \cup A_r) \cup (A_{r+1} \cup \ldots \cup A_k \cup A_{k+1})$$
$$= (A_1 \cup A_2 \cup \ldots \cup A_r) \cup [(A_{r+1} \cup \ldots \cup A_k) \cup A_{k+1}]$$
$$= [(A_1 \cup \ldots \cup A_r) \cup (A_{r+1} \cup \ldots \cup A_k)] \cup A_{k+1}$$
(by *I. H.*) $= (A_1 \cup \ldots \cup A_r \cup A_{r+1} \cup \ldots \cup A_k) \cup A_{k+1}$
$$= A_1 \cup \ldots \cup A_r \cup A_{r+1} \cup \ldots \cup A_k \cup A_{k+1}$$

∴ By the Principle of Mathematical Induction, $S(n)$ is true for all integer $n \geq 3$.

§ 4.2 Recursive Definitions

Note □[∩] **Consider** $A_1, A_2, \ldots, A_{n+1}$, **where**

$$A_i \subseteq \mathcal{U} \ \forall \ 1 \leq i \leq n+1,$$

we define their intersection recursively:

1) The intersection of A_1, A_2 **is** $A_1 \cap A_2$.

2) For $n \geq 2$, **the intersection of** $A_1, A_2, \ldots, A_n, A_{n+1}$ **is**

$$A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1} = (A_1 \cap A_2 \cap \ldots \cap A_n) \cap A_{n+1}.$$

§ 4.2 Recursive Definitions

EX4.18 □Let $n \in Z^+$ Where $n \geq 2$, and let $A_1, A_2, \dots, A_n, \subseteq \mathcal{U}$

then $\overline{A_1 \cap A_2 \cap \dots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \dots \cup \overline{A_n}$

Proof.

Let $S(n) = \overline{A_1 \cap A_2 \cap \dots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \dots \cup \overline{A_n}$, $n \in Z^+$.

① $n = 2$, $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$, ∵ the second of DeMorgan's Laws.

② Assume for some $n = k$, where $k \geq 2$:

$$\overline{A_1 \cap A_2 \cap \dots \cap A_k} = \overline{A_1} \cup \overline{A_2} \cup \dots \cup \overline{A_k}$$

Now consider $n = k + 1$ (≥ 3):

$$\overline{A_1 \cap A_2 \cap \dots \cap A_k \cap A_{k+1}} = \overline{(A_1 \cap A_2 \cap \dots \cap A_k) \cap A_{k+1}}$$

$$= \overline{(A_1 \cap A_2 \cap \dots \cap A_k)} \cup \overline{A_{k+1}} = (\overline{A_1} \cup \overline{A_2} \cup \dots \cup \overline{A_k}) \cup \overline{A_{k+1}}$$

$$= \overline{A_1} \cup \overline{A_2} \cup \dots \cup \overline{A_k} \cup \overline{A_{k+1}} \qquad \text{(by } I.\ H.\text{)}$$

∴ By the Principle of Mathematical Induction,

The generalized DeMorgan Law for $n \geq 2$ obtained.

§ 4.2 Recursive Definitions

Remark : +, ·亦可如此定義。事實上，之前已用過了(**EX4.1**, **EX4.3**) 但之後將可清楚定義。

ex : ① Define the sequence of harmonic numbers $H_1, H_2, \ldots,$ by

1) $H_1 = 1$; and
2) $\forall\, n \geq 1,\ H_{n+1} = H_n + \left(\frac{1}{n \square 1}\right)$

② Define $n!$ by

1) $0! = 1$; and
2) $\forall\, n \geq 0,\ (n+1)! = (n+1) \cdot n!$

③ The sequence $b_n = 2n,\ n \in \mathrm{N}$ can be defined recursively by

1) $b_0 = 0$; and
2) $\forall\, n \geq 0,\ b_{n+1} = b_n + 2$

§ 4.2 Recursive Definitions

EX4.19 □*The Fibonacci numbers* may be defined recursively by

1) $F_0 = 0,\ F_1 = 1$; and
2) $F_n = F_{n-1} + F_{n-2}$, for $n \in Z^+$ with $n \geq 2$.

$$F_2 = F_1 + F_0 = 1 + 0 = 1$$
$$F_3 = F_2 + F_1 = 1 + 1 = 2$$
$$F_4 = F_3 + F_2 = 2 + 1 = 3$$
$$F_5 = F_4 + F_3 = 3 + 2 = 5$$

觀察:

$$F_0^2 + F_1^2 + F_2^2 + F_3^2 + F_4^2$$
$$= 0^2 + 1^2 + 1^2 + 2^2 + 3^2 = 15 = 3 \cdot 5$$
$$F_0^2 + F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2$$
$$= 0^2 + 1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 40 = 5 \cdot 8$$

§ 4.2 Recursive Definitions

ex : $\forall\, n \in Z^+$, $\Sigma_{i=0,\,n} F_i^2 = F_n \cdot F_{n+1}$

Proof.

① For $n = 1$, $\Sigma_{i=0,\,1} F_i^2 = F_0^2 + F_1^2 = 0^2 + 1^2 = 1 = 1 \cdot 1 = F_1 \cdot F_2$

The conjecture is true.

② Assume $n = k$, $\Sigma_{i=0,\,k} F_i^2 = F_k \cdot F_{k+1}$.

Now, consider $n = k + 1$ (≥ 2):

$$\Sigma_{i=0,\,k+1} F_i^2 = \Sigma_{i=0,\,k} F_i^2 + F_{k+1}^2 = (F_k \cdot F_{k+1}) + F_{k+1}^2 \quad \text{(by } I.\ H.\text{)}$$

$$= F_{k+1} \cdot (F_k + F_{k+1}) = F_{k+1} \cdot F_{k+2}$$

∴ The truth of the case for $n = k + 1$ follows from the case for $n = k$.

By the Principle of Mathematical Induction, the given conjecture is true for all $n \in Z^+$.

§ 4.2 Recursive Definitions

<u>EX4.20</u> □***Lucas numbers*****: defined recursively by**

1) $L_0 = 2$, $L_1 = 1$; **and**

2) $L_n = L_{n-1} + L_{n-2}$, **for** $n \in Z^+$ **with** $n \geq 2$.

2, 1, 3, 4, 7, 11, 18, 29, …

<u>ex :</u> $\forall\, n \in Z^+, L_n = F_{n-1} + F_{n+1}$

Proof.(1/2)

① when $n = 1$ **and** $n = 2$ **:**

$L_1 = 1 = 0 + 1 = F_0 + F_2 = F_{1-1} + F_{1+1}$, **and**

$L_2 = 3 = 1 + 2 = F_1 + F_3 = F_{2-1} + F_{2+1}$,

$\therefore$ **The result is true for** $n = 1$ **and** $n = 2$.

Proof.(2/2)

② Assume $L_n = F_{n-1} + F_{n+1}$

$\forall\, n = 1, 2, \ldots, k-1, k$, where $k \geq 2$

and then consider L_{k+1} :

$$L_{k+1} = L_k + L_{k-1} = (F_{k-1} + F_{k+1}) + (F_{k-2} + F_k) \text{ (by } I.\ H.\text{)}$$

$$= (F_{k-1} + F_{k-2}) + (F_{k+1} + F_k)$$

$$= F_k + F_{k+2} = F_{(k+1)-1} + F_{(k+1)+1}$$

∴ By the Principle of Strong Mathematical Induction,

$L_n = F_{n-1} + F_{n+1}\ \forall\, n \in Z^+$.

§ 4.2 Recursive Definitions

EX4.21 □① **Define the binomial coefficients recursively by :**

$$\begin{cases} \binom{0}{0} = 1;\ \binom{n}{r} = 0, & \text{if } r < 0 \text{ or } r > n; \\ \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}, & \text{if } n \ge r \ge 0 \end{cases}$$

② **For $m \in Z^+$, $k \in N$, the *Eulerian number* $a_{m,k}$ are defined recursively by**

$$\begin{cases} a_{0,0} = 1;\ a_{m,k} = 0, \text{ if } k < 0 \text{ or } k \ge m; \\ a_{m,k} = (m-k)a_{m-1,k-1} + (k+1)a_{m-1,k}, \text{ if } 0 \le k \le m-1. \end{cases}$$

						Row Sum
$(m = 1)$	$a_{1,0}$ 1					$1 = 1!$
$(m = 2)$	$a_{2,0}$ 1	$a_{2,1}$ 1				$2 = 2!$
$(m = 3)$	$a_{3,0}$ 1	4	1			$6 = 3!$
$(m = 4)$	$a_{4,0}$ 1	11	11	1		$24 = 4!$
$(m = 5)$	$a_{5,0}$ 1	26	66	$a_{5,3}$ 26	1	$120 = 5!$

§ 4.2 Recursive Definitions

Conjecture : $\Sigma_{k=0}^{m-1} a_{m,k} = m! \ \forall \ m \in Z^+$

Proof.

① For $1 \le m \le 5$, it's true.

② Assume the result is true for some fixed m (≥ 1)

Now, consider $m + 1$:

$\Sigma_{k=0}^{m} a_{m+1,k} = \Sigma_{k=0}^{m} [(m-k+1)a_{m,k-1} + (k+1)a_{m,k}]$

$= [(m+1)a_{m,-1} + a_{m,0}] + [m\, a_{m,0} + 2a_{m,1}] +$

$[(m-1)a_{m,1} + 3a_{m,2}] + \ldots + [3a_{m,m-3} + (m-1)a_{m,m-2}] +$

$[2a_{m,m-2} + m\, a_{m,m-1}] + [a_{m,m-1} + (m+1)\, a_{m,m}]$

$\because a_{m,-1} = 0 = a_{m,m}$

$\therefore \Sigma_{k=0}^{m} a_{m+1,k} = [a_{m,0} + m\, a_{m,0}] + [2a_{m,1} + (m-1)a_{m,1}]$

$+ \ldots + [(m-1)a_{m,m-2} + 2a_{m,m-2}] + [m\, a_{m,m-1} + a_{m,m-1}]$

$= (m+1)\ \Sigma_{k=0}^{m-1} a_{m,k} = (m+1)\ m! = (m+1)!$ (by *I. H.*)

$\therefore$ the result is true for all $m \ge 1$ by the Principle of Math. Ind.

§ 4.2 Recursive Definitions

EX4.22 □[implicit] Define the set X recursively by

1) $1 \in X$; and
2) For each $a \in X$, $a + 2 \in X$

Claim that X consists (precisely) of all positive odd integers

Proof.(1/2)

Let $Y = \{2n + 1 \mid n \in \mathbb{N}\}$.

Claim : $X = Y$ (i.e. $X \subseteq Y$ and $Y \subseteq X$)

Proof.

① $Y \subseteq X$: $\forall\, a \in Y \Rightarrow a = 2n + 1$ for some n ($\leadsto a \in X$)

let $S(n) : 2n + 1 \in X, \forall\, n \in \mathbb{N}$.

i) $S(0) : 2 \cdot 0 + 1 = 1 \in X$ is true.

ii) Assume $S(k)$ is true for some $k \geq 0$,

i.e. $2k + 1$ is an element in X.

§ 4.2 Recursive Definitions

Proof.(2/2)

By (2) of the recursive definition of X ;

$$(2k + 1) + 2 = 2(k + 1) + 1 \in X$$

$\therefore$ $S(k + 1)$ is also true.

$\therefore$ $S(n)$ is true by the Principle of Mathematical Induction for all $n \in$ N.

② $X \subseteq Y$: (1) : $1 = 2 \cdot 0 + 1 \in Y$.

(2) : If $b \in X$ and $b \in Y$ is true,

then there exist some $k \geq 0$, s.t. $b = 2k + 1$.

Consider $b + 2 \in X$,

$$b + 2 = (2k + 1) + 2 = 2(k + 1) + 1 \in Y$$

$\therefore$ $b \in Y$ by the Principle of Mathematical Induction for all $b \in X$. So, $X \subseteq Y$.

$\therefore$ By ①, ② $X \subseteq Y$ and $Y \subseteq X \Rightarrow X = Y$.