Computer Science and Information Engineering
National Chi Nan University

Discrete Mathematics

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Chap 3 Set Theory

§ 3.1 Sets and Subsets (2)

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

EX 3.7 : ① Determine the number of subsets of the set $C = \{1, 2, 3, 4\}$.

$2 \times 2 \times 2 \times 2 = 2^4 = 16$ (include ϕ and C)

② Determine the number of subsets of two elements from C.

$C(4, 2) = 6$

③ $\therefore 2^4 = C_0^4 + C_1^4 + C_2^4 + C_3^4 + C_4^4 = \sum_{k=0,4} C(4, k)$

Def : The subset of one element ≡ the ***singleton*** subset.

Def 3.4 : The ***power set*** of A, denoted by $\mathscr{P}(A)$ (or 2^A) ≡ The collection of all subsets of A.

§ 3.1 Sets and Subsets

EX 3.8 : $C = \{1, 2, 3, 4\}$

$\mathscr{P}(C) = \{\phi,$
$\{1\}, \{2\}, \{3\}, \{4\},$
$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\},$
$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\},$
$C\}$

Remark : For any finite set A with $|A| = n$, $n \geq 0$

① $|\mathscr{P}(A)| = 2^n$

② $\forall\ 0 \leq k \leq n$, there are $C(n, k)$ subsets of size k.

③ $2^n = \sum_{k=0}^{n} C(n, k)$

EX 3.9 : Gray Code (略)

0	ϕ
1	$\{x\}$

0	0	ϕ
1	0	$\{x\}$
1	1	$\{x, y\}$
0	1	$\{y\}$

0 0	0	ϕ
1 0	0	$\{x\}$
1 1	0	$\{x, y\}$
0 1	0	$\{y\}$
0 1	1	$\{y, z\}$
1 1	1	$\{x, y, z\}$
1 0	1	$\{x, z\}$
0 0	1	$\{z\}$

0 0 0
1 0 0
1 1 0
0 1 0
0 1 1
1 1 1
1 0 1
0 0 1

0 0 0
0 1 0
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0 0 1
1 0 1
1 1 1
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1 0 0

0 0 0
0 0 1
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1 1 0
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0 1 1
1 1 1

§ 3.1 Sets and Subsets

<u>EX 3.10</u> :

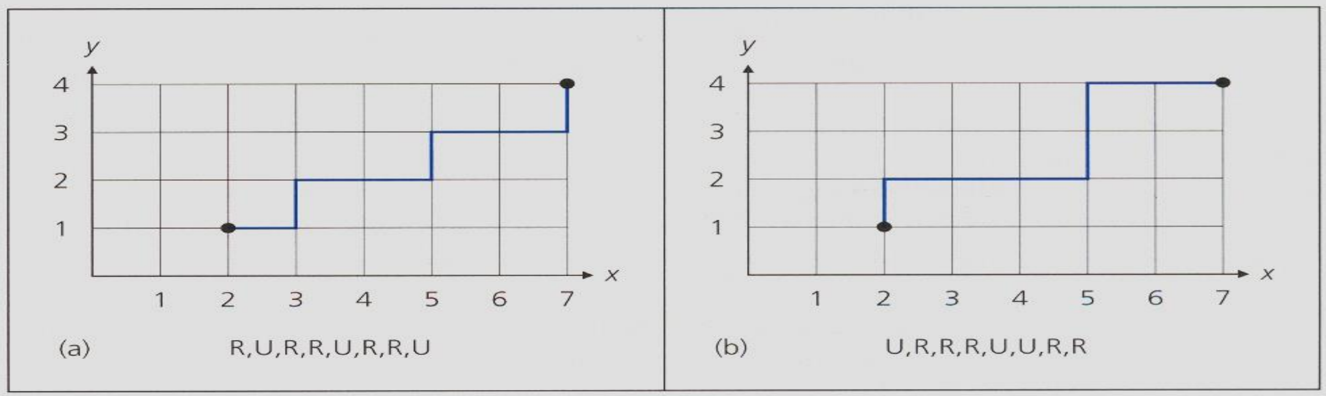

Figure 3.1

(a) R, U, R, R, U, R, R, U ⇨ {2, 5, 8} from {1, 2, 3, 4, 5, 6, 7, 8}

(b) U, R, R, R, U, U, R, R ⇨ {1, 5, 6} from {1, 2, 3, 4, 5, 6, 7, 8}

(c) U, R, U, R, R, R, U, R ⇦ {1, 3, 7} from {1, 2, 3, 4, 5, 6, 7, 8}

The number of paths equals the number of subsets *A* of {1, 2, 3, 4, 5, 6, 7, 8}, where |*A*| = 3.

$= C(8, 3) = \frac{8!}{3!5!} = 56$

(“U” 改 “R” ⇨ |*B*| = 5 ⇨ $C(8, 5) = \frac{8!}{5!3!} = 56$ **)**

EX 3.11 : There are 2^6 ways to write 7 as a sum of one or more positive integers = There are 2^6 subsets for {1, 2, 3, 4, 5, 6}

1 + 1 + 1 + 1 + 1 + 1 + 1

↓ 1st plus sign ↓ 2nd plus sign . . . ↓ 5th plus sign ↓ 6th plus sign

① {1, 4, 6} : (1 + 1) + 1 + (1 + 1) + (1 + 1) = 2 + 1 + 2 + 2

1 4 6

② {1, 2, 5, 6} : (1 + 1 + 1) + 1 + (1 + 1 + 1) = 3 + 1 + 3

1 2 5 6

③ 1 + 1 + 5 = 1 + 1 + (1 + 1 + 1 + 1 + 1) : {3, 4, 5, 6}

3 4 5 6

Table 3.1

	Composition of 7		Determining Subset of {1, 2, 3, 4, 5, 6}
(i)	1 + 1 + 1 + 1 + 1 + 1 + 1	(i)	∅
(ii)	1 + 2 + 1 + 1 + 1 + 1	(ii)	{2}
(iii)	1 + 1 + 3 + 1 + 1	(iii)	{3, 4}
(iv)	2 + 3 + 2	(iv)	{1, 3, 4, 6}
(v)	4 + 3	(v)	{1, 2, 3, 5, 6}
(vi)	7	(vi)	{1, 2, 3, 4, 5, 6}

EX 3.12 : For integers n, r with $n \geq r \geq 1$, $C(n+1, r) = C(n, r) + C(n, r-1)$.

Sol.

Let $A = \{x, a_1, a_2, \ldots, a_n\}$

① All subsets of A that contains r elements = $C(n+1, r)$.

② $C \subseteq A$, where $x \in C$ and $|C| = r$: $C(n, r-1)$.

③ $C \subseteq A$, where $x \notin C$ and $|C| = r$: $C(n, r)$.

∵ ① = ② + ③

∴ $C(n+1, r) = C(n, r) + C(n, r-1)$.

Another Sol.

使用EX3.10之方法:

視為(0, 0)到$(n+1-r, r)$之走法:共$C(n+1, r)$

= 最後一步為(i) R : $(n-r, r)$; (ii) U : $(n+1-r, r-1)$

= $C(n, r) + C(n, r-1)$.

§ 3.1 Sets and Subsets

EX 3.13 : Find the number of nonnegative integer solutions of

$$x_1 + x_2 + \ldots + x_6 < 10.$$

Sol.

$\forall\, k,\ 0 \le k \le 9$, the number of solutions to $x_1 + x_2 + \ldots + x_6 = k$ is $\binom{5+k}{k}$.

$$\therefore \text{the answer} = \binom{5}{0} + \binom{6}{1} + \binom{7}{2} + \binom{8}{3} + \ldots + \binom{14}{9}$$

$$= \left[\binom{6}{0} + \binom{6}{1}\right] + \binom{7}{2} + \binom{8}{3} + \ldots + \binom{14}{9}$$

$$= \left[\binom{7}{1} + \binom{7}{2}\right] + \binom{8}{3} + \ldots + \binom{14}{9}$$

$$= \left[\binom{8}{2} + \binom{8}{3}\right] + \ldots + \binom{14}{9}$$

$$= \ldots = \binom{14}{8} + \binom{14}{9} = \binom{15}{9} = 5005.$$

EX 3.14 : **Pascal's triangle.**

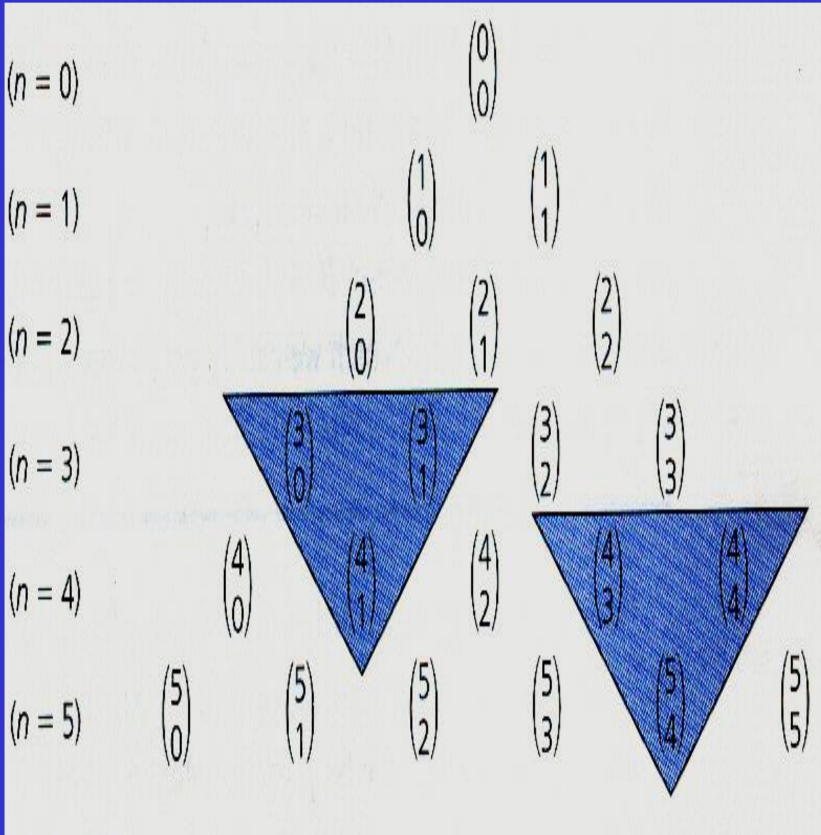

Figure 3.2

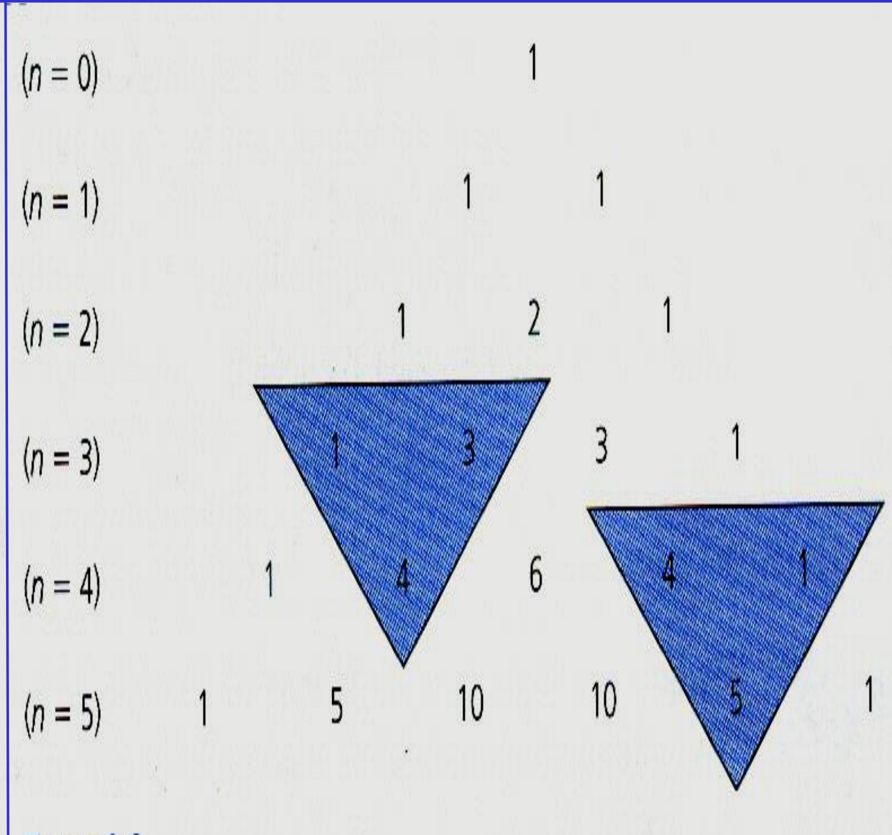

Figure 3.3

§ 3.1 Sets and Subsets

Def :

a) $\mathbf{Z}$ = the set of *integers* = $\{0, 1, -1, 2, -2, 3, -3, \ldots\}$

b) $\mathbf{N}$ = the set of *nonnegative integers* or *natural numbers* = $\{0, 1, 2, 3, \ldots\}$

c) $\mathbf{Z}^+$ = the set of *positive integers* = $\{1, 2, 3, \ldots\} = \{x \in \mathbf{Z} | x > 0\}$

d) $\mathbf{Q}$ = the set of *rational numbers* = $\{a/b | a, b \in \mathbf{Z}, b \neq 0\}$

e) $\mathbf{Q}^+$ = the set of *positive rational numbers* = $\{r \in \mathbf{Q} | r > 0\}$

f) $\mathbf{Q}^*$ = the set of *nonzero rational numbers*

g) $\mathbf{R}$ = the set of *real numbers*

h) $\mathbf{R}^+$ = the set of *positive real numbers*

i) $\mathbf{R}^*$ = the set of *nonzero real numbers*

j) $\mathbf{C}$ = the set of *complex numbers* = $\{x + yi | x, y \in \mathbf{R}, i^2 = -1\}$

k) $\mathbf{C}^*$ = the set of *nonzero complex numbers*

l) For each $n \in \mathbf{Z}^+$, $\mathbf{Z}_n = \{0, 1, 2, \ldots, n-1\}$

m) For real numbers a, b with $a < b$, $[a, b] = \{x \in \mathbf{R} | a \leq x \leq b\}$, $(a, b) = \{x \in \mathbf{R} | a < x < b\}$, $[a, b) = \{x \in \mathbf{R} | a \leq x < b\}$, $(a, b] = \{x \in \mathbf{R} | a < x \leq b\}$. The first set is called a *closed interval*, the second set an *open interval*, and the other two sets *half-open intervals*.

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Chap 3 Set Theory

§ 3.2 Set Operations and the Laws of Set Theory

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3.2 Set Operations and the Laws of Set Theory

Recall : ① ***binary operation*** : **use two** ***operands*****.**

② ***closed*** : **+ under Z^+, / under Q^+.**

Def : **For $A, B \subseteq \mathcal{U}$.**

a) $A \cup B$, **the** ***union*** **of** A **and** $B = \{x \mid x \in A \vee x \in B\}$.

b) $A \cap B$, **the** ***intersection*** **of** A **and** $B = \{x \mid x \in A \wedge x \in B\}$.

c) $A \,\Delta\, B$, **the** ***symmetric difference*** **of** A **and** B

$\equiv \{x \mid (x \in A \vee x \in B) \wedge (x \notin A \cap B)\}$

$= \{x \mid x \in A \cup B \wedge x \notin A \cap B\}$.

Note : ① $\cup, \cap, \Delta$ **are** ***closed*** **binary operations on** $\mathscr{P}(\mathcal{U})$.

i.e if $A, B \subseteq \mathcal{U}$, $A \cup B, A \cap B, A \,\Delta\, B \subseteq \mathcal{U}$

② $\mathscr{P}(\mathcal{U})$ **is** ***closed*** **under these operations.**

3.2 Set Operations and the Laws of Set Theory

EX 3.15 : $\mathcal{U} = \{1, 2, 3, \ldots, 9\}$,

$A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{7, 8, 9\}$

a) $A \cap B = \{ \quad \}$ e) $A \Delta B = \{ \quad \}$

b) $A \cup B = \{ \quad \}$ f) $A \cup C = \{ \quad \}$

c) $B \cap C = \{ \quad \}$ g) $A \Delta C = \{ \quad \}$

d) $A \cap C =$

Note : $A \cap B \subseteq A \subseteq A \cup B$

① $x \in A \cap B \Rightarrow (x \in A \wedge x \in B) \Rightarrow x \in A$.

② $x \in A \Rightarrow (x \in A \vee x \in B) \Rightarrow x \in A \cup B$.

3.2 Set Operations and the Laws of Set Theory

Def 3.6 : Let $S, T \subseteq \mathcal{U}$, S, T are called *disjoint* (or *mutually disjoint*) $\equiv S \cap T = \phi$

Thm 3.3 : If $S, T \subseteq \mathcal{U}$, then

S and T are disjoint $\Leftrightarrow S \cup T = S \Delta T$

Proof. (1/2)

($\Rightarrow$) $\forall x \in \mathcal{U}$: ① $x \in S \cup T$.

$\because S \cap T = \phi$, $\therefore x \notin S \cap T$.

$\therefore x \in S \Delta T$.

i.e. $S \cup T \subseteq S \Delta T$.

② $y \in S \Delta T \Rightarrow y \in S \cup T \wedge y \notin S \cap T$.

$\therefore y \in S \cup T$.

i.e. $S \Delta T \subseteq S \cup T$. ($\forall S, T \subseteq \mathcal{U}$ 皆成立)

by ①, ②, $S \Delta T = S \cup T$.

S and T are disjoint $\Leftrightarrow S \cup T = S \Delta T$

3.2 Set Operations and the Laws of Set Theory

Proof. (2/2)

($\Leftarrow$) Proof by contradiction.

Assume S and T are not disjoint, $S \cap T \neq \phi$.

Let $x \in S \cap T$, then $x \in S \wedge x \in T$.

$\therefore x \in S \cup T$ and $x \in S \Delta T$ $(= S \cup T)$.

But, $\because x \in S \cap T \wedge x \in S \cup T$

$\Rightarrow x \notin S \Delta T \quad \rightarrow\leftarrow$

$\therefore$ Assumption was incorrect.

i.e. S and T disjoint.

Recall : 2 – 5 = –3 ∉ N. But –3 ∈ Z. (*superset*)

minus or negative : the *unary* (or *monary*) *operation* .

3.2 Set Operations and the Laws of Set Theory

Def 3.7 : For a set $A \subseteq \mathcal{U}$, the *complement* of A, $\mathcal{U} - A$ (or $\bar{A}$)
$\equiv \{x \mid x \in \mathcal{U} \wedge x \notin A\}$

EX 3.16 : $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
$A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$, $C=\{7, 8, 9\}$.
$\overline{A} = \{6, 7, 8, 9, 10\}$.
$\overline{B} = \{1, 2, 8, 9, 10\}$.
$\overline{C} = \{1, 2, 3, 4, 5, 6, 10\}$.

Note : $\forall A \subseteq \mathcal{U}, \bar{A} \subseteq \mathcal{U}$. $\therefore \mathscr{P}(\mathcal{U})$ is closed under the unary operation.

3.2 Set Operations and the Laws of Set Theory

Def 3.8 : For $A, B \subseteq \mathcal{U}$, the (*relative*) *complement* of A in B,

$$B - A \equiv \{x \mid x \in B \wedge x \notin A\}.$$

EX 3.17 : $\mathcal{U} = \{1, 2, 3, \ldots, 9\}$,

$A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{7, 8, 9\}$

a) $B - A = \{6, 7\}$ b) $A - B = \{1, 2\}$ c) $A - C = A$

d) $C - A = C$ e) $A - A = \phi$ f) $\mathcal{U} - A = \bar{A}$

3.2 Set Operations and the Laws of Set Theory

EX 3.18 : $\mathcal{U} = \mathbf{R}$,

$A = [1, 2]$, $B = [1, 3)$

a) $A = \{x \mid 1 \le x \le 2\} \subseteq \{x \mid 1 \le x < 3\} = B$

b) $A \cup B = \{x \mid 1 \le x < 3\} = B$

c) $A \cap B = \{x \mid 1 \le x \le 2\} = A$

d) $\overline{B} = (-\infty, 1) \cup [3, +\infty) \subseteq (-\infty, 1) \cup (2, +\infty) = \bar{A}$

Thm 3.4 : For any sets $A, B \subseteq \mathcal{U}$. TFSAE:

a) $A \subseteq B$

b) $A \cup B = B$

c) $A \cap B = A$

d) $\overline{B} \subseteq \bar{A}$

3.2 Set Operations and the Laws of Set Theory

Proof. (1/2) (prove that (*a*) $\Rightarrow$ (*b*) $\Rightarrow$ (*c*) $\Rightarrow$ (*d*) $\Rightarrow$ (*a*))

i) (*a*) $\Rightarrow$ (*b*) $A \subseteq B \Rightarrow A \cup B = B$

① $B \subseteq A \cup B$: trivial.

② $A \cup B \subseteq B$: $\forall x, x \in A \cup B \Rightarrow x \in A \vee x \in B$

$\because A \subseteq B, \therefore x \in A \Rightarrow x \in B$

i.e. $x \in A \vee x \in B \Rightarrow x \in B$

$\therefore \forall x, x \in A \cup B \Rightarrow x \in B.$

by ① ②, $A \cup B = B.$

ii) (b) $\Rightarrow$ (c) $A \cup B = B \Rightarrow A \cap B = A$

① $A \cap B \subseteq A$: trivial

② $A \subseteq A \cap B$: $\forall y \in A,$

$\because A \cup B = B, \therefore y \in A \Rightarrow y \in A \cup B \Rightarrow y \in B.$

$\therefore y \in A \Rightarrow y \in A \wedge y \in B \Rightarrow y \in A \cap B.$

by ① ②, $A = A \cap B.$

Proof. (2/2)

iii) (c) ⇒ (d) : $A \cap B = A \Rightarrow \overline{B} \subseteq \overline{A}$

$\because A \cap B \subseteq B, \forall x\,[x \in A \cap B \Rightarrow x \in B].$

$\forall z\,[\,z \in \overline{B} \Leftrightarrow z \notin B\,].$

$\because z \notin B \Rightarrow z \notin A \cap B \Leftrightarrow z \notin A \Leftrightarrow z \in \overline{A},$

$\therefore \forall z\,[\,z \in \overline{B} \Rightarrow z \in \overline{A}\,]$, that is, $\overline{B} \subseteq \overline{A}$.

iv) (d) ⇒ (a) : $\overline{B} \subseteq \overline{A} \Rightarrow A \subseteq B$

$\forall w, w \in A \Leftrightarrow w \notin \overline{A}$. ------- ①

If $w \notin B$, then $w \in \overline{B}$,

$\therefore$ By (d), $w \in \overline{B} \Rightarrow w \in \overline{A}$. ------- ②

By ①②, $w \notin \overline{A} \wedge w \in \overline{A}$. $\rightarrow\leftarrow$

$\therefore w \in B$. (If)

i.e. $A \subseteq B$.

3.2 Set Operations and the Laws of Set Theory

The Laws of Set Theory

The Laws of Set Theory

For any sets A, B, and C taken from a universe $\mathcal{U}$

1)	$\overline{\overline{A}} = A$	Law of *Double Complement*
2)	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	*DeMorgan's* Laws
3)	$A \cup B = B \cup A$ $A \cap B = B \cap A$	*Commutative* Laws
4)	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	*Associative* Laws
5)	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	*Distributive* Laws

The Laws of Set Theory

6) $A \cup A = A$ $A \cap A = A$	*Idempotent* Laws
7) $A \cup \emptyset = A$ $A \cap \mathcal{U} = A$	*Identity* Laws
8) $A \cup \overline{A} = \mathcal{U}$ $A \cap \overline{A} = \emptyset$	*Inverse* Laws
9) $A \cup \mathcal{U} = \mathcal{U}$ $A \cap \emptyset = \emptyset$	*Domination* Laws
10) $A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	*Absorption* Laws

3.2 Set Operations and the Laws of Set Theory

Def 3.9 : Let s be a statement dealing with the equality of two set expressing. Each may involve ≥ 1 sets, ≥ 1 ϕ, $\mathcal{U}$, $\cap$, $\cup$.

The ***dual*** of s, s^d is obtained from s by replacing

(1) $\phi \longleftrightarrow \mathcal{U}$; (2) $\cap \longleftrightarrow \cup$.

Thm 3.5 : ***The Principle of Duality*** : Let s denote a theorem dealing with the equality of two set expressions (involving only the set operations $\cap$ and $\cup$ as described in Def 3.9). Then s^d is also a theorem .

Note : Thm 3.5 cannot be applied to particular situation.

ex : $\mathcal{U} = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 5\}$,
$C = \{1, 2\}$, $D = \{1, 3\}$

$s : A \cap B = \{1, 2, 3\} = C \cup D$.

but , s^d **cannot hold :** $A \cup B = C \cap D$.

$A \cup B = \{1, 2, 3, 4, 5\}$; $C \cap D = \{1\}$.

EX 3.19 : The dual for the statement $A \subseteq B = ?$

$A \subseteq B \Leftrightarrow A \cup B = B$, (by Thm 3.4)

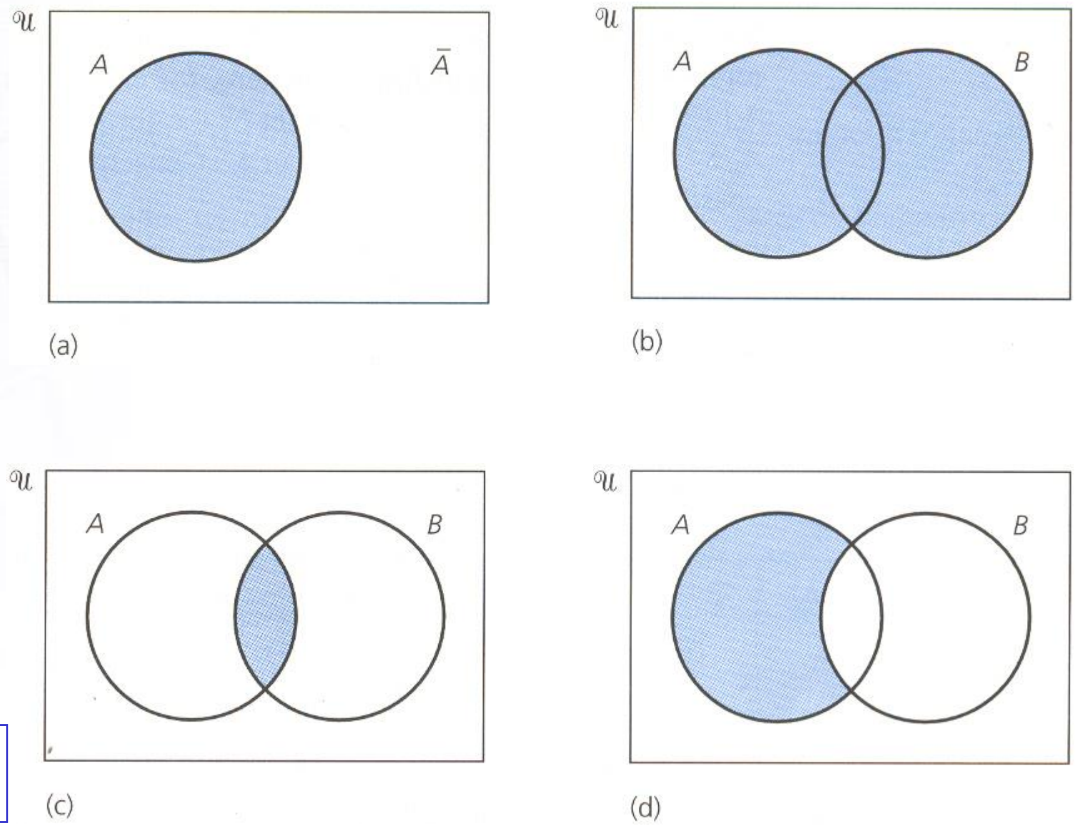
the dual for $A \cup B = B$ is $A \cap B = B$.

$A \cap B = B \Leftrightarrow B \subseteq A$,

$\therefore$ the dual for $A \subseteq B$ is $B \subseteq A$.

3.2 Set Operations and the Laws of Set Theory

<u>Def</u> : English logician John Venn (1834 – 1923) : *Venn diagram* :



$A \cup B$

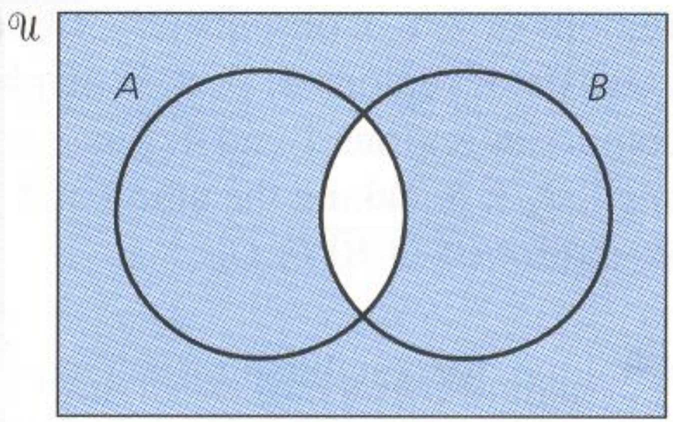
$A \cap B$

$A - B$

3.2 Set Operations and the Laws of Set Theory

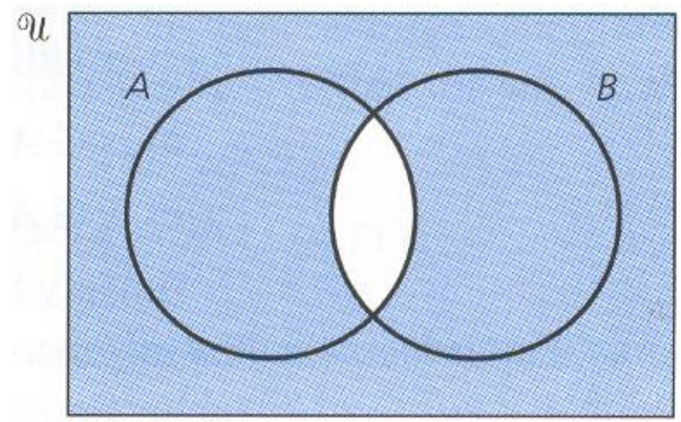
ex : ① $\overline{A \cap B} = \bar{A} \cup \bar{B}$

$\overline{A \cap B}$



(a)

$\bar{A} \cup \bar{B}$



(d)

3.2 Set Operations and the Laws of Set Theory

② $\overline{(A \cup B) \cap C} = (\bar{A} \cap \bar{B}) \cup \bar{C}$

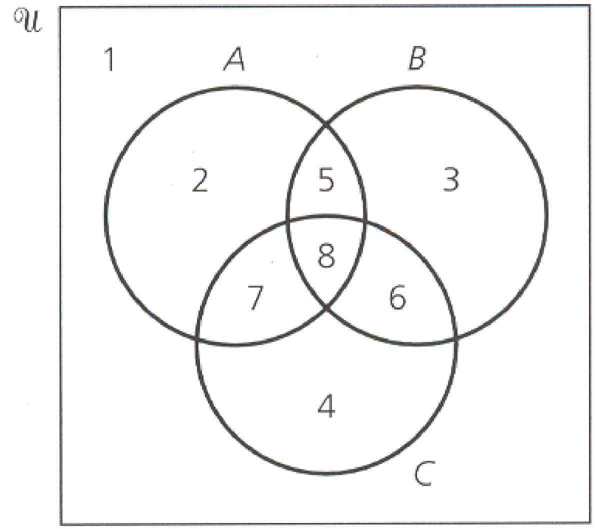


Figure 3.7

$(A \cup B) \cap C$: 6, 7, 8

$\overline{(A \cup B) \cap C}$: 1, 2, 3, 4, 5 ----- ①

$(\bar{A} \cap \bar{B})$: 1, 4

$(\bar{A} \cap \bar{B}) \cup \bar{C}$: 1, 2, 3, 4, 5 ----- ②

① = ②

3.2 Set Operations and the Laws of Set Theory

Def : ***membership table***

ex : ① $A, B \subseteq \mathcal{U}.\ \forall x \in \mathcal{U}$:

a) $x \notin A, x \notin B$ b) $x \notin A, x \in B$ c) $x \in A, x \notin B$ d) $x \in A, x \in B$:

0 0 0 1 1 0 1 1

(·) (+)

A	B	$A \cap B$	$A \cup B$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

A	$\bar{A}$
0	1
1	0

3.2 Set Operations and the Laws of Set Theory

② $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Table 3.3

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Since these columns are identical, we conclude that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Remark : ① Venn 較 membership 簡單.

② 兩者皆為引起興趣用, 尤其對不熟悉證明的讀者; 但兩者皆沒有明確說明邏輯及理由.

③ 當 set 之數過大 (> 3), 則很難畫.

④ element argument 是較其他兩者嚴謹的.

Ex 3.20 : Simplify $\overline{\overline{(A \cup B) \cap C} \cup \overline{B}}$.

Sol.

$$\overline{\overline{(A \cup B) \cap C} \cup \overline{B}}$$
$$= \overline{\overline{((A \cup B) \cap C)}} \cap \overline{\overline{B}}$$
$$= ((A \cup B) \cap C) \cap B$$
$$= (A \cup B) \cap (C \cap B)$$
$$= (A \cup B) \cap (B \cap C)$$
$$= [(A \cup B) \cap B] \cap C$$
$$= B \cap C$$

Compare : (EX 2.17)

$\neg [\neg [(p \vee q) \wedge r] \vee \neg q] \Leftrightarrow q \wedge r$

3.2 Set Operations and the Laws of Set Theory

Ex 3.21 : Express $\overline{A - B}$ in terms of $\cup$ and $\bar{\ }$.

Sol. $\overline{A - B} = \overline{A \cap \overline{B}} = \bar{A} \cup \bar{\bar{B}} = \bar{A} \cup B.$

Ex 3.22 : $\overline{A \,\Delta B} = A \,\Delta \overline{B} = \bar{A} \,\Delta B$

Proof.

$$\overline{A \,\Delta B} = \overline{(A \cup B) - (A \cap B)} = (\overline{A \cup B}) \cup (A \cap B) \text{(by Ex 3.21)}$$

$$= (A \cap B) \cup (\overline{A \cup B}) = (A \cap B) \cup (\bar{A} \cap \overline{B})$$

$$= [(A \cap B) \cup \bar{A}] \cap [(A \cap B) \cup \overline{B})]$$

$$= [(A \cup \bar{A}) \cap (B \cup \bar{A})] \cap [(A \cup \overline{B}) \cap (B \cup \overline{B})]$$

$$= [\mathcal{U} \cap (B \cup \bar{A})] \cap [(A \cup \overline{B}) \cap \mathcal{U})]$$

$$= (B \cup \bar{A}) \cap (A \cup \overline{B}) = (\bar{A} \cup B) \cap (\overline{\bar{A} \cap B})$$

$$= (\bar{A} \cup B) - (\bar{A} \cap B) = \bar{A} \,\Delta B.$$

$$= (A \cup \overline{B}) \cap (\bar{A} \cup B) = (A \cup \overline{B}) \cap (\overline{A \cap \overline{B}}) = A \,\Delta \overline{B}.$$

Def 3.10 : Let I be a nonempty set and $\mathcal{U}$ a universe. $\forall i \in I$, let $A_i \subseteq \mathcal{U}$. Then I is called an ***index set*** (or ***set of indices***), and $\forall i \in I$, i is called an ***index*** :

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for at least one } i \in I\}, \text{ and}$$

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for every } i \in I\}$$

Note : ① $x \in \bigcup_{i \in I} A_i \Leftrightarrow \exists i \in I\ (x \in A_i)$.

② $x \in \bigcap_{i \in I} A_i \Leftrightarrow \forall i \in I\ (x \in A_i)$.

③ $x \notin \bigcup_{i \in I} A_i \Leftrightarrow \forall i \in I\ (x \notin A_i)$.

④ $x \notin \bigcap_{i \in I} A_i \Leftrightarrow \exists i \in I\ (x \notin A_i)$.

3.2 Set Operations and the Laws of Set Theory

Note : ⑤ If $I = \mathbf{Z}^+$: $\bigcup_{i\in Z^+} A_i = A_1 \cup A_2 \cup \cdots = \bigcup_{i=1}^{\infty} A_i$

$$\bigcap_{i\in Z^+} A_i = A_1 \cap A_2 \cap \cdots = \bigcap_{i=1}^{\infty} A_i$$

Ex 3.23 : Let $I = \{3, 4, 5, 6, 7\}$.

$\forall i \in I$, let $A_i = \{1, 2, 3, \ldots, i\} \subseteq \mathcal{U} = \mathbf{Z}^+$.

① $\bigcup_{i\in I} A_i = \cup_{i=3}^{7} A_i = \{1, 2, 3, \ldots, 7\} = A_7$.

② $\bigcap_{i\in I} A_i = \{1, 2, 3\} = A_3$.

Ex 3.24 : Let $\mathcal{U} = \mathbf{R}$ and $I = \mathbf{R}^+$, $\forall r \in \mathbf{R}$, $A_r = [-r, r]$, then

① $\cup_{r\in I} A_r = \mathbf{R}$.

② $\cap_{r\in I} A_r = \{0\}$.

Note : Venn diagram and membership table are useless when dealing with generalized union and intersection.

Thm 3.6 : ***Generalized De Morgan's laws*** :

Let I be an index set where $\forall i \in I, A_i \subseteq \mathcal{U}$. Then

a) $\overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I} \overline{A_i}$ b) $\overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$

Proof.

(a) $\forall x \in \mathcal{U} : x \in \overline{\cup_{i\in I} A_i} \Leftrightarrow x \notin \cup_{i\in I} A_i$

$\Leftrightarrow x \notin A_i$, for all $i \in I$

$\Leftrightarrow x \in \overline{A_i}$, for all $i \in I$

$\Leftrightarrow x \in \cap_{i\in I} \overline{A_i}$

(b) exercise. 加做第20題

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Discrete Mathematics

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Chap 3 Set Theory

§ 3.3 Counting and Venn Diagrams

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5^{th} Edition)**
by Ralph P. Grimaldi

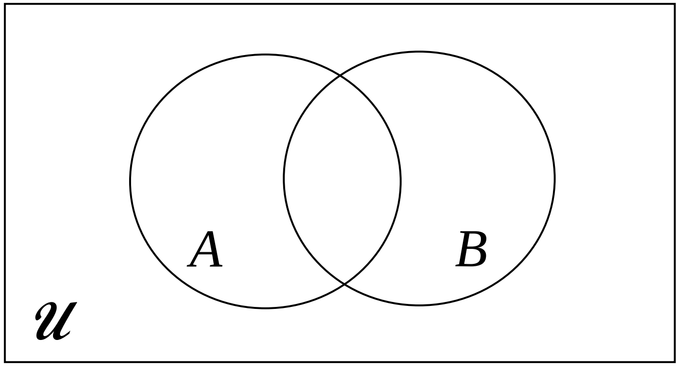
3.3 Counting and Venn Diagrams

Thm : (Chap 8)

① $|A \cup B| = |A| + |B| - |A \cap B|$;

If A **and** B **are disjoint** $\Leftrightarrow |A \cup B| = |A| + |B|$.

② $|\bar{A} \cap \bar{B}| = |\overline{A \cup B}| = |\mathcal{U}| - |A \cup B| = |\mathcal{U}| - |A| - |B| + |A \cap B|$.

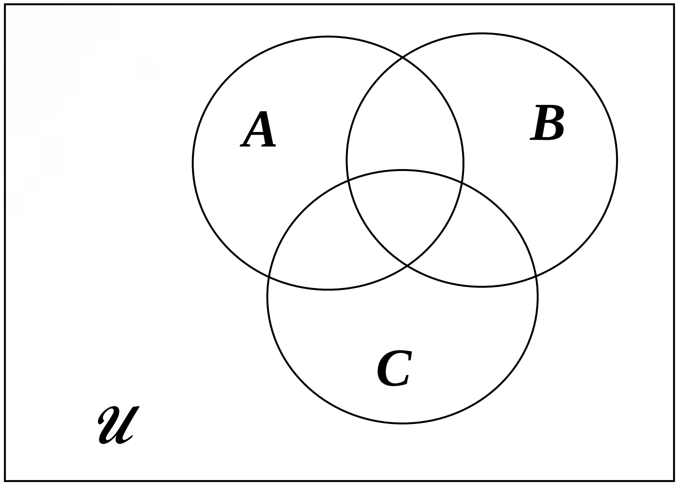


3.3 Counting and Venn Diagrams

Thm : (Chap 8)

③ $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

④ $|\bar{A} \cap \bar{B} \cap \bar{C}| = |\overline{A \cup B \cup C}| = |\mathcal{U}| - |A \cup B \cup C|$
$= |\mathcal{U}| - |A| - |B| - |C| + |A \cap B| + |B \cap C| + |C \cap A| - |A \cap B \cap C|$.



Computer Science and Information Engineering
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Chap 3 Set Theory

§ 3.4 A First Word on Probability

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§ 3.4 A First Word on Probability

Def : ① ***experiment*** $\mathcal{E}$ 、 ***sample space*** $\mathcal{S}$ 、 ***event*** A ($\subseteq \mathcal{S}$) 、 ***elementary event*** a ($\in A$). **Let** $|\mathcal{S}| = n$.

② $Pr(a)$ = ***The probability that a occurs*** $= \frac{1}{n} = \frac{|\{a\}|}{|\mathcal{S}|}$

$Pr(A)$ = ***The probability that A occurs*** $= \frac{|A|}{n} = \frac{|A|}{|\mathcal{S}|}$

Ex3.28 ~ Ex3.36: see book.

§ 3.4 A First Word on Probability

Ex3.31 : 5 cards from a standard deck of 52 cards. $\binom{52}{5} = 2598960$

What is the probability:

(a) Three aces and two jacks; (b) three aces and a pair;

(c) a full house?

Sol.

(a) $\binom{4}{3} = 4$ for aces, $\binom{4}{2} = 6$ for jacks.

Let A = the event where Tanya draws three aces and two jacks.

$\therefore |A| = \binom{4}{3}\binom{4}{2} = 4 \cdot 6$; $Pr(A) = 24 / 2598960 \approx 0.000009234$.

(b) $\binom{4}{3} = 4$ for aces, $\binom{12}{1}\binom{4}{2} = 12 \cdot 6 = 72$ for a pair.

Let B = the event where Tanya draws three aces and a pair.

$\therefore |B| = \binom{4}{3}\binom{12}{1}\binom{4}{2} = 4 \cdot 72$; $Pr(B) = 288 / 2598960 \approx 0.000110814$.

(c) $\binom{13}{1}\binom{4}{3} = 13 \cdot 4$ for three something, $\binom{12}{1}\binom{4}{2} = 12 \cdot 6 = 72$ for a pair

Let C = the event where Tanya draws a full house.

$\therefore |C| = \binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 13 \cdot 288 = 3744$;

$Pr(C) = 3744 / 2598960 \approx 0.001440576$.

§ 3.4 A First Word on Probability

Def : ③ ***Cartesian product***, or ***cross product***, of A and B = $A \times B$ $= \{(a, b) \mid a \in A, b \in B\}$.

④ ***ordered pairs*** : the element of $A \times B$. (form : (a, b))

⑤ $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Ex3.32 : $A = \{1, 2, 3\}$ and $B = \{x, y\}$, then

$A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$

$B \times A = \{(x, 1), (y, 1), (x, 2), (y, 2), (x, 3), (y, 3)\}$

$(1, x) \in A \times B$, $(1, x) \notin B \times A$

$|A \times B| = 3 \cdot 2 = 6 = |A|\,|B| = |B|\,|A| = |B \times A|$.

§ 3.4 A First Word on Probability

Ex3.37 : 120 passengers on airline:

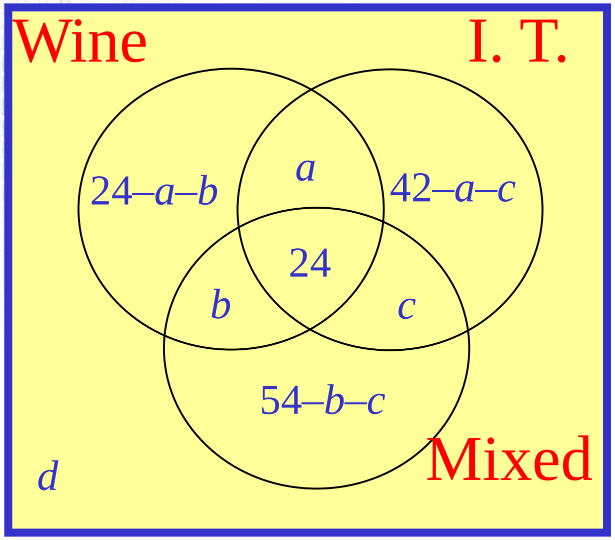
48: wine; 78: mixed drink; 66: iced tea;

36: 2 beverages; 24: 3 beverages.

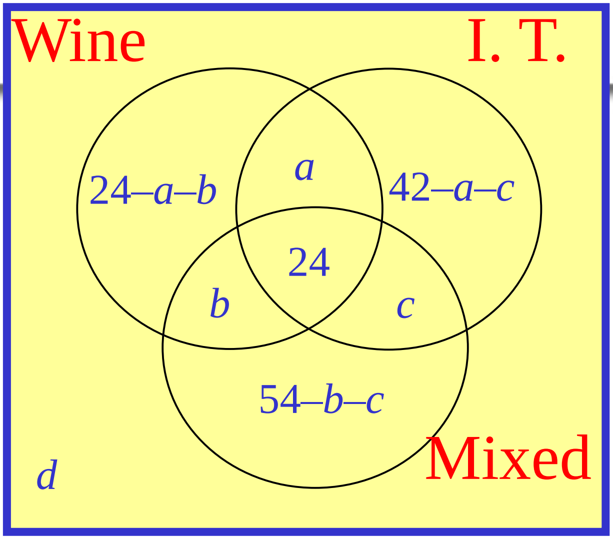
自120 位中任選2 位; what is the probability that:

a) Event *A* : they both want only iced tea?

b) Event *B* : they both enjoy exactly two of the three beverage offerings?



§ 3.4 A First Word on probability



Sol. (1/2)

$a + b + c = 36$

$24 - a - b = 24 + c - 36 = c - 12 \geq 0$

$42 - a - c = 42 + b - 36 = b + 6 \geq 0$

$54 - b - c = 54 + a - 36 = a + 18 \geq 0$

and $120 = (c - 12) + (b + 6) + (a + 18) + a + b + c + 24 + d$

$= 36 \cdot 2 + 12 + 24 + d = 108 + d$

$\therefore d = 12$

(8 unknowns 6 equations $\therefore$ infinite selected)

ex:

let a = b = 12, then c = 12, $42 - a - c = b + 6 = 18$.

let a = b = 10, then c = 16, $42 - a - c = b + 6 = 16$.

§ 3.4 A First Word on probability

Sol. (2/2)

In Book:

$|S| = \binom{120}{2} = 7140$

$|A| = \binom{18}{2} = 153$

$|B| = \binom{36}{2} = 630$

$\therefore Pr(A) = \frac{51}{2380}$, $Pr(B) = \frac{3}{34}$.