Computer Science and Information Engineering
National Chi Nan University

Discrete Mathematics

Dr. Justie Su-Tzu Juan

Chapter 2 Fundamentals of Logic

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5th Edition)**
by Ralph P. Grimaldi

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Note : In definition, an implication can be read as a biconditional, and, only in definition.

Ex 2.51 : (1/2)

a) Universe : all quadrilaterals in the plane.

A : “If a quadrilaterals is a rectangle then it has four equals angles.”

B : “If a quadrilaterals has four equal angles, then it is a rectangle.”

Let $p(x)$: x is a rectangle. $q(x)$: x has four equal angles

A : $\forall x\, [p(x) \rightarrow q(x)]$

B : $\forall x\, [q(x) \rightarrow p(x)]$

Actually, they are both intending: $\forall x\, [p(x) \leftrightarrow q(x)]$

Ex 2.51 : (2/2)

b) Universe = Z

A : "For every integer n, we call n even if it is divisible by 2."

Let $p(n)$: n is an even integer

$q(n)$: n is divisible by 2 (or, $n = 2k$, for some integer k)

(or, $\exists k\ [n = 2k]$)

A : $\forall n\ [q(n) \rightarrow p(n)]$

Actually, $\forall n\ [p(n) \leftrightarrow q(n)]$

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

<u>Ex 2.52</u> : Universe = {2, 4, 6, …, 26}

For all *n* (*n* = 2, 4, …, 26), we can write *n* as the sum of at most three perfect squares.

Sol. ***method of exhaustion*** **:** （不只一種，但我們只需要找出其中一種即可！）

2 = 1 + 1	10 = 9 + 1	18 = 16 + 1 + 1 (= 9 + 9)
4 = 4	12 = 4 + 4 + 4	20 = 16 + 4
6 = 4 + 1 + 1	14 = 9 + 4 + 1	22 = 9 + 9 + 4
8 = 4 + 4	16 =16	24 = 16 + 4 + 4
		26 = 25 + 1 (= 16 + 9 + 1)

<u>Def</u> : *Corollary* : follow immediately from a *theorem*

<u>The Rule of Universal Specification</u> **:**

$\forall x$ for a given universe, $p(x)$ is true, the $p(a)$ is true for each a in the universe.

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Ex 2.53 : (1/4)

a) Universe = all people

$m(x)$: x is a mathematics professor,

$c(x)$: x has studied calculus

All mathematics professors have studied calculus.

Leona is a mathematics professor.

Therefore Leona has studied calculus.

$$\begin{array}{l} \forall x\,[m(x) \rightarrow c(x)] \\ m(\vartheta) \\ \hline \therefore\ c(\vartheta) \end{array}$$

Let ϑ = Leona (in our universe) then :

Steps	Reason
1) $\forall x\,[m(x) \rightarrow c(x)]$	Premise
2) $m(\vartheta)$	Premise
3) $m(\vartheta) \rightarrow c(\vartheta)$	(1) & the Rule of Universal Specification
4) $\therefore\ c(\vartheta)$	(2), (3) and the Rule of Detachment

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Ex 2.53 : (2/4)

b) Universe = all triangles in the plane

$p(t)$: t has two sides of equal length

$q(t)$: t is an isosceles triangle

$r(t)$: t has two angles of equal measure

c : triangle XYZ

$$\begin{array}{l} \neg r(c) \\ \forall t\, [p(t) \rightarrow q(t)] \\ \forall t\, [q(t) \rightarrow r(t)] \\ \hline \therefore \neg p(c) \end{array}$$

See Textbook.

Step	Reasons
1) $\forall t\, [p(t) \rightarrow q(t)]$	Premise
2) $p(c) \rightarrow q(c)$	(1) and the Rule of Universal Specification
3) $\forall t\, [q(t) \rightarrow r(t)]$	Premise
4) $q(c) \rightarrow r(c)$	(3) and the Rule of Universal Specification
5) $p(c) \rightarrow r(c)$	(2), (4) and the Law of the Syllogism
6) $\neg r(c)$	Premise
7) $\therefore \neg p(c)$	(5), (6) and Modus Tollens

Ex 2.53 : (3/4)

c) Universe = student at a particular college.

m : Mary Gusberti, a student of this college.

$j(x)$: x is a junior.

$s(x)$: x is a senior.

$p(x)$: x is enrolled in a physical education class

No junior or senior is enrolled in a physical education class

Mary Gusberti is enrolled in a physical education class

Therefore Mary Gusberti is not a senior

i.e.

$$\begin{array}{l} \forall x\,[(j(x) \vee s(x)) \rightarrow \neg p(x)] \\ p(m) \\ \hline \therefore \neg s(m) \end{array}$$

$\forall x\,[(j(x) \vee s(x)) \to \neg p(x)]$
$p(m)$

$\therefore \neg s(m)$

Ex 2.53 : (4/4)

c) Sol.

Step	Reason
1) $\forall x\,[(j(x) \vee s(x)) \to \neg p(x)]$	Premise
2) $p(m)$	Premise
3) $(j(m) \vee s(m)) \to \neg p(m)$	(1) and the Rule of Universal Specification
4) $p(m) \to \neg (j(m) \vee s(m))$	(3) and $(q \to t) \Leftrightarrow (\neg t \to \neg p)$, and Law of Double Negation
5) $p(m) \to (\neg j(m) \wedge \neg s(m))$	(4) and DeMorgan's Law
6) $\neg j(m) \wedge \neg s(m)$	(2), (5) and the Rule of Detachment
7) $\therefore \neg s(m)$	(6) and the Rule of Conjunctive Simplification

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Note : the Rule of Universal Specification + Modus Ponens, Modus Tollens

c **: a member of the fixed universe**

$p(x)$, $q(x)$ **: open statements defined for this universe**

(1)
$$\begin{array}{l} \forall x\,[p(x) \rightarrow q(x)] \\ p(c) \\ \hline \therefore q(c) \end{array}$$

(2)
$$\begin{array}{l} \forall x\,[p(x) \rightarrow q(x)] \\ \neg q(c) \\ \hline \therefore \neg p(c) \end{array}$$

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Ex (1/2) : Universe = all polygons in the plane

c : quadrilateral $EFGH$, where $\angle E = 91°$

$p(x)$: x is a square

$q(x)$: x has four sides

(1') All squares have four sides,
Quadrilateral *EFGH* has four sides
Therefore quadrilateral *EFGH* is a square

(1")

$$\begin{array}{l} \forall x\,[p(x) \to q(x)] \\ q(c) \\ \hline \therefore p(c) \end{array} \longleftarrow \text{false}$$

$\because \forall x\,[p(x) \to q(x)]$ and c is a polygon in the plane

$\therefore p(c) \to q(c)$, but $[p(c) \to q(c)] \wedge q(c) \nrightarrow p(c)$

$\therefore$ invalid!!

(converse)

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

<u>Ex</u> (2/2) :

(2') All squares have four sides

Quadrilateral *EFGH* is not a square

Therefore quadrilateral *EFGH* does not have four sides

(2'')

$$\begin{array}{l} \forall x\,[p(x) \rightarrow q(x)] \\ \neg\, p(c) \\ \hline \therefore \neg\, q(c) \end{array}$$

$\because$ $\forall x\,[p(x) \rightarrow q(x)]$ and c is a polygon in the plane

$\therefore$ $p(c) \rightarrow q(c)$, but $[(p(c) \rightarrow q(c)) \wedge \neg\, p(c)] \nrightarrow \neg\, q(c)$

$\therefore$ invalid!!

(inverse)

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

***The Rule of Universal Generalization*:**

(1) If $p(c)$ is true for any arbitrarily chosen element c from our universe, then $\forall x\, p(x)$ is true.

(2) Similar results hold for the cases of two or three or more variables.

Ex 2.54 : Let $p(x)$, $r(x)$ be open statements that are defined for a given universe.

$$\begin{array}{l} \quad \forall x\,[p(x) \rightarrow q(x)] \\ \quad \forall x\,[q(x) \rightarrow r(x)] \\ \hline \therefore\ \forall x\,[p(x) \rightarrow r(x)] \end{array}$$

Steps	Reasons
(1) $\forall x\,[p(x) \rightarrow q(x)]$	Premise
(2) $p(c) \rightarrow q(c)$	(1) & the Rule of Universal Specification
(3) $\forall x\,[q(x) \rightarrow r(x)]$	Premise
(4) $q(c) \rightarrow r(c)$	(3) and the Rule of Universal Specification
(5) $p(c) \rightarrow r(c)$	(2), (4) and the Law of the Syllogism
(6) $\therefore\ \forall x\,[p(x) \rightarrow r(x)]$	(5) & the Rule of Universal Generalization

Ex 2.55 :

(a) Universe : all real number

$p(x)$: $3x - 7 = 20$, $q(x)$: $3x = 27$, $r(x)$: $x = 9$

1) If $3x - 7 = 20$, then $3x = 27$.
2) If $3x = 27$, then $x = 9$.
3) Therefore, if $3x - 7 = 20$, then $x = 9$.

$$\begin{array}{l} \forall x\, [p(x) \rightarrow q(x)] \\ \forall x\, [q(x) \rightarrow r(x)] \\ \hline \therefore \forall x\, [p(x) \rightarrow r(x)] \end{array}$$

(b) Universe : all quadrilaterals in plane geometry

"Since every square is a rectangle, and every rectangle is a parallelogram, it follows that every square is a parallelogram"

$p(x)$: x is a square $q(x)$: x is a rectangle $r(x)$: x is a parallelogram

By Ex 2.54 :

$$\begin{array}{ll} \forall x\, [p(x) \rightarrow q(x)] & ① \\ \forall x\, [q(x) \rightarrow r(x)] & ② \\ \hline \therefore \forall x\, [p(x) \rightarrow r(x)] & \therefore ③ \end{array}$$

Ex 2.56 (1/2) : $\forall x\,[p(x) \lor q(x)]$

$$\begin{array}{l} \forall x\,[p(x) \lor q(x)] \\ \forall x\,[(\neg p(x) \land q(x)) \to r(x)] \\ \hline \therefore \forall x\,[\neg r(x) \to p(x)] \end{array}$$

Let c be an element in the universe assigned for the argument.
Assume $\neg r(c)$ as an additional premise.

Steps	Reasons
(1) $\forall x\,[p(x) \lor q(x)]$	Premise
(2) $p(c) \lor q(c)$	(1) & the Rule of Universal Specification
(3) $\forall x\,[(\neg p(x) \land q(x)) \to r(x)]$	Premise
(4) $[\neg p(c) \land q(c)] \to r(c)$	(3) & the Rule of Universal Specification
(5) $\neg r(c) \to \neg [\neg p(c) \land q(c)]$	(4) and $s \to t \Leftrightarrow \neg t \to \neg s$
(6) $\neg r(c) \to [p(c) \lor \neg q(c)]$	(5), DeMorgan's Law & the Law of Double Negation

Ex 2.56 (2/2) :

Steps	Reasons
(7) $\neg r(c)$	Premise (assumed)
(8) $p(c) \vee \neg q(c)$	(7), (6) and Modus Ponens
(9) $[p(c) \vee q(c)] \wedge [p(c) \vee \neg q(c)]$	(2), (8) and the Rule of Conjunction
(10) $p(c) \vee [q(c) \wedge \neg q(c)]$	(9) & the Distributive Law of $\vee$ over $\wedge$
(11) $p(c)$	(10) & Inverse & Identity Law
(12) $\therefore \forall x\, [\neg r(x) \rightarrow p(x)]$	(7), (11) & the Rule of Universal Generalization

Remark :

1) For convenience : using the letter x instead of c

2) 將省略步驟以免過於瑣碎，除非必要

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Def 2.8 : 1) Let n be an integer. We call n *even* if n is divisible by 2. i.e. $\exists\, r \in Z$ s.t. $n = 2r$.

2) We call n *odd* if $\exists\, s \in Z$ s.t. $n = 2s + 1$.

Theorem 2.2 : $\forall\, k, l \in Z$, if k, l are both odd, then $k + l$ is even.

Proof.

1) $\because$ k, l are odd,

$\therefore$ $\exists\, a, b \in Z$ s.t. $k = 2a + 1$, $l = 2b + 1$ (by Def 2.8)

2) Then $k + l = (2a + 1) + (2b + 1) = 2(a + b + 1)$

3) $\because$ $a, b \in Z$ $\quad\therefore$ $a + b + 1 = c$ is an integer

i.e. $k + l = 2c$,

by Def 2.8, $k + l$ is even.

Remark :

1) In Step (1), k, l : chosen in an arbitrary manner.
$\therefore$ by the Rule of Universe Generalization,
the result obtained is true for all odd integers.

2) Use the Rule of Universe Specification twice in
step (1) : (l 同)
i) n is an odd integer $\rightarrow$ $n = 2r + 1$ for some integer r.
ii) k is a specific, arbitrarily chosen odd integer.
iii) Therefore $k = 2a + 1$ for some integer a.

3) $k = l \leftrightarrow \quad a = \dfrac{(k-1)}{2} = \dfrac{(l-1)}{2} = b$,
but $\because$ k may not equal to l,
$\therefore$ use the different variable a, b.

Ex 2.57 : Universe : Z

If n is an integer, then $n^2 = n$. ($\forall n\ [n^2 = n]$)

Sol.

$n = 0$, $n^2 = 0^2 = 0 = n$.

$n = 1$, $n^2 = 1^2 = 1 = n$.

But, we can not conclude $n^2 = n$, $\forall\ n$

We can not consider the choice of 0 (or 1) as an arbitrarily chosen integer!!

If $n = 2$, $n^2 = 4 \neq 2 = n$, is one counterexample!

$\therefore$ the given statement is false!!

($n = 0$ or $n = 1$ is enough to say: $\exists\ n\ [n^2 = n]$.)

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Theorem 2.3 : $\forall$ integer k, l, if k, l are both odd, then $k \cdot l$ is also odd.

Proof.

$\because$ k, l are both odd,

$\therefore$ $\exists a, b \in Z$ s.t. $k = 2a + 1$, $l = 2b + 1$ (by Def 2.8)

$\therefore$ $k \cdot l = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1$

$= 2(2ab + a + b) + 1$

where $2ab + a + b \in Z$

Therefore, by Def 2.8, $k \cdot l$ is odd.

Theorem 2.4 (1/2) : If m is an even integer, then $m + 7$ is odd.

Proof. (by three methods)

1) By a direct argument:

$\because$ m is even $\therefore$ $\exists a \in Z$ s.t. $m = 2a$.

Then $m + 7 = 2a + 7 = 2(a + 3) + 1$.

$\because$ $a + 3 \in Z$ $\therefore$ $m + 7$ is odd.

2) Prove by the contrapositive method:

Suppose $m + 7$ is not odd, hence even.

$\therefore$ $\exists b \in Z$ s.t. $m + 7 = 2b$,

then $m = 2b - 7 = 2b - 8 + 1 = 2(b - 4) + 1$.

$\because$ $b - 4 \in Z$ $\therefore$ $m = 2(b - 4) + 1$ is odd.

Therefore, If m is an even integer, then $m + 7$ is odd.

($\because$ $\forall m\, [p(m) \rightarrow q(m)] \Leftrightarrow \forall m\, [\neg q(m) \rightarrow \neg p(m)]$)

Theorem 2.4 (2/2) : If *m* is an even integer, then *m* + 7 is odd.

Proof. (by three methods)

3) Proof by "the method of proof by contradiction":

Assume *m* is even and that *m* + 7 is also even.

$\therefore$ $\exists c \in Z$ s.t. $m + 7 = 2c$,

then $m = 2c - 7 = 2(c - 4) + 1$.

$\because$ $c - 4 \in Z$ $\therefore$ *m* is odd $\rightarrow\leftarrow$

($\because$ no integer can be both even and odd!!)

i.e. *m* + 7 is even is a false assumption,

$\therefore$ *m* + 7 is odd.

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

<u>Ex</u> : If we want to prove : $\forall m\ [p(m) \rightarrow q(m)]$

1) Prove this result by the contrapositive method:

prove : $\forall m\ [\neg q(m) \rightarrow \neg p(m)]$

2) Prove by the method of proof by contradiction:

prove : assume $\forall m\ [p(m) \rightarrow q(m)]$ is false will implies F_0

i.e. $\exists m\ [p(m) \wedge \neg q(m)] \rightarrow F_0$

Compared	Assumption	Result Derived
Contrapositive	$\neg q(m)$	$\neg p(m)$
Contradiction	$p(m) \wedge \neg q(m)$	F_0

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Note : 用(2)(3)似乎較麻煩, 但當我們企圖找出一個反例時, 已等於完成(2) or (3).

Thm 2.5 : $\forall$ positive integer x, y, if $xy > 25$, then $x > 5$ or $y > 5$.

Proof.

By the method of contrapositive.
Suppose $0 < x \leq 5$ and $0 < y \leq 5$,
then $0 < xy \leq 5 \times 5 = 25$.
$\therefore$ xy does not exceed 25.
$\because$ $[\neg (x > 5) \wedge \neg (y > 5)] \rightarrow (xy \leq 25)$
$\Leftrightarrow (xy > 25) \rightarrow [(x > 5) \vee (y > 5)]$.
Hence if $xy > 25$, then $x > 5$ or $y > 5$.

Computer Science and Information Engineering
National Chi Nan University

Discrete Mathematics

Dr. Justie Su-Tzu Juan

Chap 3 Set Theory

§ 3.1 Sets and Subsets

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics **(5^{th} Edition)**
by Ralph P. Grimaldi

§ 3.1 Sets and Subsets

Def : ① ***Set*** : a well-defined collection of objects.

② ***element*** (or ***member***) : these objects.

③ ***Well-define*** : for any element, can be determined whether it is in the set or not.

ex : **The set of outstanding major league pitchers for the 1990s.**

Def : ① Capital letters : Sets.

② Lowercase letters : elements.

③ $x \in A$: x is an element of A.

④ $x \notin A$: if not.

§ 3.1 Sets and Subsets

EX 3.1 :

① $\{1, 2, 3, 4, 5\}$

$= \{x \mid x \text{ is an integer and } 1 \le x \le 5\}$

(" | " is read "such that".)

("$\{x \mid \ldots\}$" are read "the set of all x such that …")

$= \{x \mid 1 \le x \le 5\}$ where $\mathcal{U}$ = all integer

("$\mathcal{U}$" is "*universe of discourse*")

② $\{x \mid 1 \le x \le 5\}$ where $\mathcal{U}$ = all real number.

③ $\{x \mid 1 \le x \le 5\}$ where $\mathcal{U}$ = even integer.

$= \{2, 4\}$

§ 3.1 Sets and Subsets

Ex 3.2 : $\mathcal{U} = \{1, 2, \ldots\}$, positive integers.

a) $A = \{1, 4, 9, \ldots, 64, 81\} = \{x^2 \mid x \in \mathcal{U}, x^2 < 100\}$
$= \{x^2 \mid x \in \mathcal{U} \wedge x^2 < 100\}$.

b) $B = \{1, 4, 9, 16\} = \{y^2 \mid y \in \mathcal{U}, y^2 < 20\}$
$= \{y^2 \mid y \in \mathcal{U}, y^2 < 23\} = \{y^2 \mid y \in \mathcal{U} \wedge y^2 < 17\}$.

c) $C = \{2, 4, 6, 8, \ldots\} = \{2k \mid k \in \mathcal{U}\}$.

Def : ① A, B : *finite* set
C : *infinite* set
② $|A|$: *cardinality* (or *size*) : the number of elements in A.

ex : $|A| = 9$, $|B| = 4$.

§ 3.1 Sets and Subsets

Def 3.1 : C, D are sets from $\mathcal{U}$.

① C is a ***subset*** of D, $C \subseteq D$, or $D \supseteq C \equiv$
$\forall x\ [x \in C \Rightarrow x \in D]$

② C is a ***proper subset*** of D, $C \subset D$, or $D \supset C \equiv$
$C \subseteq D \wedge \exists x\ [x \in D \wedge x \notin C]$

Note : ① $C \subset D \Rightarrow C \subseteq D$
② $C \subseteq D \Rightarrow |C| \leq |D|$
③ $C \subset D \Rightarrow |C| < |D|$
④ $C \subseteq D \not\Rightarrow C \subset D$

ex : $\mathcal{U} = \{1, 2, 3, 4, 5\}$, $C = \{1, 2\}$, and $D = \{1, 2\}$.

§ 3.1 Sets and Subsets

<u>EX 3.3</u> : In ANSI FORTRAN:

① Variable : 第一個為字母, 後接續至多共5個字母或數字: $\mathcal{V}$

$$|\mathcal{V}| = 26 + 26(36) + 26(36)^2 + \ldots + 26(36)^5$$

$$= 1{,}617{,}038{,}306.$$

② Integer variable :

第一個為字母為I, J, K, L, M, N其中之一: A

$$|A| = 6 + 6(36) + 6(36)^2 + \ldots + 6(36)^5$$

$$= 373{,}162{,}686.$$

§ 3.1 Sets and Subsets

EX 3.4 : $\mathcal{U} = \{1, 2, 3, 4, 5\}$, $A = \{1, 2\}$, $B = \{x \mid x^2 \in \mathcal{U}\}$

① $B = \{1, 2\} = A$.

② $A \subseteq B$, $B \subseteq A$.

Def 3.2 : The sets C and D are said to be ***equal***, $C = D$,

$\equiv C \subseteq D$ and $D \subseteq C$.

Note : Neither **order** nor **repetition** is relevant for a general set .

ex : $\{1, 2, 3\}$

$= \{3, 1, 2\}$

$= \{2, 2, 1, 3\}$

$= \{1, 2, 1, 3, 1\}$

§ 3.1 Sets and Subsets

Remark : ① $A \not\subseteq B$, A is ***not*** a subset of $B \Leftrightarrow$

$$\exists x\,[(x \in A) \wedge (x \notin B)]$$

② $A \neq B \Leftrightarrow (A \not\subseteq B) \vee (B \not\subseteq A)$

③ $C \subset D \Leftrightarrow (C \subseteq D) \wedge (C \neq D)$

Proof.

① $A \not\subseteq B \Leftrightarrow \neg\forall x\,[x \in A \Rightarrow x \in B]$

$\Leftrightarrow \exists x\, \neg[x \in A \Rightarrow x \in B]$

$\Leftrightarrow \exists x\, \neg[\neg(x \in A) \vee (x \in B)]$

$\Leftrightarrow \exists x\, [(x \in A) \wedge \neg(x \in B)]$

$\Leftrightarrow \exists x\, [(x \in A) \wedge (x \notin B)]$

② $A \neq B \Leftrightarrow \neg(A \subseteq B \wedge B \subseteq A)$

$\Leftrightarrow \neg(A \subseteq B) \vee \neg(B \subseteq A)$

$\Leftrightarrow (A \not\subseteq B) \vee (B \not\subseteq A)$

§ 3.1 Sets and Subsets

EX 3.5 : $\mathcal{U} = \{1, 2, 3, 4, 5, 6, x, y, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$.

a) $A = \{1, 2, 3, 4\}$

$|A| = 4$

i) $A \subseteq \mathcal{U}$; ii) $A \subset \mathcal{U}$; iii) $A \in \mathcal{U}$;

iv) $\{A\} \subseteq \mathcal{U}$; v) $\{A\} \subset \mathcal{U}$; vi) $\{A\} \notin \mathcal{U}$;

b) $B = \{5, 6, x, y, A\} = \{5, 6, x, y, \{1, 2, 3, 4\}\}$

$|B| = 5$

i) $A \in B$; ii) $\{A\} \subseteq B$; iii) $\{A\} \subset B$;

iv) $\{A\} \notin B$; v) $A \nsubseteq B$; vi) $A \not\subset B$;

§ 3.1 Sets and Subsets

Thm 3.1 : Let A , B , $C \subseteq \mathcal{U}$ (that is $A \subseteq \mathcal{U}$ and $B \subseteq \mathcal{U}$ and $C \subseteq \mathcal{U}$)

a) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$.

b) $A \subset B$ and $B \subseteq C \Rightarrow A \subset C$.

c) $A \subseteq B$ and $B \subset C \Rightarrow A \subset C$.

d) $A \subset B$ and $B \subset C \Rightarrow A \subset C$.

Proof. (1/2)

(*element arguments*)

a) $\forall x \in \mathcal{U}$, $x \in A$

$\because A \subseteq B$, $\therefore x \in A \Rightarrow x \in B$

$\because B \subseteq C$, $\therefore x \in B \Rightarrow x \in C$

By the law of the syllogism, $x \in A \Rightarrow x \in C$

i. e. $A \subseteq C$

§ 3.1 Sets and Subsets

b) $A \subset B$ and $B \subseteq C \Rightarrow A \subset C$

Proof. (2/2)

b) ① $\because A \subset B, \therefore x \in A \Rightarrow x \in B$

$\because B \subseteq C, \therefore x \in B \Rightarrow x \in C$

By the law of the syllogism, $x \in A \Rightarrow x \in C$

i. e. $A \subseteq C$

② $\because A \subset B, \therefore \exists b\, [b \in B \wedge b \notin A]$

$\Rightarrow b \in B$

$\because B \subseteq C, \therefore b \in B \Rightarrow b \in C$

$\because [b \in B \wedge b \notin A] \Rightarrow b \notin A$

$\Rightarrow \exists b\, [b \in C \wedge b \notin A]$

i. e. $A \subset C$

c), d) exercise.

§ 3.1 Sets and Subsets

EX 3.6 : Let $\mathscr{U} = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$, $B = \{3, 4\}$,
$C = \{1, 2, 3, 4\}$.

a) $A \subseteq C$ b) $A \subset C$ c) $B \subset C$

d) $A \subseteq A$ e) $B \nsubseteq A$ f) $A \not\subset A$

Def 3.3 : The ***null set*** (or ***empty set***), ϕ (or $\{\}$)
$\equiv$ The set containing no elements.

Note : ① $|\phi| = 0$,
② $\{0\} \neq \phi$; $\{\phi\} \neq \phi$.

§ 3.1 Sets and Subsets

Thm 3.2 : For any universe, let $A \subseteq \mathcal{U}$. Then

① $\phi \subseteq A$.

② If $A \neq \phi$, then $\phi \subset A$.

Proof.

① proof by contradiction :

Assume $\phi \nsubseteq A$, $\therefore \exists x \in \mathcal{U}, [(x \in \phi) \wedge (x \notin A)]$

But $x \in \phi$ is impossible !!

$\therefore \phi \subseteq A$

② If $A \neq \phi$, then $\exists a \in A$.

$\because \forall x \in \mathcal{U}, x \notin \phi$, $\therefore a \notin \phi$

$\therefore \phi \subset A$

§ 3.1 Sets and Subsets

EX 3.7 : ① **Determine the number of subsets of the set $C = \{1, 2, 3, 4\}$.**

$2 \times 2 \times 2 \times 2 = 2^4 = 16$ (include ϕ and C)

② **Determine the number of subsets of two elements from C.**

$C(4, 2) = 6$

③ $\therefore\ 2^4 = C_0^4 + C_1^4 + C_2^4 + C_3^4 + C_4^4 = \sum_{k=0,4} C(4, k)$

Def : **The subset of one element ≡ the *singleton* subset.**

Def 3.4 : **The *power set* of A, denoted by $\mathscr{P}(A)$ (or 2^A) ≡ The collection of all subsets of A.**

§ 3.1 Sets and Subsets

EX 3.8 : $C = \{1, 2, 3, 4\}$

$\mathscr{P}(C) = \{\phi,$
$\{1\}, \{2\}, \{3\}, \{4\},$
$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\},$
$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\},$
$C\}$

Remark : For any finite set A with $|A| = n$, $n \geq 0$

① $|\mathscr{P}(A)| = 2^n$

② $\forall\ 0 \leq k \leq n$, there are $C(n, k)$ subsets of size k.

③ $2^n = \sum_{k=0}^{n} C(n, k)$

EX 3.9 : Gray Code (略)

0	ϕ
1	$\{x\}$

0	0	ϕ
1	0	$\{x\}$
1	1	$\{x, y\}$
0	1	$\{y\}$

0 0	0	ϕ
1 0	0	$\{x\}$
1 1	0	$\{x, y\}$
0 1	0	$\{y\}$
0 1	1	$\{y, z\}$
1 1	1	$\{x, y, z\}$
1 0	1	$\{x, z\}$
0 0	1	$\{z\}$

0 0 0
1 0 0
1 1 0
0 1 0
0 1 1
1 1 1
1 0 1
0 0 1

0 0 0
0 1 0
0 1 1
0 0 1
1 0 1
1 1 1
1 1 0
1 0 0

0 0 0
0 0 1
1 0 1
1 0 0
1 1 0
0 1 0
0 1 1
1 1 1

§ 3.1 Sets and Subsets

EX 3.10 :

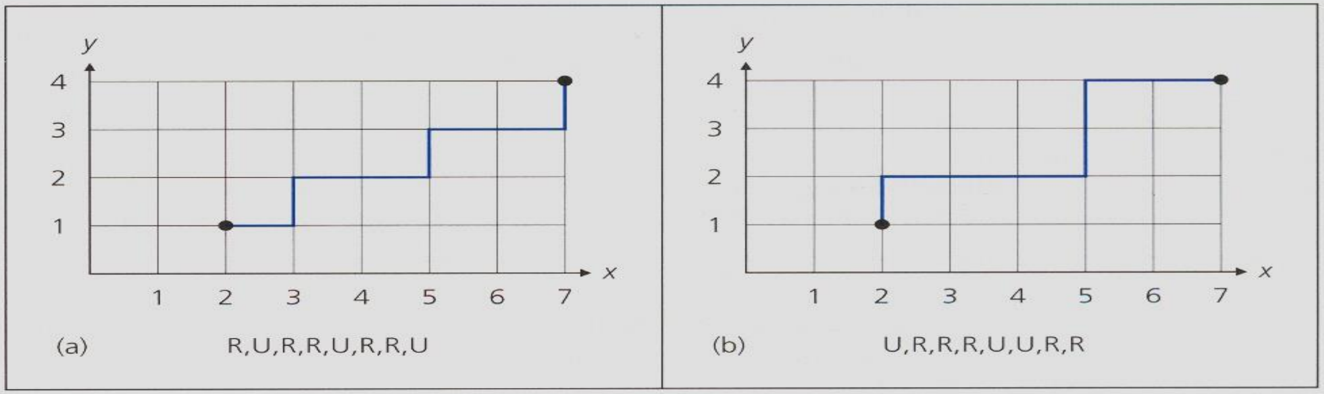

Figure 3.1

(a) R, U, R, R, U, R, R, U ⇨ {2, 5, 8} from {1, 2, 3, 4, 5, 6, 7, 8}

(b) U, R, R, R, U, U, R, R ⇨ {1, 5, 6} from {1, 2, 3, 4, 5, 6, 7, 8}

(c) U, R, U, R, R, R, U, R ⇦ {1, 3, 7} from {1, 2, 3, 4, 5, 6, 7, 8}

The number of paths equals the number of subsets *A* of {1, 2, 3, 4, 5, 6, 7, 8}, where |*A*| = 3.

$$= C(8, 3) = \frac{8!}{3!5!} = 56$$

("U" 改 "R" ⇨ $|B| = 5$ ⇨ $C(8, 5) = \frac{8!}{5!3!} = 56$)

§ 3.1 Sets and Subsets

EX 3.11 : There are 2^6 ways to write 7 as a sum of one or more positive integers = There are 2^6 subsets for {1, 2, 3, 4, 5, 6}

$$1 \underset{\downarrow}{+} 1 \underset{\downarrow}{+} 1 + 1 + 1 \underset{\downarrow}{+} 1 \underset{\downarrow}{+} 1$$

1st plus sign 2nd plus sign . . . 5th plus sign 6th plus sign

① $\{1, 4, 6\} : (1 \underset{1}{+} 1) + 1 + (1 \underset{4}{+} 1) + (1 \underset{6}{+} 1) = 2 + 1 + 2 + 2$

② $\{1, 2, 5, 6\} : (1 \underset{1}{+} 1 \underset{2}{+} 1) + 1 + (1 \underset{5}{+} 1 \underset{6}{+} 1) = 3 + 1 + 3$

③ $1 + 1 + 5 = 1 + 1 + (1 \underset{3}{+} 1 \underset{4}{+} 1 \underset{5}{+} 1 \underset{6}{+} 1) : \{3, 4, 5, 6\}$

Table 3.1

	Composition of 7		Determining Subset of {1, 2, 3, 4, 5, 6}
(i)	1 + 1 + 1 + 1 + 1 + 1 + 1	(i)	∅
(ii)	1 + 2 + 1 + 1 + 1 + 1	(ii)	{2}
(iii)	1 + 1 + 3 + 1 + 1	(iii)	{3, 4}
(iv)	2 + 3 + 2	(iv)	{1, 3, 4, 6}
(v)	4 + 3	(v)	{1, 2, 3, 5, 6}
(vi)	7	(vi)	{1, 2, 3, 4, 5, 6}

§ 3.1 Sets and Subsets

EX 3.12 : For integers n, r with $n \geq r \geq 1$, $C(n+1, r) = C(n, r) + C(n, r-1)$.

Sol.

Let $A = \{x, a_1, a_2, \ldots, a_n\}$

① All subsets of A that contains r elements $= C(n+1, r)$.

② $C \subseteq A$, where $x \in C$ and $|C| = r$: $C(n, r-1)$.

③ $C \subseteq A$, where $x \notin C$ and $|C| = r$: $C(n, r)$.

∵ ① = ② + ③

∴ $C(n+1, r) = C(n, r) + C(n, r-1)$.

Another Sol.

使用EX3.10之方法:

視為(0, 0)到$(n+1-r, r)$之走法:共$C(n+1, r)$

= 最後一步為(i) R : $(n-r, r)$; (ii) U : $(n+1-r, r-1)$

$= C(n, r) + C(n, r-1)$.

EX 3.13 : **Find the number of nonnegative integer solutions of**

$$x_1 + x_2 + \ldots + x_6 < 10.$$

Sol.

$\forall\, k,\ 0 \le k \le 9$, the number of solutions to $x_1 + x_2 + \ldots + x_6 = k$ is $\binom{5+k}{k}$.

$\therefore$ the answer $= \binom{5}{0} + \binom{6}{1} + \binom{7}{2} + \binom{8}{3} + \ldots + \binom{14}{9}$

$= [\binom{6}{0} + \binom{6}{1}] + \binom{7}{2} + \binom{8}{3} + \ldots + \binom{14}{9}$

$= [\binom{7}{1} + \binom{7}{2}] + \binom{8}{3} + \ldots + \binom{14}{9}$

$= [\binom{8}{2} + \binom{8}{3}] + \ldots + \binom{14}{9}$

$= \ldots = \binom{14}{8} + \binom{14}{9} = \binom{15}{9} = 5005.$

<u>EX 3.14</u> : Pascal's triangle.

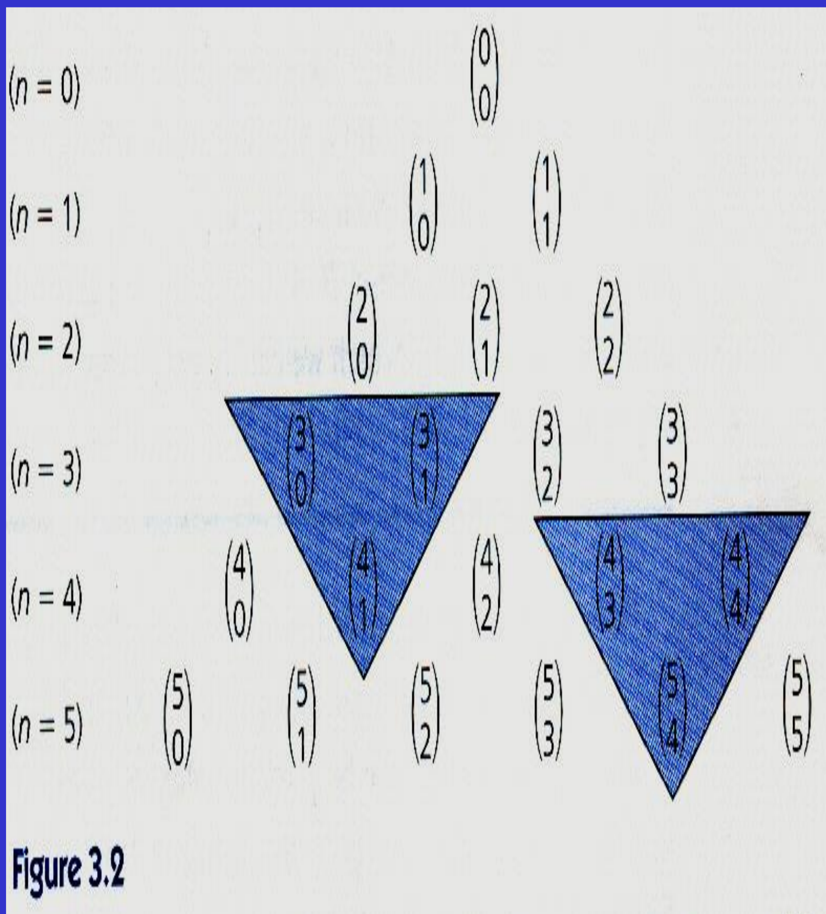

Figure 3.2

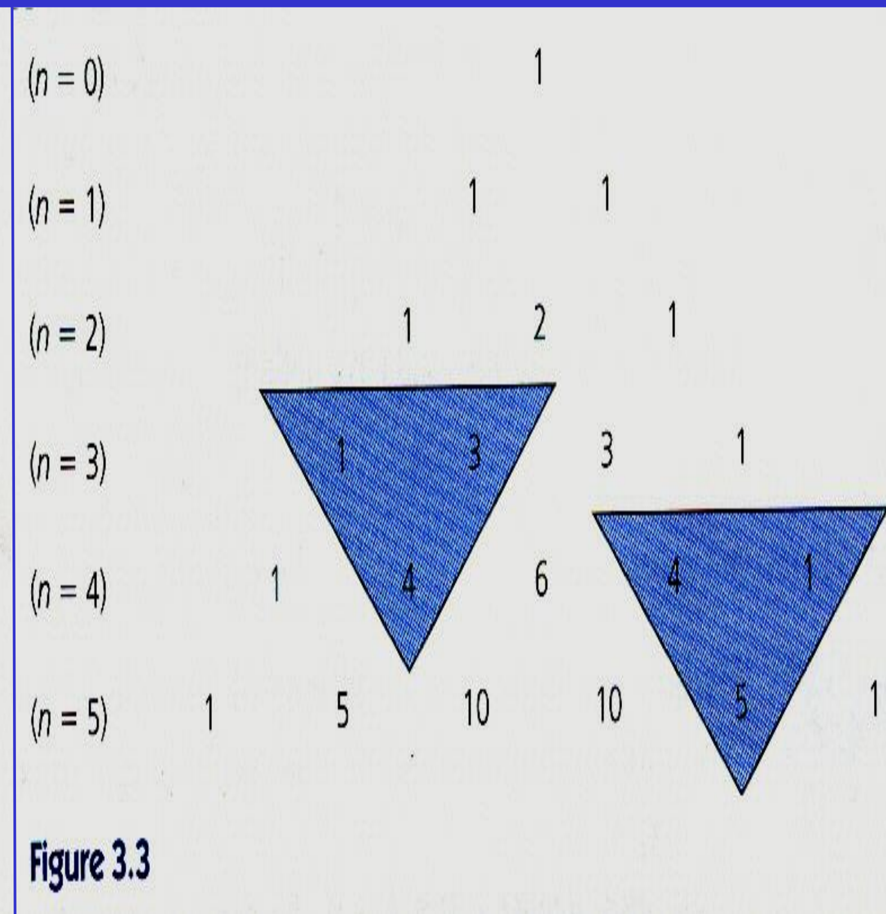

Figure 3.3

§ 3.1 Sets and Subsets

Def :

a) $\mathbf{Z}$ = the set of *integers* = $\{0, 1, -1, 2, -2, 3, -3, \ldots\}$

b) $\mathbf{N}$ = the set of *nonnegative integers* or *natural numbers* = $\{0, 1, 2, 3, \ldots\}$

c) $\mathbf{Z}^+$ = the set of *positive integers* = $\{1, 2, 3, \ldots\} = \{x \in \mathbf{Z} \mid x > 0\}$

d) $\mathbf{Q}$ = the set of *rational numbers* = $\{a/b \mid a, b \in \mathbf{Z}, b \neq 0\}$

e) $\mathbf{Q}^+$ = the set of *positive rational numbers* = $\{r \in \mathbf{Q} \mid r > 0\}$

f) $\mathbf{Q}^*$ = the set of *nonzero rational numbers*

g) $\mathbf{R}$ = the set of *real numbers*

h) $\mathbf{R}^+$ = the set of *positive real numbers*

i) $\mathbf{R}^*$ = the set of *nonzero real numbers*

j) $\mathbf{C}$ = the set of *complex numbers* = $\{x + yi \mid x, y \in \mathbf{R}, i^2 = -1\}$

k) $\mathbf{C}^*$ = the set of *nonzero complex numbers*

l) For each $n \in \mathbf{Z}^+$, $\mathbf{Z}_n = \{0, 1, 2, \ldots, n - 1\}$

m) For real numbers a, b with $a < b$, $[a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$, $(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$, $[a, b) = \{x \in \mathbf{R} \mid a \leq x < b\}$, $(a, b] = \{x \in \mathbf{R} \mid a < x \leq b\}$. The first set is called a *closed interval*, the second set an *open interval*, and the other two sets *half-open intervals*.