Chapter 2 Fundamentals of Logic

§ 2.4 The Use of Quantifiers (2)

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics (5th Edition)
by Ralph P. Grimaldi

(c) Fall 2018, Justie Su-Tzu Juan
§ 2.4 The Use of Quantifiers

Def : 1) free variable: the variable $x$ in each of open statement $p(x)$
    2) bound variable: the variable $x$ in the statement $\exists x \ p(x)$ or $\forall x \ p(x)$

Ex 2.36 : The universe = $R$

$p(x) : x \geq 0 \quad q(x) : x^2 \geq 0 \quad r(x) : x^2 - 3x - 4 = 0 \quad s(x) : x^2 - 3 > 0$

1) $\exists x \ [p(x) \land r(x)]$
   
   Let $x = 4$, then $p(4)$ is true and $r(4)$ is true
   
   $\therefore [p(4) \land r(4)]$ is true $\Rightarrow \exists x \ [p(x) \land r(x)]$ is true.

2) $\forall x \ [p(x) \rightarrow q(x)]$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$p(x)$</th>
<th>$q(x)$</th>
<th>$p(x) \rightarrow q(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; 0$</td>
<td>F</td>
<td>T</td>
<td></td>
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<tr>
<td>$\geq 0$</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>
$\p(x) : x \geq 0 \quad \q(x) : x^2 \geq 0 \quad \r(x) : x^2 - 3x - 4 = 0 \quad \s(x) : x^2 - 3 > 0$

### § 2.4 The Use of Quantifiers

Note: “$\forall x [p(x) \rightarrow q(x)]$” 可能被敘述成：

a) For every real number $x$, if $x \geq 0$, then $x^2 \geq 0$.

b) Every nonnegative real number has a nonnegative square.

c) The square of any nonnegative real number is a nonnegative real number.

d) All nonnegative real numbers have nonnegative squares.

3) $\exists x [p(x) \rightarrow q(x)]$ is true

1’) $\forall x [q(x) \rightarrow s(x)] \quad (1) \quad \exists x [p(x) \land r(x)]$

Let $x = 1$, $q(1)$ is true, but $s(1)$: $1 - 3 > 0$ is false

∴ $q(1) \rightarrow s(1)$ is false, i.e. $x = 1$ is a counterexample

$\Rightarrow \forall x [q(x) \rightarrow s(x)]$ is false.

(不只一個反例，$\forall -\sqrt{3} < x < \sqrt{3}$ 皆是)
§ 2.4 The Use of Quantifiers

2’) \( \forall x [r(x) \lor s(x)] \) ( 2)\( \forall x [p(x) \rightarrow q(x)]\)

Let \( x = 1 \) (or \( \frac{1}{2}, -\frac{3}{2}, 0, \ldots \)), then \( r(1) \) is false and \( s(1) \) is false, too.

\[ \therefore r(1) \lor s(1) \text{ is false} \]

\[ \Rightarrow \forall x [r(x) \lor s(x)] \text{ is false.} \]

2”’) \( \exists x [r(x) \lor s(x)] \) is true.

3’) \( \forall x [r(x) \rightarrow p(x)] \) ( 3)\( \exists x [p(x) \rightarrow q(x)]\)

Let \( x = -1, r(-1) = (-1)^2 - 3(-1) - 4 = 0 \) is true, but \( p(-1) \) is false.

\[ \therefore r(-1) \rightarrow p(-1) \text{ is false. (the unique counterexample)} \]

\[ \Rightarrow \forall x [r(x) \rightarrow p(x)] \text{ is false.} \]

Note: “\( \forall x [r(x) \rightarrow p(x)] \)” 可被敘述成：
a) For every real number \( x \), if \( x^2 - 3x - 4 = 0 \), then \( x \geq 0 \).
b) For every real number \( x \), if \( x \) is a solution of the equation \( x^2 - 3x - 4 = 0 \), then \( x \geq 0 \).
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Remark: 1) $\forall x p(x) \Rightarrow \exists x p(x)$, for the universe $\neq \emptyset$.
   2) $\exists x p(x) \not\Rightarrow \forall x p(x)$

Ex 2.37:
   a) The universe $= R$
      1) If a number is rational, then it is a real number.
      2) If $x$ is rational, then $x$ is real.
         (The use of the universal quantifier is *implicit* as opposed to *explicit*)
         Let $p(x) : x$ is a rational number
         $q(x) : x$ is a real number
         $\Rightarrow (1) = (2) = \forall x [p(x) \rightarrow q(x)]$. 
§ 2.4 The Use of Quantifiers

Ex 2.37:

b) The universe = All triangles in the plane.
   “An equilateral triangle has three angles of 60°, and conversely.”
   Let \( e(t) \) : Triangle \( t \) is equilateral
   \( a(t) \) : Triangle \( t \) has three angles of 60°
   \[ \Rightarrow \quad \text{“”} = \forall t \ [e(t) \iff a(t)] \]

c) \( \sin^2 x + \cos^2 x = 1 \) (for all real number \( x \))
   \[ \Rightarrow \quad \text{The universe of } x = \mathbb{R}, \forall x \ [\sin^2 x + \cos^2 x = 1] \]

d) The universe = \( \mathbb{N} \)
   “The integer 41 is equal to the sum of two perfect squares.”
   \[ \Rightarrow \exists m \exists n \ [41 = m^2 + n^2] \]
§ 2.4 The Use of Quantifiers

Ex 2.38 : \( p(x) : x^2 \geq 0 \)

1) The universe = \( R \) : \( \forall x \ p(x) \) is true \quad (\exists x \ p(x) \) is true
2) The universe = \( C \) : \( \forall x \ p(x) \) is false \quad (\exists x \ p(x) \) is true

\[ \therefore \text{let } x = i, \text{ then } p(i) : i^2 (= -1) \geq 0 \text{ is false.} \]

Ex 2.39 : See Textbook

Table 2.21 : See Textbook

Def 2.6 : Let \( p(x), q(x) \) be open statements defined for a given universe.

1) \( p(x) \) is (logically) equivalent to \( q(x) \) : \( \forall x \ [p(x) \iff q(x)] \) :
   \( p(x) \iff q(x) \) for each \( x \) in the universe.
2) \( p(x) \) logically implies \( q(x) \) : \( \forall x \ [p(x) \Rightarrow q(x)] \) :
   \( p(x) \Rightarrow q(x) \) for each \( x \) in the universe.
The Use of Quantifiers

ex: The universe: all triangles in the plane.

Let \( p(x) \): \( x \) is equiangular, \( q(x) \): \( x \) is equilateral

\[ \therefore \text{for all particular triangle } a, \ p(a) \leftrightarrow q(a) \text{ is true.} \]

\[ \therefore \forall x \ [p(x) \iff q(x)] \]

Note: 1) \( \forall x \ [p(x) \iff q(x)] \) iff \( \forall x \ [p(x) \implies q(x)] \land \forall x \ [q(x) \implies p(x)] \)

2) Def 2.6 can be given for two open statements that involve \( \geq 2 \) variable.

Def 2.7:

1) The **contrapositive** of \( \forall x \ [p(x) \implies q(x)] \) is \( \forall x \ [\neg q(x) \implies \neg p(x)] \)

2) The **converse** of \( \forall x \ [p(x) \implies q(x)] \) is \( \forall x \ [q(x) \implies p(x)] \)

3) The **inverse** of \( \forall x \ [p(x) \implies q(x)] \) is \( \forall x \ [\neg p(x) \implies \neg q(x)] \)
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Ex 2.40: The universe = all quadrilaterals in the plane
Let \( s(x) : x \) is a square, \( e(x) : x \) is equilateral

a) \( \forall x [s(x) \rightarrow e(x)] \) is a true statement
   \( \iff \forall x [\neg e(x) \rightarrow \neg s(x)] \) (the contrapositive)

b) \( \forall x [e(x) \rightarrow s(x)] \) is a false statement (the converse)
   \( \iff \forall x [\neg s(x) \rightarrow \neg e(x)] \) (the inverse)

Ex 2.41: The universe = \( R \)
Let \( p(x) : |x| > 3, \ q(x) : x > 3 \)

a) \( \forall x [p(x) \rightarrow q(x)] \) is false. (let \( x = -5, p(-5) : T, q(-5) : F \))

b) The converse of (a) = Every real number greater than 3 has magnitude (or, absolute value) greater than 3.
   \( \forall x [q(x) \rightarrow p(x)] \) is true.
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Ex 2.41:

c) The inverse of (a) is also true: \( \forall x [\neg p(x) \rightarrow \neg q(x)] \):

“If the magnitude of a real number is less than or equal to 3, then the number itself is less than or equal to 3.”

(b) \( \iff \) (c)

(d) The contrapositive of (a) = \( \forall x [\neg q(x) \rightarrow \neg p(x)] \) (is false):

“If a real number is less than or equal to 3, then so is its magnitude.”

(d) \( \iff \) (a)

e) Let \( r(x) : x < -3 \), defined for the universe of all real number:

Statement: \( \forall x [p(x) \rightarrow (r(x) \lor q(x))] \)

Contrapositive: \( \forall x [\neg (r(x) \lor q(x)) \rightarrow \neg p(x)] \)

Converse: \( \forall x [(r(x) \lor q(x)) \rightarrow p(x)] \)

Inverse: \( \forall x [\neg p(x) \rightarrow \neg (r(x) \lor q(x))] \)

\( \Rightarrow \forall x [p(x) \iff (r(x) \lor q(x)))] \)
§ 2.4 The Use of Quantifiers

Ex 2.42 : The universe = \( \mathbb{Z} \)

Let \( r(x) : 2x + 1 = 5 \), \( s(x) : x^2 = 9 \).

1) \( \exists x \ [r(x) \land s(x)] \) is false
   
   \therefore no integer \( a \) such that \( 2a + 1 = 5 \) and \( a^2 = 9 \)

2) \( \exists x \ r(x) \land \exists x \ s(x) \) is true
   
   \therefore \( \exists b = 2, r(b) : 2b + 1 = 5 \) is true
   
   \exists c = 3, s(c) : c^2 = 9 is true

3) \( \exists x \ [r(x) \land s(x)] \Leftrightarrow [\exists x \ r(x) \land \exists x \ s(x)] \)
   
   \[ \exists x \ r(x) \land \exists x \ s(x) \] \nleftrightarrow \exists x \ [r(x) \land s(x)]

Def : 1) \( \Leftrightarrow \) is read “is not logically equivalent to”

2) \( \nleftrightarrow \) is read “does not logically imply”
Note: $\exists x [p(x) \land q(x)] \Rightarrow [\exists x p(x) \land \exists x q(x)]$

Proof.

If $\exists x [p(x) \land q(x)]$ is true then there is at least one element $c$ in the universe s.t. $p(c) \land q(c)$ is true.

$\therefore [p(c) \land q(c)] \Rightarrow p(c)$ and $[p(c) \land q(c)] \Rightarrow q(c)$

(by Conjunctive Simplification)

i.e. $\exists x p(x)$ is true and $\exists x q(x)$ is true

$\therefore [\exists x p(x) \land \exists x q(x)]$ is true.

Table 2.22 : See Textbook
Ex 2.43: Let $p(x)$, $q(x)$ and $r(x)$ denote open statements for a given universe.

1) $\forall x [p(x) \land (q(x) \land r(x))] \iff \forall x [(p(x) \land q(x)) \land r(x)]$

   $\therefore$ For each $a$ in the universe,

   $p(a) \land (q(a) \land r(a)) \iff (p(a) \land q(a)) \land r(a)$

   $\therefore \forall x [p(x) \land (q(x) \land r(x))] \iff \forall x [(p(x) \land q(x)) \land r(x)]$

2) $\exists x [p(x) \to q(x)] \iff \exists x [\neg p(x) \lor q(x)]$

   $\therefore$ For each $c$ in the universe,

   $[p(c) \to q(c)] \iff [\neg p(c) \lor q(c)]$

   $\therefore \exists x [p(x) \to q(x)]$ is true iff $\exists x [\neg p(x) \lor q(x)]$ is true

   i.e. $\exists x [p(x) \to q(x)] \iff \exists x [\neg p(x) \lor q(x)]$
§ 2.4 The Use of Quantifiers

Ex 2.43:

3) a) $\forall x \neg \neg p(x) \iff \forall x p(x)$
    b) $\forall x \neg [p(x) \wedge q(x)] \iff \forall x [\neg p(x) \vee \neg q(x)]$
    c) $\forall x \neg [p(x) \vee q(x)] \iff \forall x [\neg p(x) \wedge \neg q(x)]$

4) a) $\exists x \neg \neg p(x) \iff \exists x p(x)$
    b) $\exists x \neg [p(x) \wedge q(x)] \iff \exists x [\neg p(x) \vee \neg q(x)]$
    c) $\exists x \neg [p(x) \vee q(x)] \iff \exists x [\neg p(x) \wedge \neg q(x)]$
Remark: \( \neg [\forall x \ p(x)] \iff \exists x \ \neg p(x) \)

Proof.

\( \neg [\forall x \ p(x)] \) : It is not the case that for all \( x \), \( p(x) \) holds.

1) “\( \neg [\forall x \ p(x)] \)” is true = “\( \forall x \ p(x) \)” is false
   = For some \( a \) of the universe, \( \neg p(a) \) is true
   = “\( \exists x \ \neg p(x) \)” is true

2) “\( \exists x \ \neg p(x) \)” is true = For some \( b \) of the universe,
   \( \neg p(b) \) is true
   = For some \( b \) of the universe, \( p(b) \) is false
   = “\( \forall x \ p(x) \)” is false
   = “\( \neg [\forall x \ p(x)] \)” is true

by (1), (2) “\( \neg [\forall x \ p(x)] \)” is true iff “\( \exists x \ \neg p(x) \)” is true

Similar, “\( \neg [\forall x \ p(x)] \)” is false iff “\( \exists x \ \neg p(x) \)” is false
§ 2.4 The Use of Quantifiers

Table 2.23:

| ¬ [∀x p(x)] | ⇔ | ∃x ¬ p(x) |
| ¬ [∃x p(x)] | ⇔ | ∀x ¬ p(x) |
| ¬ [∀x ¬ p(x)] | ⇔ | ∃x ¬ ¬ p(x) ⇔ ∃x p(x) |
| ¬ [∃x ¬ p(x)] | ⇔ | ∀x ¬ ¬ p(x) ⇔ ∀x p(x) |

Rules of Negating Statements with One Quantifier.
§ 2.4 The Use of Quantifiers

Ex. 2.44 : The universe = \( \mathbb{Z} \)

1) Let \( p(x) : x \) is odd, \( q(x) : x^2 - 1 \) is even

“If \( x \) is odd, then \( x^2 - 1 \) is even” = \( \forall x [p(x) \rightarrow q(x)] \) (T)

\( \neg [\forall x (p(x) \rightarrow q(x))] \)

\( \iff \exists x [\neg (p(x) \rightarrow q(x))] \iff \exists x [\neg (\neg p(x) \lor q(x))] \)

\( \iff \exists x [\neg \neg p(x) \land \neg q(x)] \iff \exists x [p(x) \land \neg q(x)] \)

= “There exists an integer \( x \) such that \( x \) is odd and \( x^2 - 1 \) is odd” (F)

2) \( r(x) : 2x + 1 = 5, s(x) : x^2 = 9 \) (In Ex 2.42)

\( \exists x [r(x) \land s(x)] \) is false, \( \neg [\exists x (r(x) \land s(x))] \) is true?

\( \neg [\exists x (r(x) \land s(x))] \iff \forall x [\neg (r(x) \land s(x))] \)

\( \iff \forall x [\neg r(x) \lor \neg s(x)] \)

= “For every integer \( x \) , \( 2x + 1 \neq 5 \) or \( x^2 \neq 9 \).”
§ 2.4 The Use of Quantifiers

More than one quantifier (More than one variable)

Ex 2.45 : $\forall x \, \forall y \, p(x, y) \iff \forall y \, \forall x \, p(x, y)$

ex : The universe = $R$

1) $\forall x \, \forall y \, (x + y = y + x) \iff \forall y \, \forall x \, (x + y = y + x)$
2) $\forall x \, \forall y \, (xy = yx) \iff \forall y \, \forall x \, (xy = yx)$

Ex 2.46 : $\forall x, y, z \, p(x, y, z) \iff \forall y, x, z \, p(x, y, z)$

$\iff \forall x, z, y \, p(x, y, z)$ ...六種

$(\forall x, y, z \equiv \forall x, \forall y, \forall z )$

ex : The universe = $R$

$\forall x \, \forall y \, \forall z \, [x + (y + z) = (x + y) + z]$

$\iff \forall y \, \forall x \, \forall z \, [x + (y + z) = (x + y) + z]$
\textbf{Ex 2.47} : \( \exists x \ \exists y \ p(x, y) \iff \exists y \ \exists x \ p(x, y) \)

\textbf{ex} : For the universe of all integer

"There exist integer \( x, y \) such that \( x + y = 6 \)"

\( \exists x \ \exists y \ (x + y = 6) \iff \exists y \ \exists x \ (x + y = 6) \)
§ 2.4 The Use of Quantifies

Both \( \exists \) and \( \forall \)

Ex 2.48 : The universe = \( \mathbb{Z} \), let \( p(x, y) : x + y = 17 \)

1) \( \forall x \exists y p(x, y) : \)
   “For every integer \( x \), there exist an integer \( y \) such that \( x + y = 17 \)” (T)
   \( \therefore \forall x : \text{let } y = 17 - x \text{ is a integer} \quad \therefore x + y = x + (17 - x) = 17 \)
   (Each \( x \) gives rise to a different value of \( y \))

2) \( \exists y \forall x p(x, y) : \)
   “There exist an integer \( y \) so that for all integer \( x \), \( x + y = 17 \)” (F)
   \( \therefore \text{Once an integer } y \text{ is selected, only } x \text{ satisfy } x + y = 17 \text{ is } y = 17 \rightarrow \leftarrow \)

\( \therefore \forall x \exists y p(x, y) \iff \exists y \forall x p(x, y) \)

Note : See Textbook (1. careful & precise, 2. related statements)
§ 2.4 The Use of Quantifies

Ex 2.49: The universe of $x, y$ are equal. What is the negation of

$$\forall x \exists y [(p(x, y) \land q(x, y)) \rightarrow r(x, y)]$$

Sol.

$$\neg [\forall x \exists y [(p(x, y) \land q(x, y)) \rightarrow r(x, y)]]$$

$$\iff \exists x [\neg [\exists y [(p(x, y) \land q(x, y)) \rightarrow r(x, y)]]]$$

$$\iff \exists x \forall y [\neg [(p(x, y) \land q(x, y)) \rightarrow r(x, y)]]$$

$$\iff \exists x \forall y [\neg [(p(x, y) \land q(x, y)) \lor \neg r(x, y)]]$$

$$\iff \exists x \forall y [(p(x, y) \land q(x, y)) \land \neg r(x, y)]$$

$$\Rightarrow$$ When we want to prove “$\forall x \exists y [(p(x, y) \land q(x, y)) \rightarrow r(x, y)]$” as the conclusion, use the method of Proof by Contradiction. We would assume the negation of this conclusion as an additional premise. i.e. the additional premise = “$\exists x \forall y [(p(x, y) \land q(x, y)) \land \neg r(x, y)]$”
Ex 2.50 : In calculus, the definition of \textit{limit}:

Let $I$ be an open interval containing the real number $a$ and suppose the function $f$ is defined throughout $I$, except possibly at $a$. We say $f$ has the limit $L$ as $x$ approaches $a$, and write $\lim_{x \to a} f(x) = L$, iff for every $\varepsilon > 0$, there exist a $\delta > 0$ so that, for all $x$ in $I$, $(0 < |x - a| < \delta) \implies (|f(x) - L| < \varepsilon)$.

This can be expressed in symbolic form as:

$$ \lim_{x \to a} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0$$

$$ \forall x \ [(0 < |x - a| < \delta) \implies (|f(x) - L| < \varepsilon)]$$

The universe = the real number in the open interval $I$, except possibly $a$.

To negate this definition:
§ 2.4 The Use of Quantifiers

Sol.

\[ \lim_{x \to a} f(x) \neq L \]
\[ \iff \neg \forall \varepsilon > 0, \exists \delta > 0 \forall x \ [(0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \varepsilon)] \]
\[ \iff \exists \varepsilon > 0, \forall \delta > 0 \exists x \neg [(0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \varepsilon)] \]
\[ \iff \exists \varepsilon > 0, \forall \delta > 0 \exists x \neg [\neg (0 < |x - a| < \delta) \lor (|f(x) - L| < \varepsilon)] \]
\[ \iff \exists \varepsilon > 0, \forall \delta > 0 \exists x \ [(0 < |x - a| < \delta) \land \neg (|f(x) - L| < \varepsilon)] \]
\[ \iff \exists \varepsilon > 0, \forall \delta > 0 \exists x \ [(0 < |x - a| < \delta) \land (|f(x) - L| \geq \varepsilon)] \]

\[ \therefore \lim_{x \to a} f(x) \neq L \iff \text{there exist a positive real number } \varepsilon \text{ such that for every positive real number } \delta, \text{ there is an } x \text{ in } I \]

\[ \text{such that } 0 < |x - a| < \delta \text{ (this is, } x \neq a \text{ and its distance from } a \text{ is less than } \delta \text{), but } |f(x) - L| \geq \varepsilon \text{ (this is, the value of } f(x) \text{ differs form } L \text{ by at least } \varepsilon). \]
Chapter 2 Fundamentals of Logic
§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

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(c) Fall 2018, Justie Su-Tzu Juan
Note: In definition, an implication can be read as a biconditional, and, only in definition.

Ex 2.51: (1/2)

a) Universe: all quadrilaterals in the plane.
   A: “If a quadrilateral is a rectangle then it has four equal angles.”
   B: “If a quadrilateral has four equal angles, then it is a rectangle.”

Let $p(x): x$ is a rectangle. $q(x): x$ has four equal angles
A: $\forall x [p(x) \rightarrow q(x)]$
B: $\forall x [q(x) \rightarrow p(x)]$
Actually, they are both intending: $\forall x [p(x) \leftrightarrow q(x)]$
§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Ex 2.51 : (2/2)

b) Universe = \( \mathbb{Z} \)

A : “For every integer \( n \), we call \( n \) even if it is divisible by 2.”

Let \( p(n) : n \) is an even integer

\( q(n) : n \) is divisible by 2 \begin{align*}
\text{(or, } n &= 2k, \text{ for some integer } k) \\
\text{(or, } \exists k \ [n = 2k])
\end{align*}

A : \( \forall n \ [q(n) \rightarrow p(n)] \)

Actually, \( \forall n \ [p(n) \leftrightarrow q(n)] \)
Ex 2.52: Universe = \{2, 4, 6, \ldots, 26\}

For all \( n \) \((n = 2, 4, \ldots, 26)\), we can write \( n \) as the sum of at most three perfect squares.

Sol. \textit{method of exhaustion}: (不只一種，但我們只需要找出其中一種即可！)

\[
\begin{align*}
2 &= 1 + 1 & 10 &= 9 + 1 & 18 &= 16 + 1 + 1 \ (= 9 + 9) \\
4 &= 4 & 12 &= 4 + 4 + 4 & 20 &= 16 + 4 \\
6 &= 4 + 1 + 1 & 14 &= 9 + 4 + 1 & 22 &= 9 + 9 + 4 \\
8 &= 4 + 4 & 16 &= 16 & 24 &= 16 + 4 + 4 \\
26 &= 25 + 1 \ (= 16 + 9 + 1)
\end{align*}
\]

Def: \textit{Corollary}: follow immediately from a \textit{theorem}

\textbf{The Rule of Universal Specification:}

\( \forall x \) for a given universe, \( p(x) \) is true, the \( p(a) \) is true for each \( a \) in the universe.
§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Ex 2.53 : (1/4)

a) Universe = all people

\[ m(x) : x \text{ is a mathematics professor} , \quad c(x) : x \text{ has studied calculus} \]

All mathematics professors have studied calculus.
Leona is a mathematics professor.
Therefore Leona has studied calculus.

Let \( \mathcal{I} = \text{Leona} \) (in our universe) then :

<table>
<thead>
<tr>
<th>Steps</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) ( \forall x \ [m(x) \rightarrow c(x)] )</td>
<td>Premise</td>
</tr>
<tr>
<td>2) ( m(\mathcal{I}) )</td>
<td>Premise</td>
</tr>
<tr>
<td>3) ( m(\mathcal{I}) \rightarrow c(\mathcal{I}) )</td>
<td>(1) &amp; the Rule of Universal Specification</td>
</tr>
<tr>
<td>4) ( \therefore c(\mathcal{I}) )</td>
<td>(2), (3) and the Rule of Detachment</td>
</tr>
</tbody>
</table>
§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Ex 2.53 : (2/4)

b) Universe = all triangles in the plane

\[ p(t) : t \text{ has two sides of equal length} \]
\[ q(t) : t \text{ is an isosceles triangle} \]
\[ r(t) : t \text{ has two angles of equal measure} \]

\[ c : \text{triangle } XYZ \]

See Textbook.

<table>
<thead>
<tr>
<th>Step</th>
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<tbody>
<tr>
<td>1) ( \forall t [p(t) \rightarrow q(t)] )</td>
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<tr>
<td>2) ( p(c) \rightarrow q(c) )</td>
<td>(1) and the Rule of Universal Specification</td>
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<tr>
<td>3) ( \forall t [q(t) \rightarrow r(t)] )</td>
<td>Premise</td>
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<td>4) ( q(c) \rightarrow r(c) )</td>
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<tr>
<td>5) ( p(c) \rightarrow r(c) )</td>
<td>(2), (4) and the Law of the Syllogism</td>
</tr>
<tr>
<td>6) ( \neg r(c) )</td>
<td>Premise</td>
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<tr>
<td>7) ( \therefore \neg p(c) )</td>
<td>(5), (6) and Modus Tollens</td>
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§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Ex 2.53 : (3/4)

c) Universe = student at a particular college.

\[ m : \text{Mary Gusberti, a student of this college.} \]
\[ j(x) : x \text{ is a junior.} \]
\[ s(x) : x \text{ is a senior.} \]
\[ p(x) : x \text{ is enrolled in a physical education class} \]

\begin{align*}
\text{No junior or senior is enrolled in a physical education class} \\
\text{Mary Gusberti is enrolled in a physical education class} \\
\text{Therefore Mary Gusberti is not a senior}
\end{align*}

\[ \forall x [(j(x) \lor s(x)) \rightarrow \neg p(x)] \]
\[ p(m) \]
\[ \therefore \neg s(m) \]
Ex 2.53 : (4/4)

c) Sol.

<table>
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<tr>
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<tbody>
<tr>
<td>1) $\forall x [(j(x) \lor s(x)) \rightarrow \neg p(x)]$</td>
<td>Premise</td>
</tr>
<tr>
<td>2) $p(m)$</td>
<td>Premise</td>
</tr>
<tr>
<td>3) $(j(m) \lor s(m)) \rightarrow \neg p(m)$</td>
<td>(1) and the Rule of Universal Specification</td>
</tr>
<tr>
<td>4) $p(m) \rightarrow \neg (j(m) \lor s(m))$</td>
<td>(3) and $(q \rightarrow t) \Leftrightarrow (\neg t \rightarrow \neg p)$, and Law of Double Negation</td>
</tr>
<tr>
<td>5) $p(m) \rightarrow (\neg j(m) \land \neg s(m))$</td>
<td>(4) and DeMorgan’s Law</td>
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<tr>
<td>6) $\neg j(m) \land \neg s(m)$</td>
<td>(2), (5) and the Rule of Detachment</td>
</tr>
<tr>
<td>7) $\therefore \neg s(m)$</td>
<td>(6) and the Rule of Conjunctive Simplification</td>
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</table>
§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Note: the Rule of Universal Specification + Modus Ponens, Modus Tollens

c: a member of the fixed universe

p(x), q(x): open statements defined for this universe

(1) \( \forall x \ [p(x) \rightarrow q(x)] \)

\[
\begin{align*}
p(c) & \quad \therefore q(c)
\end{align*}
\]

(2) \( \forall x \ [p(x) \rightarrow q(x)] \)

\[
\begin{align*}
\neg q(c) & \quad \therefore \neg p(c)
\end{align*}
\]
§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Ex (1/2) : Universe = all polygons in the plane
   c : quadrilateral $EFGH$, where $\angle E = 91^\circ$
   $p(x) : x$ is a square
   $q(x) : x$ has four sides

(1’) All squares have four sides,
   Quadrilateral $EFGH$ has four sides
   Therefore quadrilateral $EFGH$ is a square

(1’’)
   $\forall x [p(x) \rightarrow q(x)]$
   $q(c)$
   $\therefore p(c)$ ←— false

   $\therefore \forall x [p(x) \rightarrow q(x)]$ and $c$ is a polygon in the plane
   $\therefore p(c) \rightarrow q(c)$, but $[p(c) \rightarrow q(c)] \land q(c) \not\Rightarrow p(c)$
   $\therefore$ invalid!!
   (converse)
§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Ex (2/2) :
(2’) All squares have four sides
Quadrilateral $EFGH$ is not a square
Therefore quadrilateral $EFGH$ does not have four sides

(2’’)
\( \forall x \ [p(x) \rightarrow q(x)] \)
\[
\neg p(c)
\]
\[
\therefore \neg q(c)
\]
\[
\therefore \forall x \ [p(x) \rightarrow q(x)] \text{ and } c \text{ is a polygon in the plane}
\]
\[
\therefore p(c) \rightarrow q(c), \text{ but } [(p(c) \rightarrow q(c)) \land \neg p(c)] \nRightarrow \neg q(c)
\]
\[
\therefore \text{invalid}!!
\]
(inverse)
§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

The Rule of Universal Generalization:

(1) If \( p(c) \) is true for any arbitrarily chosen element \( c \) from our universe, then \( \forall x \ p(x) \) is true.

(2) Similar results hold for the cases of two or three or more variables.

Ex 2.54: Let \( p(x) \), \( r(x) \) be open statements that are defined for a given universe.

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<tr>
<td>(2) ( p(c) \rightarrow q(c) )</td>
<td>(1) &amp; the Rule of Universal Specification</td>
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<tr>
<td>(3) ( \forall x \ [q(x) \rightarrow r(x)] )</td>
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<tr>
<td>(4) ( q(c) \rightarrow r(c) )</td>
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<td>(5) ( p(c) \rightarrow r(c) )</td>
<td>(2), (4) and the Law of the Syllogism</td>
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<tr>
<td>(6) ( \therefore \ \forall x \ [p(x) \rightarrow r(x)] )</td>
<td>(5) &amp; the Rule of Universal Generalization</td>
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§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Ex 2.55:
(a) Universe: all real number
   \( p(x) : 3x - 7 = 20, \quad q(x) : 3x = 27, \quad r(x) : x = 9 \)
   1) If \( 3x - 7 = 20 \), then \( 3x = 27 \). \( \forall x [p(x) \rightarrow q(x)] \)
   2) If \( 3x = 27 \), then \( x = 9 \). \( \forall x [q(x) \rightarrow r(x)] \)
   3) Therefore, if \( 3x - 7 = 20 \), then \( x = 9 \). \( \therefore \forall x [p(x) \rightarrow r(x)] \)

(b) Universe: all quadrilaterals in plane geometry
   “Since every square is a rectangle, and every rectangle is a parallelogram, it follows that every square is a parallelogram”
   \( p(x) : x \) is a square \( q(x) : x \) is a rectangle \( r(x) : x \) is a parallelogram
   \( \forall x [p(x) \rightarrow q(x)] \) \( \quad \therefore \quad 1 \)
   By Ex 2.54:
   \( \forall x [q(x) \rightarrow r(x)] \) \( \quad \therefore \quad 2 \)
   \( \therefore \forall x [p(x) \rightarrow r(x)] \) \( \quad \therefore \quad 3 \)
Let $c$ be an element in the universe assigned for the argument. Assume $\neg r(c)$ as an additional premise.

### Steps

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<td>(1) $\forall x \ [p(x) \lor q(x)]$</td>
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<td>(2) $p(c) \lor q(c)$</td>
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<tr>
<td>(3) $\forall x \ [\neg p(x) \land q(x)) \rightarrow r(x)]$</td>
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<tr>
<td>(4) $\neg p(c) \land q(c) \rightarrow r(c)$</td>
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<tr>
<td>(5) $\neg r(c) \rightarrow \neg \neg p(c) \land q(c)$</td>
<td>(4) and $s \rightarrow t \iff \neg t \rightarrow \neg s$</td>
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<tr>
<td>(6) $\neg r(c) \rightarrow [p(c) \lor \neg q(c)]$</td>
<td>(5), DeMorgan’s Law &amp; the Law of Double Negation</td>
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**Ex 2.56 (2/2):**

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<td>Premise (assumed)</td>
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<tr>
<td>(8) ( p(c) \lor \neg q(c) )</td>
<td>(7), (6) and Modus Ponens</td>
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<tr>
<td>(9) ([p(c) \lor q(c)] \land [p(c) \lor \neg q(c)])</td>
<td>(2), (8) and the Rule of Conjunction</td>
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<tr>
<td>(10) ( p(c) \lor [q(c) \land \neg q(c)] )</td>
<td>(9) &amp; the Distributive Law of ( \lor ) over ( \land )</td>
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<tr>
<td>(11) ( p(c) )</td>
<td>(10) &amp; Inverse &amp; Identity Law</td>
</tr>
<tr>
<td>(12) ( \therefore \forall x [\neg r(x) \rightarrow p(x)] )</td>
<td>(7), (11) &amp; the Rule of Universal Generalization</td>
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</table>

**Remark:**

1) For convenience: using the letter \( x \) instead of \( c \)

2) 將省略步驟以免過於瑣碎，除非必要
Def 2.8 : 1) Let \( n \) be an integer. We call \( n \) **even** if \( n \) is divisible by 2. i.e. \( \exists r \in \mathbb{Z} \text{ s.t. } n = 2r \).

2) We call \( n \) **odd** if \( \exists s \in \mathbb{Z} \text{ s.t. } n = 2s + 1 \).

Theorem 2.2 : \( \forall k, l \in \mathbb{Z}, \) if \( k, l \) are both odd, then \( k + l \) is even.

**Proof.**

1) \( \therefore k, l \) are odd,

   \( \therefore \exists a, b \in \mathbb{Z} \text{ s.t. } k = 2a + 1, l = 2b + 1 \) (by **Def 2.8**)

2) Then \( k + l = (2a + 1) + (2b + 1) = 2(a + b + 1) \)

3) \( \therefore a, b \in \mathbb{Z} \therefore a + b + 1 = c \) is an integer

   i.e. \( k + l = 2c \),

   by Def 2.8, \( k + l \) is even.
Remark:

1) In Step (1), $k, l :$ chosen in an arbitrary manner. ∴ by the Rule of Universe Generalization, the result obtained is true for all odd integers.

2) Use the Rule of Universe Specification twice in step (1): $(l \text{ 同})$

   i) $n$ is an odd integer $\rightarrow n = 2r + 1$ for some integer $r$.
   ii) $k$ is a specific, arbitrarily chosen odd integer.
   iii) Therefore $k = 2a + 1$ for some integer $a$.

3) $k = l \iff a = \frac{(k - 1)}{2} = \frac{(l - 1)}{2} = b,$

   but ∴ $k$ may not equal to $l,$
   ∴ use the different variable $a, b.$
§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Ex 2.57 : Universe : Z

If $n$ is an integer, then $n^2 = n$. ($\forall n \ [n^2 = n]$)

Sol.

$n = 0$, $n^2 = 0^2 = 0 = n$.

$n = 1$, $n^2 = 1^2 = 1 = n$.

But, we can not conclude $n^2 = n$, $\forall n$.

We can not consider the choice of 0 (or 1) as an arbitrarily chosen integer!!

If $n = 2$, $n^2 = 4 \neq 2 = n$, is one counterexample!

$\therefore$ the given statement is false!!

$(n = 0 \ or \ n = 1 \ is \ enough \ to \ say: \ \exists \ n \ [n^2 = n].)$
Theorem 2.3: \( \forall \text{ integer } k, l, \text{ if } k, l \text{ are both odd, then } k \cdot l \text{ is also odd.} \)

Proof.

\[ \because k, l \text{ are both odd,} \]
\[ \therefore \exists a, b \in \mathbb{Z} \text{ s.t. } k = 2a + 1, l = 2b + 1 \text{ (by Def 2.8)} \]
\[ \therefore k \cdot l = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 \]
\[ = 2(2ab + a + b) + 1 \]

where \( 2ab + a + b \in \mathbb{Z} \)

Therefore, by Def 2.8, \( k \cdot l \) is odd.
Theorem 2.4 (1/2) : If \( m \) is an even integer, then \( m + 7 \) is odd.

Proof. (by three methods)

1) By a direct argument:
\[
\therefore \ m \text{ is even} \quad \therefore \ \exists a \in \mathbb{Z} \text{ s.t. } m = 2a.
\]
Then \( m + 7 = 2a + 7 = 2(a + 3) + 1. \)
\[
\therefore \ a + 3 \in \mathbb{Z} \quad \therefore \ m + 7 \text{ is odd.}
\]

2) Prove by the contrapositive method:
Suppose \( m + 7 \) is not odd, hence even.
\[
\therefore \ \exists b \in \mathbb{Z} \text{ s.t. } m + 7 = 2b,
\]
then \( m = 2b - 7 = 2b - 8 + 1 = 2(b - 4) + 1. \)
\[
\therefore \ b - 4 \in \mathbb{Z} \quad \therefore \ m = 2(b - 4) + 1 \text{ is odd.}
\]
Therefore, If \( m \) is an even integer, then \( m + 7 \) is odd.

\[
(\therefore \ \forall m [p(m) \rightarrow q(m)] \iff \forall m [\neg q(m) \rightarrow \neg p(m)])
\]
Theorem 2.4 (2/2) : If $m$ is an even integer, then $m + 7$ is odd.
 Proof. (by three methods)

3) Proof by “the method of proof by contradiction”:
Assume $m$ is even and that $m + 7$ is also even.

$\therefore \exists c \in \mathbb{Z} \text{ s.t. } m + 7 = 2c,$
then $m = 2c - 7 = 2(c - 4) + 1.$

$\therefore c - 4 \in \mathbb{Z} \quad \therefore m$ is odd $\quad \rightarrow \leftarrow$

($\therefore$ no integer can be both even and odd!!)
i.e. $m + 7$ is even is a false assumption,
$\therefore m + 7$ is odd.
Ex: If we want to prove: $\forall m [p(m) \rightarrow q(m)]$

1) Prove this result by the contrapositive method:
   \[\text{prove}: \forall m [\neg q(m) \rightarrow \neg p(m)]\]

2) Prove by the method of proof by contradiction:
   \[\text{prove}: \text{assume } \forall m [p(m) \rightarrow q(m)]\] is false will implies $F_0$
   
   i.e. $\exists m [p(m) \land \neg q(m)] \rightarrow F_0$

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<th>Result Derived</th>
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<td>$\neg q(m)$</td>
<td>$\neg p(m)$</td>
</tr>
<tr>
<td>Contradiction</td>
<td>$p(m) \land \neg q(m)$</td>
<td>$F_0$</td>
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§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems
§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Note: 用(2)(3)似乎較麻煩，但當我們企圖找出一個反例時，已等於完成(2) or (3).

Thm 2.5: ∀ positive integer x, y, if xy > 25, then x > 5 or y > 5.
Proof.

By the method of contrapositive.
Suppose 0 < x ≤ 5 and 0 < y ≤ 5,
then 0 < xy ≤ 5 × 5 = 25.
∴ xy does not exceed 25.
∴ [¬ (x > 5) ∧ ¬ (y > 5)] → (xy ≤ 25)
⇔ (xy > 25) → [(x > 5) ∨ (y > 5)].
Hence if xy > 25, then x > 5 or y > 5.