Computer Science and Information Engineering
National Chi Nan University

Discrete Mathematics

Dr. Justie Su-Tzu Juan

Chapter 2 Fundamentals of Logic

§ 2.4 The Use of Quantifiers

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics (5th Edition)
by Ralph P. Grimaldi

§ 2.4 The Use of Quantifiers

Def : ***open statement*** **:**

1) contains ≥ 1 variables;

2) not statement;

3) becomes a statement when the variables are replaced by certain allowable choices.

Def : ***universe*** **(*of discourse*) for the open statement :**

The set of all allowable choices.

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ex : 1) "The number $x + 2$ is an even integer" not statement

2) "The number $x + 2$ is an even integer" is an open statement

3) "The number $x + 2$ is an even integer" is denoted by $p(x)$

then $\neg p(x)$: "The number $x + 2$ is not an even integer"

4) Let $q(x, y)$: The numbers $y + 2$, $x - y$, and $x + 2y$ are even integers.

Hence, $p(5)$ is F: The number 7 is an even integer.

$\neg p(7)$ is T: The number 9 is not an even integer.

$q(4, 2)$ is T: The numbers 4, 2 and 8 are even integers.

$q(5, 2)$, $q(4, 7)$ are F:

$\neg q(5, 2)$, $\neg q(4, 7)$ are T:

$$\Rightarrow \begin{cases} \text{For some } x,\ p(x). \\ \text{For some } x, y,\ q(x, y). \end{cases} \quad \text{or} \quad \begin{cases} \text{For some } x,\ \neg p(x). \\ \text{For some } x, y,\ \neg q(x, y). \end{cases}$$

並不是negation

§ 2.4 The Use of Quantifiers

<u>Def</u> : 1) *existential quantifier* : $\exists x$:

"for some *x*", "For at least one *x*", "there exists an *x* such that".

2) *universal quantifier* : $\forall x$:

"for all *x*", "for every *x*", "for each *x*", "for any *x*".

3) *quantified statement* : open statement with quantifier.

<u>Note</u> : 1) $\exists x \exists y \; q(x, y) = \exists x, y \; q(x, y)$

2) $\forall x \forall y \; q(x, y) = \forall x, y \; q(x, y)$

<u>ex</u> : $r(x)$: "2*x* is an even integer", universe = all integers

1) $\forall x \; r(x)$ **2) $\exists x \; r(x)$**

3) $\forall x \neg r(x)$ **3) $\exists x \neg r(x)$**

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Def : 1) ***free variable***: the variable x in each of open statement $p(x)$

2) ***bound variable***: the variable x in the statement $\exists x\, p(x)$ or $\forall x\, p(x)$

Ex 2.36 : The universe = R

$p(x) : x \geq 0$ $q(x) : x^2 \geq 0$ $r(x) : x^2 - 3x - 4 = 0$ $s(x) : x^2 - 3 > 0$

1) $\exists x\, [p(x) \wedge r(x)]$

Let $x = 4$, then $p(4)$ is true and $r(4)$ is true

$\therefore [p(4) \wedge r(4)]$ is true $\Rightarrow \exists x\, [p(x) \wedge r(x)]$ is true.

2) $\forall x\, [p(x) \rightarrow q(x)]$

x	$p(x)$	$q(x)$	$p(x) \rightarrow q(x)$
< 0	F		T
≥ 0	T	T	T

$\Rightarrow \forall x\, [p(x) \rightarrow q(x)]$ is true.

$p(x): x \geq 0$ $q(x): x^2 \geq 0$ $r(x): x^2 - 3x - 4 = 0$ $s(x): x^2 - 3 > 0$

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Note : "$\forall x\ [p(x) \rightarrow q(x)]$"可能被敘述成：

a) For every real number x, if $x \geq 0$, then $x^2 \geq 0$.

b) Every nonnegative real number has a nonnegative square.

c) The square of any nonnegative real number is a nonnegative real number.

d) All nonnegative real numbers have nonnegative squares.

3) $\exists x\ [p(x) \rightarrow q(x)]$ is true

1') $\forall x\ [q(x) \rightarrow s(x)]$ (1) $\exists x\ [p(x) \wedge r(x)]$)

Let $x = 1$, $q(1)$ is true, but $s(1)$: $1 - 3 > 0$ is false

$\therefore$ $q(1) \rightarrow s(1)$ is false, i.e. $x = 1$ is a counterexample

$\Rightarrow \forall x\ [q(x) \rightarrow s(x)]$ is false.

(不只一個反例，$\forall -\sqrt{3} < x < \sqrt{3}$ 皆是)

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2') $\forall x\, [r(x) \vee s(x)]$ (2) $\forall x\, [p(x) \rightarrow q(x)]$)

Let $x = 1$ (or ½, –3/2 , 0, …), then $r(1)$ is false and $s(1)$ is false, too. ∴ $r(1) \vee s(1)$ is false

⇒ $\forall x\, [r(x) \vee s(x)]$ is false.

2") $\exists x\, [r(x) \vee s(x)]$ is true.

3') $\forall x\, [r(x) \rightarrow p(x)]$ (3) $\exists x\, [p(x) \rightarrow q(x)]$)

Let $x = -1$, $r(-1) = (-1)^2 - 3(-1) - 4 = 0$ is true, but $p(-1)$ is false.

∴ $r(-1) \rightarrow p(-1)$ is false. (the unique counterexample)

⇒ $\forall x\, [r(x) \rightarrow p(x)]$ is false.

Note : "$\forall x\, [r(x) \rightarrow p(x)]$"可被敘述成：

a) For every real number x, if $x^2 - 3x - 4 = 0$, then $x \geq 0$.

b) For every real number x, if x is a solution of the equation $x^2 - 3x - 4 = 0$, then $x \geq 0$.

§ 2.4 The Use of Quantifiers

Remark : 1) $\forall x\, p(x) \Rightarrow \exists x\, p(x)$, for the universe $\neq \phi$.

2) $\exists x\, p(x) \not\Rightarrow \forall x\, p(x)$

Ex 2.37 :

a) The universe = R

1) If a number is rational, then it is a real number.

2) If x is rational, then x is real.

(The use of the universal quantifier is ***implicit*** as opposed to ***explicit***)

Let $p(x)$: x is a rational number

$q(x)$: x is a real number

$\Rightarrow$ (1) = (2) = $\forall x\, [p(x) \rightarrow q(x)]$.

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Ex 2.37 :

b) The universe = All triangles in the plane.

"An equilateral triangle has three angles of 60°, and conversely."

Let $e(t)$: Triangle t is equilateral

$a(t)$: Triangle t has three angles of 60°

$\Rightarrow$ " " $= \forall t\, [e(t) \leftrightarrow a(t)]$

c) $\sin^2 x + \cos^2 x = 1$ (for all real number x)

$\Rightarrow$ The universe of $x = R$, $\forall x\, [\sin^2 x + \cos^2 x = 1]$

d) The universe = N

"The integer 41 is equal to the sum of two perfect squares."

$\Rightarrow \exists m\, \exists n\, [41 = m^2 + n^2]$

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Ex 2.38 : $p(x) : x^2 \geq 0$

(1) The universe = R : $\forall x\, p(x)$ is true ($\exists x\, p(x)$ is true)

(2) The universe = C : $\forall x\, p(x)$ is false ($\exists x\, p(x)$ is true)

$\because$ let $x = i$, then $p(i) : i^2\ (= -1) \geq 0$ is false.

Ex 2.39 : See Textbook

Table 2.21 : See Textbook

Def 2.6 : Let $p(x)$, $q(x)$ be open statements defined for a given universe.

1) $p(x)$ is (*logically*) *equivalent* to $q(x)$: $\forall x\, [p(x) \Leftrightarrow q(x)]$:

$p(x) \Leftrightarrow q(x)$ for each x in the universe.

2) $p(x)$ *logically implies* $q(x)$: $\forall x\, [p(x) \Rightarrow q(x)]$:

$p(x) \Rightarrow q(x)$ for each x in the universe.

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<u>ex</u> : The universe : all triangles in the plane.

Let $p(x)$: x is equiangular, $q(x)$: x is equilateral

$\because$ for all particular triangle a, $p(a) \leftrightarrow q(a)$ is true.

$\therefore$ $\forall x\, [p(x) \Leftrightarrow q(x)]$

<u>Note</u> : 1) $\forall x\, [p(x) \Leftrightarrow q(x)]$ iff $\forall x\, [p(x) \Rightarrow q(x)] \wedge \forall x\, [q(x) \Rightarrow p(x)]$

2) <u>Def 2.6</u> can be given for two open statements that involve $\geq$ 2 variable.

<u>Def 2.7</u> :

1) The *contrapositive* of $\forall x\, [p(x) \rightarrow q(x)]$ is $\forall x\, [\neg q(x) \rightarrow \neg p(x)]$

2) The *converse* of $\forall x\, [p(x) \rightarrow q(x)]$ is $\forall x\, [q(x) \rightarrow p(x)]$

3) The *inverse* of $\forall x\, [p(x) \rightarrow q(x)]$ is $\forall x\, [\neg p(x) \rightarrow \neg q(x)]$

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Ex 2.40 : The universe = all quadrilaterals in the plane

Let $s(x)$: x is a square, $e(x)$: x is equilateral

a) $\forall x\ [s(x) \rightarrow e(x)]$ is a true statement

$\Leftrightarrow \forall x\ [\neg\, e(x) \rightarrow \neg\, s(x)]$ (the contrapositive)

b) $\forall\ x\ [e(x) \rightarrow s(x)]$ is a false statement (the converse)

$\Leftrightarrow \forall x\ [\neg\, s(x) \rightarrow \neg\, e(x)]$ (the inverse)

Ex 2.41 : The universe = R

Let $p(x)$: $|x| > 3$, $q(x)$: $x > 3$

a) $\forall x\ [p(x) \rightarrow q(x)]$ is false. (let $x = -5$, $p(-5)$: T, $q(-5)$: F)

b) The converse of (a) = Every real number greater than 3 has magnitude (or, absolute value) greater than 3.

$\forall x\ [q(x) \rightarrow p(x)]$ is true.

$p(x): |x| > 3, \; q(x): x > 3$

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Ex 2.41 :

c) The inverse of (a) is also true : $\forall x\, [\neg p(x) \rightarrow \neg q(x)]$:

"If the magnitude of a real number is less than or equal to 3, then the number itself is less than or equal to 3."

(b) ⇔ (c)

d) The contrapositive of (a) = $\forall x\, [\neg q(x) \rightarrow \neg p(x)]$ (is false) :

"If a real number is less than or equal to 3, then so is its magnitude."

(d) ⇔ (a)

e) Let $r(x): x < -3$, defined for the universe of all real number :

Statement : $\forall x\, [p(x) \rightarrow (r(x) \vee q(x))]$

Contrapositive : $\forall x\, [\neg (r(x) \vee q(x)) \rightarrow \neg p(x)]$

Converse : $\forall x\, [(r(x) \vee q(x)) \rightarrow p(x)]$

Inverse : $\forall x\, [\neg p(x) \rightarrow \neg (r(x) \vee q(x))]$

} all true

$\Rightarrow \forall x\, [p(x) \Leftrightarrow (r(x) \vee q(x))]$

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Ex 2.42 : The universe = Z

Let $r(x) : 2x + 1 = 5$, $s(x) : x^2 = 9$.

1) $\exists x\, [r(x) \wedge s(x)]$ is false

$\because$ no integer a such that $2a + 1 = 5$ and $a^2 = 9$

2) $\exists x\, r(x) \wedge \exists x\, s(x)$ is true

$\because$ $\exists b = 2$, $r(b) : 2b + 1 = 5$ is true

$\exists c = 3$, $s(c) : c^2 = 9$ is true

3) $\exists x\, [r(x) \wedge s(x)] \not\Leftrightarrow [\exists x\, r(x) \wedge \exists x\, s(x)]$

$[\exists x\, r(x) \wedge \exists x\, s(x)] \not\Rightarrow \exists x\, [r(x) \wedge s(x)]$

Def : 1) $\not\Leftrightarrow$ is read "*is not logically equivalent to*"

2) $\not\Rightarrow$ is read "*does not logically imply*"

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Note : $\exists x\,[p(x) \wedge q(x)] \Rightarrow [\exists x\, p(x) \wedge \exists x\, q(x)]$

Proof.

If $\exists x\,[p(x) \wedge q(x)]$ is true then

there is at least one element c in the universe

s.t. $p(c) \wedge q(c)$ is true.

$\because$ $[p(c) \wedge q(c)] \Rightarrow p(c)$ and $[p(c) \wedge q(c)] \Rightarrow q(c)$

(by Conjunctive Simplification)

i.e. $\exists x\, p(x)$ is true and $\exists x\, q(x)$ is true

$\therefore$ $[\exists x\, p(x) \wedge \exists x\, q(x)]$ is true.

Table 2.22 : See Textbook

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Ex 2.43 : Let $p(x)$, $q(x)$ and $r(x)$ denote open statements for a given universe.

1) $\forall x\, [p(x) \wedge (q(x) \wedge r(x))] \Leftrightarrow \forall x\, [(p(x) \wedge q(x)) \wedge r(x)]$

$\because$ For each a in the universe,

$p(a) \wedge (q(a) \wedge r(a)) \Leftrightarrow (p(a) \wedge q(a)) \wedge r(a)$

$\therefore \forall x\, [p(x) \wedge (q(x) \wedge r(x))] \Leftrightarrow \forall x\, [(p(x) \wedge q(x)) \wedge r(x)]$

2) $\exists x\, [p(x) \rightarrow q(x)] \Leftrightarrow \exists x\, [\neg p(x) \vee q(x)]$

$\because$ For each c in the universe,

$[p(c) \rightarrow q(c)] \Leftrightarrow [\neg p(c) \vee q(c)]$

$\therefore \exists x\, [p(x) \rightarrow q(x)]$ is true iff $\exists x\, [\neg p(x) \vee q(x)]$ is true

i.e. $\exists x\, [p(x) \rightarrow q(x)] \Leftrightarrow \exists x\, [\neg p(x) \vee q(x)]$

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Ex 2.43 :

3) a) $\forall x \neg \neg p(x) \Leftrightarrow \forall x\, p(x)$

b) $\forall x \neg [p(x) \wedge q(x)] \Leftrightarrow \forall x [\neg p(x) \vee \neg q(x)]$

c) $\forall x \neg [p(x) \vee q(x)] \Leftrightarrow \forall x [\neg p(x) \wedge \neg q(x)]$

4) a) $\exists x \neg \neg p(x) \Leftrightarrow \exists x\, p(x)$

b) $\exists x \neg [p(x) \wedge q(x)] \Leftrightarrow \exists x [\neg p(x) \vee \neg q(x)]$

c) $\exists x \neg [p(x) \vee q(x)] \Leftrightarrow \exists x [\neg p(x) \wedge \neg q(x)]$

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Remark : $\neg [\forall x\, p(x)] \Leftrightarrow \exists x \neg p(x)$

Proof.

$\neg [\forall x\, p(x)]$: It is not the case that for all x, $p(x)$ holds.

1) "$\neg [\forall x\, p(x)]$" is true = "$\forall x\, p(x)$" is false

= For some a of the universe, $\neg p(a)$ is true

= "$\exists x \neg p(x)$" is true

2) "$\exists x \neg p(x)$" is true = For some b of the universe, $\neg p(b)$ is true

= For some b of the universe, $p(b)$ is false

= "$\forall x\, p(x)$" is false

= "$\neg [\forall x\, p(x)]$" is true

by (1), (2) "$\neg [\forall x\, p(x)]$" is true iff "$\exists x \neg p(x)$" is true

Similar, "$\neg [\forall x\, p(x)]$" is false iff "$\exists x \neg p(x)$" is false

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Table 2.23 :

$\neg [\forall x\ p(x)]$	$\Leftrightarrow$	$\exists x \neg p(x)$		
$\neg [\exists x\ p(x)]$	$\Leftrightarrow$	$\forall x \neg p(x)$		
$\neg [\forall x \neg p(x)]$	$\Leftrightarrow$	$\exists x \neg\neg p(x)$	$\Leftrightarrow$	$\exists x\ p(x)$
$\neg [\exists x \neg p(x)]$	$\Leftrightarrow$	$\forall x \neg\neg p(x)$	$\Leftrightarrow$	$\forall x\ p(x)$

Rules of Negating Statements with One Quantifier.

Ex. 2.44 : The universe $= Z$

1) Let $p(x)$: x is odd, $q(x)$: $x^2 - 1$ is even

"If x is odd, then $x^2 - 1$ is even" $= \forall x\, [p(x) \rightarrow q(x)]$ (T)

$\neg\, [\forall\, x\, (p(x) \rightarrow q(x))]$

$\Leftrightarrow \exists x\, [\neg\, (p(x) \rightarrow q(x))] \Leftrightarrow \exists x\, [\neg\, (\neg\, p(x) \vee q(x))]$

$\Leftrightarrow \exists x\, [\neg \neg\, p(x) \wedge \neg\, q(x)] \Leftrightarrow \exists x\, [p(x) \wedge \neg\, q(x)]$

= "There exists an integer x such that x is odd and $x^2 - 1$ is odd" (F)

2) $r(x)$: $2x + 1 = 5$, $s(x)$: $x^2 = 9$ (In Ex 2.42)

$\exists x\, [r(x) \wedge s(x)]$ is false, $\neg\, [\exists x\, (r(x) \wedge s(x))]$ is true?

$\neg\, [\exists x\, (r(x) \wedge s(x))] \Leftrightarrow \forall x\, [\neg\, (r(x) \wedge s(x))]$

$\Leftrightarrow \forall x\, [\neg\, r(x) \vee \neg\, s(x)]$

= "For every integer x, $2x + 1 \neq 5$ or $x^2 \neq 9$."

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More than one quantifier (More than one variable)

Ex 2.45 : $\forall x\, \forall y\, p(x, y) \Leftrightarrow \forall y\, \forall x\, p(x, y)$

ex : The universe = R

1) $\forall x\, \forall y\, (x + y = y + x) \Leftrightarrow \forall y\, \forall x\, (x + y = y + x)$

2) $\forall x\, \forall y\, (xy = yx) \Leftrightarrow \forall y\, \forall x\, (xy = yx)$

Ex 2.46 : $\forall x, y, z\, p(x, y, z) \Leftrightarrow \forall y, x, z\, p(x, y, z)$

$\Leftrightarrow \forall x, z, y\, p(x, y, z)$ …六種

($\forall x, y, z \equiv \forall\, x, \forall\, y, \forall\, z$)

ex : The universe = R

$\forall x\, \forall y\, \forall z\, [x + (y + z) = (x + y) + z]$

$\Leftrightarrow \forall y\, \forall x\, \forall z\, [x + (y + z) = (x + y) + z]$

§ 2.4 The Use of Quantifies

Ex 2.47 : $\exists x\, \exists y\, p(x, y) \Leftrightarrow \exists y\, \exists x\, p(x, y)$

ex : For the universe of all integer

"There exist integer x, y such that $x + y = 6$"

$\exists x\, \exists y\, (x + y = 6) \Leftrightarrow \exists y\, \exists x\, (x + y = 6)$

§ 2.4 The Use of Quantifies

Both ∃ and ∀

Ex 2.48 : The universe = Z, let $p(x, y)$: $x + y = 17$

1) $\forall x \, \exists y \, p(x, y)$:

"For every integer x, there exist an integer y such that $x + y = 17$" (T)

$\because$ $\forall x$: let $y = 17 - x$ is a integer $\therefore$ $x + y = x + (17 - x) = 17$

(Each x gives rise to a different value of y)

2) $\exists y \, \forall x \, p(x, y)$:

"There exist an integer y so that for all integer x, $x + y = 17$" (F)

$\because$ Once an integer y is selected, only x satisfy $x + y = 17$ is $y - 17$ →←

$\therefore$ $\forall x \, \exists y \, p(x, y) \not\Leftrightarrow \exists y \, \forall x \, p(x, y)$

Note : See Textbook (1. careful & precise, 2. related statements)

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Ex 2.49 : The universe of x, y are equal. What is the negation of

$$\forall x\, \exists y\, [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]$$

Sol. $\neg [\forall x\, \exists y\, [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]]$

$\Leftrightarrow \exists x\, [\neg [\, \exists y\, [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]]]$

$\Leftrightarrow \exists x\, \forall y\, \neg [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]$

$\Leftrightarrow \exists x\, \forall y\, \neg [\neg (p(x, y) \wedge q(x, y)) \vee r(x, y)]$

$\Leftrightarrow \exists x\, \forall y\, [\neg \neg (p(x, y) \wedge q(x, y)) \wedge \neg r(x, y)]$

$\Leftrightarrow \exists x\, \forall y\, [(p(x, y) \wedge q(x, y)) \wedge \neg r(x, y)]$

$\Rightarrow$ When we want to prove "$\forall x\, \exists y\, [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]$" as the conclusion, use the method of Proof by Contradiction.

We would assume the negation of this conclusion as an additional premise. i.e. the additional premise =

"$\exists x\, \forall y\, [(p(x, y) \wedge q(x, y)) \wedge \neg r(x, y)]$"

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Ex 2.50 : In calculus, the definition of *limit* :

Let I be an open interval containing the real number a and suppose the function f is defined throughout I, except possibly at a. We say f has the limit L as x approaches a, and write $lim_{x \to a} f(x) = L$, iff for every $\varepsilon > 0$, there exist a $\delta > 0$ so that, for all x in I, $(0 < |x - a| < \delta) \to (|f(x) - L| < \varepsilon)$.

This can be expressed in symbolic form as :

$$\lim_{x \to a} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$$
$$\forall x\, [(0 < |x - a| < \delta) \to (|f(x) - L| < \varepsilon)]$$

The universe = the real number in the open interval I, except possibly a.

To negate this definition :

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Sol.

$\lim_{x\to a} f(x) \neq L$

$\Leftrightarrow \neg\, [\forall \varepsilon > 0, \exists \delta > 0\ \forall x\ [(0 < |x - a| < \delta) \to (|f(x) - L| < \varepsilon)]]$

$\Leftrightarrow \exists \varepsilon > 0, \forall \delta > 0\ \exists x\ \neg\, [(0 < |x - a| < \delta) \to (|f(x) - L| < \varepsilon)]$

$\Leftrightarrow \exists \varepsilon > 0, \forall \delta > 0\ \exists x\ \neg\, [\neg\, (0 < |x - a| < \delta) \vee (|f(x) - L| < \varepsilon)]$

$\Leftrightarrow \exists \varepsilon > 0, \forall \delta > 0\ \exists x\ [\neg\neg\, (0 < |x - a| < \delta) \wedge \neg\, (|f(x) - L| < \varepsilon)]$

$\Leftrightarrow \exists \varepsilon > 0, \forall \delta > 0\ \exists x\ [(0 < |x - a| < \delta) \wedge (|f(x) - L| \geq \varepsilon)]$

$\therefore$ $\lim_{x\to a} f(x) \neq L$ iff there exist a positive real number ε such that for every positive real number δ, there is an x in I such that $0 < |x - a| < \delta$ (this is, $x \neq a$ and its distance from a is less than δ), but $|f(x) - L| \geq \varepsilon$ (this is, the value of $f(x)$ differs form L by at least ε).

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Dr. Justie Su-Tzu Juan

Chapter 2 Fundamentals of Logic

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems (1)

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Note : In definition, an implication can be read as a biconditional, and, only in definition.

Ex 2.51 : (1/2)

a) Universe : all quadrilaterals in the plane.

A : "If a quadrilaterals is a rectangle then it has four equals angles."

B : "If a quadrilaterals has four equal angles, then it is a rectangle."

Let $p(x)$: x is a rectangle. $q(x)$: x has four equal angles

A : $\forall x\ [p(x) \rightarrow q(x)]$

B : $\forall x\ [q(x) \rightarrow p(x)]$

Actually, they are both intending: $\forall x\ [p(x) \leftrightarrow q(x)]$

<u>Ex 2.51</u> : (2/2)

b) Universe = Z

A : "For every integer *n*, we call *n* even if it is divisible by 2."

Let $p(n)$: n is an even integer

$q(n)$: n is divisible by 2 (or, $n = 2k$, for some integer k)

(or, $\exists k\ [n = 2k]$)

A : $\forall n\ [q(n) \rightarrow p(n)]$

Actually, $\forall n\ [p(n) \leftrightarrow q(n)]$

<u>Ex 2.52</u> : Universe = {2, 4, 6, …, 26}

For all *n* (*n* = 2, 4, …, 26), we can write *n* as the sum of at most three perfect squares.

Sol. ***method of exhaustion*** **:**（不只一種，但我們只需要找出其中一種即可！）

2 = 1 + 1	10 = 9 + 1	18 = 16 + 1 + 1 (= 9 + 9)
4 = 4	12 = 4 + 4 + 4	20 = 16 + 4
6 = 4 + 1 + 1	14 = 9 + 4 + 1	22 = 9 + 9 + 4
8 = 4 + 4	16 =16	24 = 16 + 4 + 4
		26 = 25 + 1 (= 16 + 9 + 1)

<u>Def</u> : ***Corollary*** **: follow immediately from a** ***theorem***

<u>The Rule of Universal Specification</u> **:**

$\forall x$ for a given universe, $p(x)$ is true, the $p(a)$ is true for each a in the universe.

Ex 2.53 : (1/4)

$$\begin{array}{l} \forall x\,[m(x) \rightarrow c(x)] \\ m(\vartheta) \\ \hline \therefore\; c(\vartheta) \end{array}$$

a) Universe = all people

$m(x)$: x is a mathematics professor,

$c(x)$: x has studied calculus

All mathematics professors have studied calculus.

Leona is a mathematics professor.

Therefore Leona has studied calculus.

Let ϑ = Leona (in our universe) then :

Steps	Reason
1) $\forall x\,[m(x) \rightarrow c(x)]$	Premise
2) $m(\vartheta)$	Premise
3) $m(\vartheta) \rightarrow c(\vartheta)$	(1) & the Rule of Universal Specification
4) $\therefore\; c(\vartheta)$	(2), (3) and the Rule of Detachment

Ex 2.53 : (2/4)

b) Universe = all triangles in the plane

$p(t)$: t has two sides of equal length

$q(t)$: t is an isosceles triangle

$r(t)$: t has two angles of equal measure

c : triangle XYZ

$$\begin{array}{l} \neg r(c) \\ \forall t\,[p(t) \rightarrow q(t)] \\ \forall t\,[q(t) \rightarrow r(t)] \\ \hline \therefore \neg p(c) \end{array}$$

See Textbook.

Step	Reasons
1) $\forall t\,[p(t) \rightarrow q(t)]$	Premise
2) $p(c) \rightarrow q(c)$	(1) and the Rule of Universal Specification
3) $\forall t\,[q(t) \rightarrow r(t)]$	Premise
4) $q(c) \rightarrow r(c)$	(3) and the Rule of Universal Specification
5) $p(c) \rightarrow r(c)$	(2), (4) and the Law of the Syllogism
6) $\neg r(c)$	Premise
7) $\therefore \neg p(c)$	(5), (6) and Modus Tollens

Ex 2.53 : (3/4)

c) Universe = student at a particular college.

m : Mary Gusberti, a student of this college.

$j(x)$: x is a junior.

$s(x)$: x is a senior.

$p(x)$: x is enrolled in a physical education class

No junior or senior is enrolled in a physical education class
Mary Gusberti is enrolled in a physical education class
Therefore Mary Gusberti is not a senior

i.e.

$$\forall x\,[(j(x) \vee s(x)) \rightarrow \neg p(x)]$$
$$p(m)$$
$$\therefore \neg s(m)$$

$\forall x\ [(j(x) \vee s(x)) \rightarrow \neg p(x)]$
$p(m)$
$\therefore \neg s(m)$

Ex 2.53 : (4/4)

c) Sol.

Step	Reason
1) $\forall x\ [(j(x) \vee s(x)) \rightarrow \neg p(x)]$	Premise
2) $p(m)$	Premise
3) $(j(m) \vee s(m)) \rightarrow \neg p(m)$	(1) and the Rule of Universal Specification
4) $p(m) \rightarrow \neg (j(m) \vee s(m))$	(3) and $(q \rightarrow t) \Leftrightarrow (\neg t \rightarrow \neg p)$, and Law of Double Negation
5) $p(m) \rightarrow (\neg j(m) \wedge \neg s(m))$	(4) and DeMorgan's Law
6) $\neg j(m) \wedge \neg s(m)$	(2), (5) and the Rule of Detachment
7) $\therefore \neg s(m)$	(6) and the Rule of Conjunctive Simplification

§ 2.5 Quantifiers, Definitions, and the Proofs of Theorems

Note : **the Rule of Universal Specification + Modus Ponens, Modus Tollens**

c : **a member of the fixed universe**

$p(x)$, $q(x)$: **open statements defined for this universe**

(1)

$$\begin{array}{l} \forall x\,[p(x) \rightarrow q(x)] \\ p(c) \\ \hline \therefore\ q(c) \end{array}$$

(2)

$$\begin{array}{l} \forall x\,[p(x) \rightarrow q(x)] \\ \neg\, q(c) \\ \hline \therefore\ \neg\, p(c) \end{array}$$

Ex (1/2) : Universe = all polygons in the plane

c : quadrilateral $EFGH$, where $\angle E = 91°$

$p(x)$: x is a square

$q(x)$: x has four sides

(1') All squares have four sides,
Quadrilateral *EFGH* has four sides
Therefore quadrilateral *EFGH* is a square

(1")

$$\begin{array}{l} \forall x\,[p(x) \rightarrow q(x)] \\ q(c) \\ \hline \therefore p(c) \end{array} \longleftarrow \text{false}$$

$\because$ $\forall x\,[p(x) \rightarrow q(x)]$ and c is a polygon in the plane

$\therefore$ $p(c) \rightarrow q(c)$, but $[p(c) \rightarrow q(c)] \wedge q(c) \nrightarrow p(c)$

$\therefore$ invalid!!

(converse)

Ex (2/2) :

(2') All squares have four sides

Quadrilateral *EFGH* is not a square

Therefore quadrilateral *EFGH* does not have four sides

(2")

$$\forall x\,[p(x) \rightarrow q(x)]$$

$$\neg\, p(c)$$

$$\therefore \neg\, q(c)$$

$\because$ $\forall x\,[p(x) \rightarrow q(x)]$ and c is a polygon in the plane

$\therefore$ $p(c) \rightarrow q(c)$, but $[(p(c) \rightarrow q(c)) \wedge \neg\, p(c)] \nrightarrow \neg\, q(c)$

$\therefore$ invalid!!

(inverse)