Computer Science and Information Engineering
National Chi Nan University

Discrete Mathematics

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Chapter 2 Fundamentals of Logic

§ 2.3 Logical Implication: Rules of Inference

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics (5^{th} Edition)
by Ralph P. Grimaldi

§ 2.3 Logical Implication: Rules of Inference

Def : For $n \in N$, in the implication $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$,
we called $p_1, p_2, \ldots, p_n$: ***premises***, q : ***conclusion***

Ex 2.19 : p : Roger studies
q : Roger plays tennis
r : Roger passes discrete math.
Let p_1 : If Roger studies, then he will pass discrete math.
p_2 : If Roger doesn't play tennis, then he'll study.
p_3 : Roger failed discrete math.
Determine $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$ is valid or not.

Sol. (1/2)

$p_1 : p \rightarrow r, \quad p_2 : \neg q \rightarrow p, \quad p_3 : \neg r$

$(p_1 \wedge p_2 \wedge p_3) \rightarrow q \Leftrightarrow [(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$

§ 2.3 Logical Implication: Rules of Inference

$p_1 : p \rightarrow r, \quad p_2 : \neg q \rightarrow p, \quad p_3 : \neg r$

$[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$

Sol. (2/2)

			p_1	p_2	p_3	$(p_1 \wedge p_2 \wedge p_3) \rightarrow q$
p	q	r	$p \rightarrow r$	$\neg q \rightarrow p$	$\neg r$	$[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$
0	0	0	1	0	1	1
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	0	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

§ 2.3 Logical Implication: Rules of Inference

<u>Ex 2.20</u> : $[p \wedge ((p \wedge r) \rightarrow s)] \rightarrow (r \rightarrow s)$

$p_1 : p$ $p_2 : (p \wedge r) \rightarrow s$ $q : r \rightarrow s$

p_1				p_2	q	$(p_1 \wedge p_2) \rightarrow q$
p	r	s	$p \wedge r$	$(p \wedge r) \rightarrow s$	$r \rightarrow s$	$[p \wedge ((p \wedge r) \rightarrow s)] \rightarrow (r \rightarrow s)$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	0	1
0	1	1	0	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

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Def : If $(p_1 \wedge p_2) \rightarrow q$, then q is ***deduced*** or ***inferred*** from the truth of the premises p_1, p_2.

Def 2.4 : If $p \rightarrow q$ is a tautology, then say p ***logically implies*** q and write $p \Rightarrow q$ to denote this situation.

Note :
1. $p \Rightarrow q \equiv p \rightarrow q$ is a tautology $\equiv p \rightarrow q$ is a ***logical implication***
2. $p \Leftrightarrow q \equiv p \leftrightarrow q$ is a tautology;
   $\equiv p \rightarrow q$ and $q \rightarrow p$ are tautologies;
   $\equiv p \Rightarrow q$ and $q \Rightarrow p$.
3. $p \not\Rightarrow q \equiv p \rightarrow q$ is not a tautology

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Ex 2.21 :

$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

$\neg(p \wedge q) \Rightarrow (\neg p \vee \neg q)$ and $(\neg p \vee \neg q) \Rightarrow \neg(p \wedge q)$

$\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$ and $(\neg p \vee \neg q) \rightarrow \neg(p \wedge q)$ are tautologies

$[\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)] \Leftrightarrow T_0$ and $[(\neg p \vee \neg q) \rightarrow \neg(p \wedge q)] \Leftrightarrow T_0$

Remark :

1. 當 p的個數增加， “表”將越來越大：2^5, 2^6, …以致不易check.
2. check $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ 之類問題時，
   只需check “當 $p_1 = p_2 = \ldots = p_n = 1$時，$q$ 是否 =1” 即可。
   (Ex 2.19中第3列, Ex 2.20中第5, 6, 8列)

Def : *rules of inference* : p. 78

§ 2.3 Logical Implication: Rules of Inference

Ex 2.22 : *Modus Ponens* (*Rule of Detachment*): $[p \wedge (p \rightarrow q)] \Rightarrow q$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

p

$p \rightarrow q$

$\therefore q$

ex :

(a) 1) Lydia wins a ten-million-dollar lottery.

p

2) If Lydia wins a ten-million-dollar lottery, then Kay will quit her job.

$p \rightarrow q$

$\therefore q$

3) Therefore Kay will quit her job.

$p \rightarrow q$

(b) 1) If Allison vacations in Paris, then she will have to win a scholarship

p

$\therefore q$

2) Allison is vacationing in Paris

3) Therefore Allison won a scholarship

§ 2.3 Logical Implication: Rules of Inference

$$\begin{array}{r} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Note : 1. By the first substitution rule : p or q may be replaced by compound statements.

ex : $[(r \vee s) \wedge [(r \vee s) \to (\neg t \wedge u)]] \Rightarrow (\neg t \wedge u)$

2. We can apply the first substitution rule for each of the rules of inference we shall study.

Ex 2.23 : *Law of the Syllogism* : $[(p \to q) \wedge (q \to r)] \Rightarrow (p \to r)$

$$\begin{array}{r} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

ex : 1) If $396 \mid 35244$, then $66 \mid 35244$.

2) If $66 \mid 35244$, then $3 \mid 35244$.

3) Therefore, If $396 \mid 35244$, then $3 \mid 35244$.

§ 2.3 Logical Implication: Rules of Inference

Ex 2.24 : 1) Rita is baking a cake.

2) If Rita is baking a cake, then she is not practicing her flute.

3) If Rita is not practicing her flute, then her father will not buy her a car.

4) Therefore Rita's father will not buy her a car

$$\begin{array}{l} p \\ p \to \neg q \\ \neg q \to \neg r \\ \hline \therefore \neg r \end{array}$$

Sol. 1

Steps	Reasons
1) $p \to \neg q$	Premise
2) $\neg q \to \neg r$	Premise
3) $p \to \neg r$	By (1), (2) and the Law of the Syllogism
4) p	Premise
5) $\therefore \neg r$	By (3), (4) and the Rule of Detachment

§ 2.3 Logical Implication: Rules of Inference

Sol. 2

$$\begin{array}{l} p \\ p \rightarrow \neg q \\ \neg q \rightarrow \neg r \\ \hline \therefore \neg r \end{array}$$

Steps	Reasons
1) p	Premise
2) $p \rightarrow \neg q$	Premise
3) $\neg q$	By (1), (2) and the Rule of Detachment
4) $\neg q \rightarrow \neg r$	Premise
5) $\therefore \neg r$	By (3), (4) and the Rule of Detachment

§ 2.3 Logical Implication: Rules of Inference

Ex 2.25 : *Modus Tollens (Method of denying)* :

$$[(p \to q) \land (\neg q)] \Rightarrow \neg p$$

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

ex : 1) If Connie is elected president of Phi Delta sorority, then Helen will pledge that sorority.

2) Helen did not pledge Phi Delta sorority.

3) Therefore Connie was not elected president of Phi Delta sorority.

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Recall : $p \to q \Leftrightarrow \neg p \lor q$

ex :

$p \rightarrow r$
$r \rightarrow s$
$t \vee \neg s$
$\neg t \vee u$
$\neg u$
―――――
$\therefore \neg p$

Sol.

Steps	Reasons
1) $p \rightarrow r,\ r \rightarrow s$	Premises
2) $p \rightarrow s$	(1) and Law of the Syllogism
3) $t \vee \neg s$	Premise
4) $\neg s \vee t$	(3) and the Commutative Law
5) $s \rightarrow t$	(4) and $\neg s \vee t \Leftrightarrow s \rightarrow t$
6) $p \rightarrow t$	(2), (5) and the Law of the Syllogism
7) $\neg t \vee u$	Premise
8) $t \rightarrow u$	(7) and $\neg t \vee u \Leftrightarrow t \rightarrow u$
9) $p \rightarrow u$	(6), (8) and the Law of the Syllogism
10) $\neg u$	Premise
11) $\therefore \neg p$	(9), (10) and Modus Tollens

§ 2.3 Logical Implication: Rules of Inference

Note : $[(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u] \Rightarrow \neg p$

but $[(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u] \not\Leftrightarrow \neg p$

$\because$ when $p = 0, u = 1,$

then 右 $= \neg p = 1$ and 左 $= 0$ ($\because \neg u = 0$)

Note : *Modus Ponens*: *Modus Tollens*:

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array} \qquad \begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array} \qquad \begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array} \qquad \begin{array}{l} p \to q \\ \neg p \\ \hline \therefore \neg q \end{array}$$

ex : 舉例說明 (p. 74) $[(p \to q) \wedge q] \not\Rightarrow p$

(p. 75) $[(p \to q) \wedge \neg p] \not\Rightarrow \neg q$

§ 2.3 Logical Implication: Rules of Inference

Ex 2.26 : *Rule of Conjunction* :

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Ex 2.27 : *Rule of Disjunctive Syllogism* : $[(p \vee q) \wedge \neg p] \Rightarrow q$

$\because p \vee q \Leftrightarrow \neg p \rightarrow q$ and Modus Ponens : $[(p \rightarrow q) \wedge p] \Rightarrow q$

ex : 1) Bart's wallet is in his back pocket or it is on his desk.

2) Bart's wallet is not in his back pocket.

3) Therefore Bart's wallet is on his deck

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

§ 2.3 Logical Implication: Rules of Inference

Ex 2.28 : ***Rule of Contradiction*** : $(\neg p \to F_0) \Rightarrow p$

$$\frac{\neg p \to F_0}{\therefore p}$$

p	$\neg p$	F_0	$\neg p \to F_0$	$(\neg p \to F_0) \to p$
0	1	0	0	1
1	0	0	1	1

Def : **The method of *Proof by Contradiction* (or *Reductio ad Absurdum*).**

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to q \Leftrightarrow (p_1 \wedge p_2 \wedge \ldots \wedge p_n \wedge \neg q) \to F_0$$

(Let $p = p_1 \wedge p_2 \wedge \ldots \wedge p_n$) $\quad p \to q \Leftrightarrow (p \wedge \neg q) \to F_0$

p	q	F_0	$p \wedge \neg q$	$(p \wedge \neg q) \to F_0$	$p \to q$	$(p \to q) \leftrightarrow [(p \wedge \neg q) \to F_0]$
0	0	0	0	1	1	1
0	1	0	0	1	1	1
1	0	0	1	0	0	1
1	1	0	0	1	1	1

§ 2.3 Logical Implication: Rules of Inference

Table 2.19 (1/2):

Table 2.19

Rule of Inference	Related Logical Implication	Name of Rule
1) p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Rule of Detachment (Modus Ponens)
2) $p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Law of the Syllogism
3) $p \to q$ $\neg q$ $\therefore \neg p$	$[(p \to q) \wedge \neg q] \to \neg p$	Modus Tollens
4) p q $\therefore p \wedge q$		Rule of Conjunction
5) $p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Rule of Disjunctive Syllogism
6) $\neg p \to F_0$ $\therefore p$	$(\neg p \to F_0) \to p$	Rule of Contradiction

§ 2.3 Logical Implication: Rules of Inference

Table 2.19 (2/2):

Table 2.19

Rule of Inference	Related Logical Implication	Name of Rule
7) $p \wedge q$ / $\therefore p$	$(p \wedge q) \rightarrow p$	Rule of Conjunctive Simplification
8) p / $\therefore p \vee q$	$p \rightarrow p \vee q$	Rule of Disjunctive Amplification
9) $p \wedge q$, $p \rightarrow (q \rightarrow r)$ / $\therefore r$	$[(p \wedge q) \wedge [p \rightarrow (q \rightarrow r)]] \rightarrow r$	Rule of Conditional Proof
10) $p \rightarrow r$, $q \rightarrow r$ / $\therefore (p \vee q) \rightarrow r$	$[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$	Rule for Proof by Cases
11) $p \rightarrow q$, $r \rightarrow s$, $p \vee r$ / $\therefore q \vee s$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$	Rule of the Constructive Dilemma
12) $p \rightarrow q$, $r \rightarrow s$, $\neg q \vee \neg s$ / $\therefore \neg p \vee \neg r$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s)] \rightarrow (\neg p \vee \neg r)$	Rule of the Destructive Dilemma

Ex 2.29 : Sol. 1

$$\begin{array}{r} p \rightarrow r \\ \neg p \rightarrow q \\ q \rightarrow s \\ \hline \therefore \neg r \rightarrow s \end{array}$$

Steps	Reasons
1) $p \rightarrow r$	Premise
2) $\neg r \rightarrow \neg p$	(1) and $p \rightarrow r \Leftrightarrow \neg r \rightarrow \neg p$
3) $\neg p \rightarrow q$	Premise
4) $\neg r \rightarrow q$	(2), (3) and the Law of the Syllogism
5) $q \rightarrow s$	Premise
6) $\therefore \neg r \rightarrow s$	(4), (5) and the Law of the Syllogism

§ 2.3 Logical Implication: Rules of Inference

Ex 2.29 : Sol. 2

$$\begin{array}{r} p \to r \\ \neg p \to q \\ q \to s \\ \hline \therefore \neg r \to s \end{array}$$

Steps	Reasons
1) $p \to r$	Premise
2) $q \to s$	Premise
3) $\neg p \to q$	Premise
4) $p \vee q$	(3) and $(\neg p \to q) \Leftrightarrow (\neg\neg p \vee q) \Leftrightarrow (p \vee q)$
5) $r \vee s$	(1), (2), (4) and the Rule of the Constructive Dilemma
6) $\therefore \neg r \to s$	(5) and $(r \vee s) \Leftrightarrow (\neg\neg r \vee s) \Leftrightarrow (\neg r \to s)$

§ 2.3 Logical Implication: Rules of Inference

Ex 2.30 :

$$\begin{array}{l} p \to q \\ q \to (r \wedge s) \\ \neg r \vee (\neg t \vee u) \\ p \wedge t \\ \hline \therefore u \end{array}$$

Steps	Reasons
1) $p \to q$	Premise
2) $q \to (r \wedge s)$	Premise
3) $p \to (r \wedge s)$	(1), (2) and the Law of the Syllogism
4) $p \wedge t$	Premise
5) p	(4) and the Rule of Conjunctive Simplification
6) $r \wedge s$	(3), (5) and the Rule of Detachment
7) r	(6) and the Rule of Conjunctive Simplification
8) $\neg r \vee (\neg t \vee u)$	Premise
9) $\neg (r \wedge t) \vee u$	(8), the Associative Law of $\vee$ and DeMorgan's Laws
10) t	(4) and the Rule of Conjunctive Simplification
11) $r \wedge t$	(7), (10) and the Rule of Conjunction
12) $\therefore u$	(9), (11) and the Rule of Disjunctive Syllogism

§ 2.3 Logical Implication: Rules of Inference

Ex 2.31 : **If the band could not play rock music or the refreshments were not delivered on time, then the New Year's party would have been canceled and Alicia would have been angry. If the party were canceled, the refunds would have had to be made. No refunds were made. Therefore the band could play rock music.**

Sol. (1/2)

p : The band could play rock music

q : The refreshments were delivered on time

r : The New Year's party was canceled

s : Alicia was angry

t : Refunds had to be made

The argument above now become:

$$(\neg p \vee \neg q) \rightarrow r \wedge s$$
$$r \rightarrow t$$
$$\neg t$$
$$\overline{\qquad\qquad\qquad\qquad}$$
$$\therefore p$$

$$\begin{array}{l} (\neg p \vee \neg q) \rightarrow (r \wedge s) \\ r \rightarrow t \\ \neg t \\ \hline \therefore p \end{array}$$

Sol. (2/2)

Steps	Reasons
1) $r \rightarrow t$	Premise
2) $\neg t$	Premise
3) $\neg r$	(1), (2) and Modus Tollens
4) $\neg r \vee \neg s$	(3) and the Rule of Disjunctive Amplification
5) $\neg(r \wedge s)$	(4) and DeMorgan's Laws
6) $(\neg p \vee \neg q) \rightarrow (r \wedge s)$	Premise
7) $\neg(\neg p \vee \neg q)$	(5), (6) and Modus Tollens
8) $p \wedge q$	(7), DeMorgan's Laws, and the Law of Double Negation
9) $\therefore p$	(8), and the Rule of Conjunctive Simplification

§ 2.3 Logical Implication: Rules of Inference

Ex 2.32 : **Prove by Contradiction:**

$$\begin{array}{l} \neg p \leftrightarrow q \\ q \rightarrow r \\ \neg r \\ \hline \therefore p \end{array} \quad \rightarrow \quad \begin{array}{l} \neg p \leftrightarrow q \\ q \rightarrow r \\ \neg r \\ \neg p \\ \hline \therefore F_0 \end{array}$$

Steps	Reasons
1) $\neg p \leftrightarrow q$	Premise
2) $(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)$	(1) and $(\neg p \leftrightarrow q) \Leftrightarrow [(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)]$
3) $\neg p \rightarrow q$	(2) and the Rule of Conjunctive Simplification
4) $q \rightarrow r$	Premise
5) $\neg p \rightarrow r$	(3), (4) and the Law of the Syllogism
6) $\neg p$	Premise (the one assumed)
7) r	(5), (6) and the Rule of Detachment
8) $\neg r$	Premise
9) $r \wedge \neg r$ $(\Leftrightarrow F_0)$	(7), (8), and the Rule Conjunction
10) $\therefore p$	(6), (9) and the method of Proof by Contradiction

§ 2.3 Logical Implication: Rules of Inference

Note : 1) $[(\neg p \leftrightarrow q) \wedge (q \rightarrow r) \wedge \neg r \wedge \neg p] \Rightarrow F_0$

$\therefore [(\neg p \leftrightarrow q) \wedge (q \rightarrow r) \wedge \neg r \wedge \neg p]$ is false (= 0)

But $\because$ $(\neg p \leftrightarrow q)$, $q \rightarrow r$, $\neg r$ are the given premises (= 1)

$\therefore \neg p$ is false (= 0), i.e. p is true (= 1).

2) $[p \rightarrow (q \rightarrow r)] \Leftrightarrow [(p \wedge q) \rightarrow r]$

p	q	r	$p \wedge q$	$[(p \wedge q) \rightarrow r$	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	0	1
0	1	1	0	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Let $p = p_1 \wedge p_2 \wedge \ldots \wedge p_n$

$\Rightarrow [(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow (q \rightarrow r)] \Leftrightarrow [(p_1 \wedge p_2 \wedge \ldots \wedge p_n \wedge q) \rightarrow r]$

§ 2.3 Logical Implication: Rules of Inference

(★) $u \to r$
$(r \wedge s) \to (p \vee t)$
$q \to (u \wedge s)$
$\neg t$

$\therefore q \to p$

(★★) $u \to r$
$(r \wedge s) \to (p \vee t)$
$q \to (u \wedge s)$
$\neg t$
q

$\therefore p$

Ex 2.33 :

Steps	Reasons
1) q	Premise
2) $q \to (u \wedge s)$	Premise
3) $u \wedge s$	(1), (2) and the Rule of the Detachment
4) u	(3) and the Rule of Conjunctive Simplification
5) $u \to r$	Premise
6) r	(4), (5) and the Rule of Detachment
7) s	(3) and the Rule of Conjunctive Simplification
8) $r \wedge s$	(6), (7) and the Rule of Conjunction
9) $(r \wedge s) \to (p \vee t)$	Premise
10) $p \vee t$	(8), (9) and the Rule of Detachment
11) $\neg t$	Premise
12) $\therefore p$	(10), (11) and the Rule of Disjunctive Syllogism

$\therefore [(u \to r) \wedge [(r \wedge s) \to (p \vee t)] \wedge [q \to (u \wedge s)] \wedge \neg t \wedge q] \Rightarrow p$

$\therefore [(u \to r) \wedge [(r \wedge s) \to (p \vee t)] \wedge [q \to (u \wedge s)] \wedge \neg t] \Rightarrow (q \to p)$

§ 2.3 Logical Implication: Rules of Inference

Ex 2.34 :

p

$p \vee q$

$q \rightarrow (r \rightarrow s)$

$t \rightarrow r$

$\therefore \neg s \rightarrow \neg t$

To show this is an invalid argument:

need one assignment of truth values for p, q, r, s, t

s.t. p, $p \vee q$, $q \rightarrow (r \rightarrow s)$, $t \rightarrow r$ are all true (= 1),

but $\neg s \rightarrow \neg t$ is false (= 0)

想法： $\neg s \rightarrow \neg t$ is false $\Rightarrow \neg s : 1 \wedge \neg t : 0$

i.e. $s : 0 \wedge t : 1$

$\because t \rightarrow r$ is true $\Rightarrow r : 1$

p is true $\Rightarrow p : 1 \Rightarrow p \vee q$ is true

$q \rightarrow (r \rightarrow s)$ is true $\Rightarrow q : 0$

$\therefore$ when p, r, t are true; q, s are false :

p, $p \vee q$, $q \rightarrow (r \rightarrow s)$, $t \rightarrow r$ are all have the truth value 1.

while the conclusion $\neg s \rightarrow \neg t$ has the truth value 0

$\therefore$ the given argument invalid.

§ 2.3 Logical Implication: Rules of Inference

Note : **Prove : consider *all* cases.**

Disprove : provide *one* case called *counterexample*.

Ex 2.35 : **validity or invalidity?**

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow s \\ r \rightarrow \neg s \\ \neg p \veebar r \\ \hline \therefore \neg p \end{array}$$

Sol. **If $\neg p$ be false while the 4 premises are all true, then $p : 1$**

$\because p \rightarrow q$ is true $\quad \therefore q : 1$

$\because q \rightarrow s$ is true $\quad \therefore s : 1$

$\because r \rightarrow \neg s$ is true $\quad \therefore r : 0$

But $p : 1$, and $\because \neg p \veebar r$ is true $\quad \therefore r : 1 \quad \rightarrow\leftarrow$

i.e. $p \Rightarrow (\neg r \wedge r)$ $(\Leftrightarrow F_0)$

$\therefore$ the given argument is valid. (Prove by contradiction)

期中考注意事項:

1. 考試時間: 9:10 ~ 12:00.
2. 將分兩教室考試, 請前一天先看moodle助教將會公布, 並請於9:00至教室, 按座位表(將張貼於前後門)入座.
3. 10:00後才能交卷；10:00後禁止進入教室考試。
4. 禁止使用**計算機**, **翻譯機**, 手機請**關機**. 禁止攜帶**計算紙**.
5. 當然不可以作弊.
6. 請於答案卷左上角填上題號以方便閱卷.
7. 請正確使用答案卷.
8. 可不按題號順序作答, 但**題號**請標示清楚.
9. 有任何疑問或需第二張答案卷者請舉手問助教, 但**不可**請助教翻譯題目.
10. 考試期間欲上廁所者將由助教陪同.
11. 使用兩張答案卷的同學記得兩張都要寫名字, 並將之合併後一起交回.

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Discrete Mathematics

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Chapter 2 Fundamentals of Logic

§ 2.4 The Use of Quantifiers (1)

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§ 2.4 The Use of Quantifiers

Def : ***open statement*** **:**

1) contains ≥ 1 variables;

2) not statement;

3) becomes a statement when the variables are replaced by certain allowable choices.

Def : ***universe*** **(*of discourse*) for the open statement :**

The set of all allowable choices.

§ 2.4 The Use of Quantifiers

ex : 1) "The number $x + 2$ is an even integer" not statement

2) "The number $x + 2$ is an even integer" is an open statement

3) "The number $x + 2$ is an even integer" is denoted by $p(x)$

then $\neg p(x)$: "The number $x + 2$ is not an even integer"

4) Let $q(x, y)$: The numbers $y + 2$, $x - y$, and $x + 2y$ are even integers.

Hence, $p(5)$ is F: The number 7 is an even integer.

$\neg p(7)$ is T: The number 9 is not an even integer.

$q(4, 2)$ is T: The numbers 4, 2 and 8 are even integers.

$q(5, 2)$, $q(4, 7)$ are F:

$\neg q(5, 2)$, $\neg q(4, 7)$ are T:

$\Rightarrow$ $\begin{cases} \text{For some } x, p(x). \\ \text{For some } x, y, q(x, y). \end{cases}$ or $\begin{cases} \text{For some } x, \neg p(x). \\ \text{For some } x, y, \neg q(x, y). \end{cases}$

並不是negation

§ 2.4 The Use of Quantifiers

Def : 1) ***existential quantifier*** : $\exists x$:

"for some x", "For at least one x", "there exists an x such that".

2) ***universal quantifier*** : $\forall x$:

"for all x", "for every x", "for each x", "for any x".

3) ***quantified statement*** : open statement with quantifier.

Note : 1) $\exists x\, \exists y\;\; q(x, y) = \exists x, y\; q(x, y)$

2) $\forall x\, \forall y\; q(x, y) = \forall x, y\; q(x, y)$

ex : $r(x)$: "$2x$ is an even integer", universe = all integers

1) $\forall x\; r(x)$ 2) $\exists x\; r(x)$

3) $\forall x \neg r(x)$ 3) $\exists x \neg r(x)$

§ 2.4 The Use of Quantifiers

Def : 1) ***free variable***: the variable x in each of open statement $p(x)$

2) ***bound variable***: the variable x in the statement $\exists x\, p(x)$ or $\forall x\, p(x)$

Ex 2.36 : The universe = R

$p(x) : x \geq 0$ $q(x) : x^2 \geq 0$ $r(x) : x^2 - 3x - 4 = 0$ $s(x) : x^2 - 3 > 0$

1) $\exists x\, [p(x) \wedge r(x)]$

Let $x = 4$, then $p(4)$ is true and $r(4)$ is true

$\therefore$ $[p(4) \wedge r(4)]$ is true $\Rightarrow$ $\exists x\, [p(x) \wedge r(x)]$ is true.

2) $\forall x\, [p(x) \rightarrow q(x)]$

x	$p(x)$	$q(x)$	$p(x) \rightarrow q(x)$
< 0	F		T
≥ 0	T	T	T

$\Rightarrow \forall x\, [p(x) \rightarrow q(x)]$ is true.

$p(x): x \geq 0$ $q(x): x^2 \geq 0$ $r(x): x^2 - 3x - 4 = 0$ $s(x): x^2 - 3 > 0$

§ 2.4 The Use of Quantifiers

Note : "$\forall x\,[p(x) \rightarrow q(x)]$"可能被敘述成：

a) For every real number x, if $x \geq 0$, then $x^2 \geq 0$.

b) Every nonnegative real number has a nonnegative square.

c) The square of any nonnegative real number is a nonnegative real number.

d) All nonnegative real numbers have nonnegative squares.

3) $\exists x\,[p(x) \rightarrow q(x)]$ is true

1') $\forall x\,[q(x) \rightarrow s(x)]$ (1) $\exists x\,[p(x) \wedge r(x)]$)

Let $x = 1$, $q(1)$ is true, but $s(1)$: $1 - 3 > 0$ is false

$\therefore q(1) \rightarrow s(1)$ is false, i.e. $x = 1$ is a counterexample

$\Rightarrow \forall x\,[q(x) \rightarrow s(x)]$ is false.

(不只一個反例，$\forall -\sqrt{3} < x < \sqrt{3}$皆是)

$p(x) : x \geq 0$ $q(x) : x^2 \geq 0$ $r(x) : x^2 - 3x - 4 = 0$ $s(x) : x^2 - 3 > 0$

§ 2.4 The Use of Quantifiers

2') $\forall x\, [r(x) \vee s(x)]$ (2) $\forall x\, [p(x) \rightarrow q(x)]$)

Let $x = 1$ (or ½, –3/2 , 0, …), then $r(1)$ is false and $s(1)$ is false, too. $\therefore r(1) \vee s(1)$ is false

$\Rightarrow \forall x\, [r(x) \vee s(x)]$ is false.

2") $\exists x\, [r(x) \vee s(x)]$ is true.

3') $\forall x\, [r(x) \rightarrow p(x)]$ (3) $\exists x\, [p(x) \rightarrow q(x)]$)

Let $x = -1$, $r(-1) = (-1)^2 - 3(-1) - 4 = 0$ is true, but $p(-1)$ is false.

$\therefore r(-1) \rightarrow p(-1)$ is false. (the unique counterexample)

$\Rightarrow \forall x\, [r(x) \rightarrow p(x)]$ is false.

Note : "$\forall x\, [r(x) \rightarrow p(x)]$"可被敘述成：

a) For every real number x, if $x^2 - 3x - 4 = 0$, then $x \geq 0$.

b) For every real number x, if x is a solution of the equation $x^2 - 3x - 4 = 0$, then $x \geq 0$.

§ 2.4 The Use of Quantifiers

Remark : 1) $\forall x\, p(x) \Rightarrow \exists x\, p(x)$, for the universe $\neq \phi$.

2) $\exists x\, p(x) \not\Rightarrow \forall x\, p(x)$

Ex 2.37 :

a) The universe = R

1) If a number is rational, then it is a real number.

2) If x is rational, then x is real.

(The use of the universal quantifier is ***implicit*** as opposed to ***explicit***)

Let $p(x)$: x is a rational number

$q(x)$: x is a real number

$\Rightarrow$ (1) = (2) = $\forall x\, [p(x) \rightarrow q(x)]$.

§ 2.4 The Use of Quantifiers

Ex 2.37 :

b) The universe = All triangles in the plane.

"An equilateral triangle has three angles of 60°, and conversely."

Let $e(t)$: Triangle t is equilateral

$a(t)$: Triangle t has three angles of 60°

⇒ " " $= \forall t\ [e(t) \leftrightarrow a(t)]$

c) $\sin^2 x + \cos^2 x = 1$ (for all real number x)

⇒ The universe of $x = R$, $\forall x\ [\sin^2 x + \cos^2 x = 1]$

d) The universe = N

"The integer 41 is equal to the sum of two perfect squares."

⇒ $\exists m\ \exists n\ [41 = m^2 + n^2]$

§ 2.4 The Use of Quantifiers

Ex 2.38 : $p(x) : x^2 \geq 0$

(1) The universe = R : $\forall x\, p(x)$ is true (∃$x\, p(x)$ is true)

(2) The universe = C : $\forall x\, p(x)$ is false (∃$x\, p(x)$ is true)

∵ let $x = i$, then $p(i) : i^2\ (= -1) \geq 0$ is false.

Ex 2.39 : See Textbook

Table 2.21 : See Textbook

Def 2.6 : Let $p(x)$, $q(x)$ be open statements defined for a given universe.

1) $p(x)$ is (*logically*) *equivalent* to $q(x)$: $\forall x\, [p(x) \Leftrightarrow q(x)]$: $p(x) \Leftrightarrow q(x)$ for each x in the universe.

2) $p(x)$ *logically implies* $q(x)$: $\forall x\, [p(x) \Rightarrow q(x)]$: $p(x) \Rightarrow q(x)$ for each x in the universe.

§ 2.4 The Use of Quantifiers

ex : The universe : all triangles in the plane.

Let $p(x)$: x is equiangular, $q(x)$: x is equilateral

$\because$ for all particular triangle a, $p(a) \leftrightarrow q(a)$ is true.

$\therefore \forall x\, [p(x) \Leftrightarrow q(x)]$

Note : 1) $\forall x\, [p(x) \Leftrightarrow q(x)]$ iff $\forall x\, [p(x) \Rightarrow q(x)] \wedge \forall x\, [q(x) \Rightarrow p(x)]$

2) Def 2.6 can be given for two open statements that involve $\geq$ 2 variable.

Def 2.7 :

1) The *contrapositive* of $\forall x\, [p(x) \rightarrow q(x)]$ is $\forall x\, [\neg q(x) \rightarrow \neg p(x)]$

2) The *converse* of $\forall x\, [p(x) \rightarrow q(x)]$ is $\forall x\, [q(x) \rightarrow p(x)]$

3) The *inverse* of $\forall x\, [p(x) \rightarrow q(x)]$ is $\forall x\, [\neg p(x) \rightarrow \neg q(x)]$

§ 2.4 The Use of Quantifiers

Ex 2.40 : The universe = all quadrilaterals in the plane

Let $s(x)$: x is a square, $e(x)$: x is equilateral

a) $\forall x\ [s(x) \rightarrow e(x)]$ is a true statement

$\Leftrightarrow \forall x\ [\neg\, e(x) \rightarrow \neg\, s(x)]$ (the contrapositive)

b) $\forall\ x\ [e(x) \rightarrow s(x)]$ is a false statement (the converse)

$\Leftrightarrow \forall x\ [\neg\, s(x) \rightarrow \neg\, e(x)]$ (the inverse)

Ex 2.41 : The universe = R

Let $p(x)$: $|x| > 3$, $q(x)$: $x > 3$

a) $\forall x\ [p(x) \rightarrow q(x)]$ is false. (let $x = -5$, $p(-5)$: T, $q(-5)$: F)

b) The converse of (a) = Every real number greater than 3 has magnitude (or, absolute value) greater than 3.

$\forall x\ [q(x) \rightarrow p(x)]$ is true.

$p(x) : |x| > 3, \;\; q(x) : x > 3$

§ 2.4 The Use of Quantifiers

Ex 2.41 :

c) The inverse of (a) is also true : $\forall x\, [\neg p(x) \rightarrow \neg q(x)]$:

"If the magnitude of a real number is less than or equal to 3, then the number itself is less than or equal to 3."

(b) ⇔ (c)

d) The contrapositive of (a) = $\forall x\, [\neg q(x) \rightarrow \neg p(x)]$ (is false) :

"If a real number is less than or equal to 3, then so is its magnitude."

(d) ⇔ (a)

e) Let $r(x) : x < -3$, defined for the universe of all real number :

Statement :	$\forall x\, [p(x) \rightarrow (r(x) \vee q(x))]$	all true
Contrapositive :	$\forall x\, [\neg (r(x) \vee q(x)) \rightarrow \neg p(x)]$	
Converse :	$\forall x\, [(r(x) \vee q(x)) \rightarrow p(x)]$	
Inverse :	$\forall x\, [\neg p(x) \rightarrow \neg (r(x) \vee q(x))]$	

$\Rightarrow \forall x\, [p(x) \Leftrightarrow (r(x) \vee q(x))]$

§ 2.4 The Use of Quantifiers

Ex 2.42 : The universe = Z

Let $r(x) : 2x + 1 = 5$, $s(x) : x^2 = 9$.

1) $\exists x\, [r(x) \wedge s(x)]$ is false

$\because$ no integer a such that $2a + 1 = 5$ and $a^2 = 9$

2) $\exists x\, r(x) \wedge \exists x\, s(x)$ is true

$\because$ $\exists b = 2$, $r(b) : 2b + 1 = 5$ is true

$\exists c = 3$, $s(c) : c^2 = 9$ is true

3) $\exists x\, [r(x) \wedge s(x)] \not\Leftrightarrow [\exists x\, r(x) \wedge \exists x\, s(x)]$

$[\exists x\, r(x) \wedge \exists x\, s(x)] \not\Rightarrow \exists x\, [r(x) \wedge s(x)]$

Def : 1) $\not\Leftrightarrow$ is read "*is not logically equivalent to*"

2) $\not\Rightarrow$ is read "*does not logically imply*"

§ 2.4 The Use of Quantifiers

Note : $\exists x\,[p(x) \wedge q(x)] \Rightarrow [\exists x\, p(x) \wedge \exists x\, q(x)]$

Proof.

If $\exists x\,[p(x) \wedge q(x)]$ is true then

there is at least one element c in the universe

s.t. $p(c) \wedge q(c)$ is true.

$\because [p(c) \wedge q(c)] \Rightarrow p(c)$ and $[p(c) \wedge q(c)] \Rightarrow q(c)$

(by Conjunctive Simplification)

i.e. $\exists x\, p(x)$ is true and $\exists x\, q(x)$ is true

$\therefore [\exists x\, p(x) \wedge \exists x\, q(x)]$ is true.

Table 2.22 : See Textbook

§ 2.4 The Use of Quantifiers

Ex 2.43 : Let $p(x)$, $q(x)$ and $r(x)$ denote open statements for a given universe.

1) $\forall x\, [p(x) \wedge (q(x) \wedge r(x))] \Leftrightarrow \forall x\, [(p(x) \wedge q(x)) \wedge r(x)]$

$\because$ For each a in the universe,

$$p(a) \wedge (q(a) \wedge r(a)) \Leftrightarrow (p(a) \wedge q(a)) \wedge r(a)$$

$\therefore \forall x\, [p(x) \wedge (q(x) \wedge r(x))] \Leftrightarrow \forall x\, [(p(x) \wedge q(x)) \wedge r(x)]$

2) $\exists x\, [p(x) \rightarrow q(x)] \Leftrightarrow \exists x\, [\neg p(x) \vee q(x)]$

$\because$ For each c in the universe,

$$[p(c) \rightarrow q(c)] \Leftrightarrow [\neg p(c) \vee q(c)]$$

$\therefore \exists x\, [p(x) \rightarrow q(x)]$ is true iff $\exists x\, [\neg p(x) \vee q(x)]$ is true

i.e. $\exists x\, [p(x) \rightarrow q(x)] \Leftrightarrow \exists x\, [\neg p(x) \vee q(x)]$

§ 2.4 The Use of Quantifiers

Ex 2.43 :

3) a) $\forall x \neg \neg p(x) \Leftrightarrow \forall x\, p(x)$

b) $\forall x \neg [p(x) \wedge q(x)] \Leftrightarrow \forall x\, [\neg p(x) \vee \neg q(x)]$

c) $\forall x \neg [p(x) \vee q(x)] \Leftrightarrow \forall x\, [\neg p(x) \wedge \neg q(x)]$

4) a) $\exists x \neg \neg p(x) \Leftrightarrow \exists x\, p(x)$

b) $\exists x \neg [p(x) \wedge q(x)] \Leftrightarrow \exists x\, [\neg p(x) \vee \neg q(x)]$

c) $\exists x \neg [p(x) \vee q(x)] \Leftrightarrow \exists x\, [\neg p(x) \wedge \neg q(x)]$

§ 2.4 The Use of Quantifiers

Remark : $\neg [\forall x\, p(x)] \Leftrightarrow \exists x \neg p(x)$

Proof.

$\neg [\forall x\, p(x)]$: It is not the case that for all x, $p(x)$ holds.

1) "$\neg [\forall x\, p(x)]$" is true = "$\forall x\, p(x)$" is false

= For some a of the universe, $\neg p(a)$ is true

= "$\exists x \neg p(x)$" is true

2) "$\exists x \neg p(x)$" is true = For some b of the universe,

$\neg p(b)$ is true

= For some b of the universe, $p(b)$ is false

= "$\forall x\, p(x)$" is false

= "$\neg [\forall x\, p(x)]$" is true

by (1), (2) "$\neg [\forall x\, p(x)]$" is true iff "$\exists x \neg p(x)$" is true

Similar, "$\neg [\forall x\, p(x)]$" is false iff "$\exists x \neg p(x)$" is false

§ 2.4 The Use of Quantifiers

Table 2.23 :

$\neg [\forall x\, p(x)]$	$\Leftrightarrow$	$\exists x \neg p(x)$		
$\neg [\exists x\, p(x)]$	$\Leftrightarrow$	$\forall x \neg p(x)$		
$\neg [\forall x \neg p(x)]$	$\Leftrightarrow$	$\exists x \neg\neg p(x)$	$\Leftrightarrow$	$\exists x\, p(x)$
$\neg [\exists x \neg p(x)]$	$\Leftrightarrow$	$\forall x \neg\neg p(x)$	$\Leftrightarrow$	$\forall x\, p(x)$

Rules of Negating Statements with One Quantifier.

§ 2.4 The Use of Quantifiers

Ex. 2.44 : The universe $= Z$

1) Let $p(x)$: x is odd, $q(x)$: $x^2 - 1$ is even

"If x is odd, then $x^2 - 1$ is even" $= \forall x\, [p(x) \rightarrow q(x)]$ (T)

$\neg\, [\forall\, x\, (p(x) \rightarrow q(x))]$

$\Leftrightarrow \exists x\, [\neg\, (p(x) \rightarrow q(x))] \Leftrightarrow \exists x\, [\neg\, (\neg\, p(x) \vee q(x))]$

$\Leftrightarrow \exists x\, [\neg \neg\, p(x) \wedge \neg\, q(x))] \Leftrightarrow \exists x\, [p(x) \wedge \neg\, q(x))]$

= "There exists an integer x such that x is odd and $x^2 - 1$ is odd" (F)

2) $r(x)$: $2x + 1 = 5$, $s(x)$: $x^2 = 9$ (In Ex 2.42)

$\exists x\, [r(x) \wedge s(x)]$ is false, $\neg\, [\exists x\, (r(x) \wedge s(x))]$ is true?

$\neg\, [\exists x\, (r(x) \wedge s(x))] \Leftrightarrow \forall x\, [\neg\, (r(x) \wedge s(x))]$

$\Leftrightarrow \forall x\, [\neg\, r(x) \vee \neg\, s(x))]$

= "For every integer x , $2x + 1 \neq 5$ or $x^2 \neq 9$."