Computer Science and Information Engineering
National Chi Nan University

Discrete Mathematics

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Chapter 1 Fundamental Principles of Counting

§ 1.3 Combinations: The Binomial Theorem

Slides for a Course Based on the Text
Discrete & Combinatorial Mathematics (5th Edition)
by Ralph P. Grimaldi

§ 1.3 Combinations: The Binomial Theorem

See textbook from p.15 to p.19 by yourself.

If there are n distinct objects, each selection, or combination, of r of these objects, with no reference to order, corresponds to $r!$ permutations of size r from the n objects. Thus the number of combinations of size r from a collection of size n, denoted $C(n, r)$, where $0 \le r \le n$, satisfies $(r!) \times C(n, r) = P(n, r)$ and

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}, \quad 0 \le r \le n.$$

Also denoted by $\binom{n}{r}$ (= $({}^{n}{}_{r})$), read as "n choose r".

Def : 1. For $r = 0$, $C(n, 0) = 1$, for all $n \ge 0$.

2. For all $0 \le r \le n$, $C(n, r) = n!/[r!\,(n - r)!]$.

§ 1.3 Combinations: The Binomial Theorem

<u>EX 1.25</u> : Select 5 cards from 52 cards (5/52):

a) no clubs $\binom{39}{5} = 575{,}757$.

b) at least one club ≡ not in (*a*) $= \binom{52}{5} - \binom{39}{5} = 2{,}023{,}203$.

c) obtain (*b*) in another way ? Select one club, then 4/51 ?

$$\binom{13}{1} \cdot \binom{51}{4} = 13 \times 249{,}900 = 3{,}248{,}700.$$

→ ***wrong!!***

<u>ex</u> : 3C : 5C-KC-7H-JS

5C : 3C-KC-7H-JS

KC : 3C-5C-7H-JS

→ all the same!! But count 3 times!!

§ 1.3 Combinations: The Binomial Theorem

d) another way :

Number of Clubs		Number of Cards that are not Clubs	
1	$\binom{13}{1}$	4	$\binom{39}{4}$
2	$\binom{13}{2}$	3	$\binom{39}{3}$
3	$\binom{13}{3}$	2	$\binom{39}{2}$
4	$\binom{13}{4}$	1	$\binom{39}{1}$
5	$\binom{13}{5}$	0	$\binom{39}{0}$

$$\Rightarrow \sum_{i=1}^{5} \binom{13}{i}\binom{39}{5-i} = 2{,}023{,}203.$$

§ 1.3 Combinations: The Binomial Theorem

<u>Note</u> : $\binom{n}{r} = \binom{n}{n-r}$, $\forall\ n \geq r \geq 0$.

<u>Thm 1.1</u> : The Binomial Theorem

If x and y are variables and n is a positive integer , then

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{n-k} x^k y^{n-k}$$

<u>ex</u> : $(x + y)^4 = (x + y)\,(x + y)\,(x + y)\,(x + y) \rightarrow x^2 y^2$

Proof.

$$(x + y)^n = (x + y)\,(x + y) \ldots (x + y)$$

nth factor

the coefficient of $x^k y^{n-k}$, where $0 \leq k \leq n$, is the number of different ways in which we can select k x's from the n available factors

§ 1.3 Combinations: The Binomial Theorem

EX 1.26 : a) $(x + y)^7$: the coefficient of $x^5 y^2$ = ? $\binom{7}{2} = \binom{7}{5} = 21$

b) $(2a - 3b)^7$: the coefficient of $a^5 b^2$ = ?

$$(2a - 3b)^7 = \sum_{k=0}^{7} \binom{7}{k} (2a)^k (-3b)^{7-k}$$

$$= \sum_{k=0}^{7} \binom{7}{k} 2^k \cdot (-3)^{7-k} \cdot a^k \cdot b^{7-k}$$

$$k = 5 \Rightarrow \binom{7}{5} \cdot 2^5 \cdot (-3)^{7-5} = 21 \cdot 32 \cdot 9 = 6048$$

Corollary 1.1 : $\forall\, n \in N$, (a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$

(b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0$

Proof.

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k \cdot y^{n-k}$$

(a) Let $x = y = 1 \Rightarrow 2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$

(b) Let $x = -1, y = 1 \Rightarrow (-1 + 1)^n = 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1)^{n-k}$

$$= \binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n \binom{n}{n}$$

§ 1.3 Combinations: The Binomial Theorem

Theorem 1.2 : **multinomial theorem**

$$\forall\ n, t \in \mathbb{N}$$

$$(x_1 + x_2 + \ldots + x_t)^n = \sum_{\substack{n_i \in \mathbb{Z}\ 0 \le n_i \le n\ \forall\ 1 \le i \le t \\ \text{and } n_1 + n_2 + \ldots + n_t = n}} \frac{n!}{n_1! n_2! \cdots n_t!} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$$

Proof.

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{t-1}}{n_t}$$

$$= \frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{(n-n_1-n_2-\cdots-n_{t-1})!}{n_t!(n-n_1-n_2-\cdots-n_t)!}$$

$$= \frac{n!}{n_1! n_2! \cdots n_t!}$$

§ 1.3 Combinations: The Binomial Theorem

Def : $\dfrac{n!}{n_1!\,n_2!\cdots n_t!} = \dbinom{n}{n_1 n_2 \cdots n_t}$ (= $\binom{n}{n_1\, n_2\, \ldots\, n_t}$) called **multinomial coefficient**.

EX 1.27 : a) $(x + y + z)^7$:

the coefficient of $x^2\, y^2\, z^3 = 7! \,/\, (2!\; 2!\; 3!) = 210$

the coefficient of $x\, y\, z^5 = 7! \,/\, 5! = 42$

the coefficient of $x^3\, z^4 = 7! \,/\, (3!\; 4!) = 35$

b) $(a + 2b - 3c + 2d + 5)^{16}$:

the coefficient of $a^2\, b^3\, c^2\, d^5$ = ?

Sol.

General item : $\binom{16}{i\, j\, k\, l\, m}\, a^i\, (2b)^j\, (-3c)^k\, (2d)^l\, (5)^m$

Let $i = 2;\ j = 3;\ k = 2;\ l = 5$

$\Rightarrow m = 16 - 2 - 3 - 2 - 5 = 4$

$\Rightarrow$ the coefficient of $a^2\, b^3\, c^2\, d^5$

$= \binom{16}{2\,3\,2\,5\,4}\, 2^3\, (-3)^2\, 2^5\, 5^4 = 435{,}891{,}456{,}000{,}000$

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Chapter 1 Fundamental Principles of Counting

§ 1.4 Combinations With Repetition

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§ 1.4 Combinations With Repetition

Note : n : distinct object, r : arrangement size $\Rightarrow n^r$, $\forall\ r \geq 0$.

EX 1.28 : 7 high school freshmen,
4 purchases : cheeseburger, hot dog, taco, fish sandwich.
How many purchases are possible?

Sol.

http://zh.wikipedia.org/wiki/%E5%A2%A8%E8%A5%BF%E5%93%A5%E5%8D%B7%E9%A5%BC

see Table 1.6 : c, c, h, h, t, t, f $\Rightarrow xx \mid xx \mid xx \mid x$

t, t, t, t, t, f, f $\Rightarrow \mid \mid xxxxx \mid xx$

$\Rightarrow (3 + 7)! / (7!\ 3!) = \binom{10}{7}$

The number of combinations of n objects taken r at a time, with repetition, is $C(n + r - 1, r) = \dfrac{(n+r-1)!}{r!(n-1)!} = \dbinom{n+r-1}{r}$

§ 1.4 Combinations With Repetition

EX 1.29 : **20 kinds of donuts, select a dozen:**

$$\binom{20+12-1}{12} = \binom{31}{12} = 141{,}120{,}525.$$

EX 1.30 : **4 vice presidents / $1000 (each multiple of $100)**

(a) $C(4 + 10 - 1, 10) = C(13, 10) = 286$

(b) each receive at least $100:

$$C(4 + 6 - 1, 6) = 84$$

(c) each receive at least $100, and Mona get at least $500:

$$C(4 + 2 - 1, 2)$$
$$= C(3 + 2 - 1, 2) + C(3 + 1 - 1, 1) + C(3 + 0 - 1, 0)$$
$$= 10$$

§ 1.4 Combinations With Repetition

EX 1.31 : 7 bananas, 6 oranges / 4 children : each at least one banana

Sol.

see Table 1.7 :

1)	1, 2, 3	$b \mid b \mid b \mid$
2)	1, 3, 3	$b \mid \mid b\ b \mid$

$\Rightarrow 6! / (3!\ 3!) = \binom{6}{3}$

see Table 1.8 :

1)	1, 2, 2, 3, 3, 4	$o \mid oo \mid oo \mid o$
3)	2, 2, 2, 3, 3, 3	$\mid ooo \mid ooo \mid$

$\Rightarrow (6 + 3)! / (6!\ 3!) = \binom{9}{6}$

$\Rightarrow C(4 + 3 - 1, 3) \cdot C(4 + 6 - 1, 6) = 1680.$

§ 1.4 Combinations With Repetition

EX 1.32 : message : 12 different symbols + 45 spaces with at least 3 spaces between each pair of consecutive symbols How many?

Sol.

$$45 - (3 \cdot 11) = 12$$

$$12! \cdot C(11 + 12 - 1, 12) \doteqdot 3.097 \cdot 10^{14}$$

EX 1.33 : All integer solutions : $x_1 + x_2 + x_3 + x_4 = 7$ where $x_i \geq 0$, $\forall\ 1 \leq i \leq 4$.

Sol.

$$C(4 + 7 - 1, 7) = 120.$$

§ 1.4 Combinations With Repetition

Note : The following are the equivalence:

(a) The number of integer solution of the equation

$$x_1 + x_2 + \ldots + x_n = r,\ \ x_i \geq 0,\ \ 1 \leq i \leq n.$$

(b) The number of selections, with repetitions, of size *r* from a collection of size *n*.

(c) The number of ways *r* identical objects can be distributed among *n* distinct containers.

EX 1.34 : 10 identical white marbles (彈珠) among six distinct containers!

Sol.

$$x_1 + x_2 + \ldots + x_6 = 10$$

$$C^{6+10-1}_{10} = C^{15}_{10} \Rightarrow 3003$$

§ 1.4 Combinations With Repetition

EX 1.35 : $x_1 + x_2 + \ldots + x_6 < 10$, $x_i \geq 0$, $1 \leq i \leq 6$.

Sol.

($\Rightarrow x_1 + x_2 + \ldots + x_6 = k$, $k \in$ N and $0 \leq k \leq 9$)

$\Rightarrow x_1 + x_2 + \ldots + x_6 + x_7 = 10, \quad x_i \geq 0, 1 \leq i \leq 6, \; x_7 > 0$

Let $x_7 - 1 = y_7$

$\Rightarrow x_1 + x_2 + \ldots + x_6 + y_7 = 9, \quad x_i \geq 0, \; y_7 \geq 0$

$\Rightarrow C(7 + 9 - 1, 9) = 5005$.

EX 1.36 : $(x + y)^n$: total number of terms = ?

$\Rightarrow C(2 + n - 1, n) = n + 1$.

$(w + x + y + z)^{10}$: total number of terms = ?

$\Rightarrow C(4 + 10 - 1, 10) = 286$

§ 1.4 Combinations With Repetition

EX 1.37 : (a) **compositions** of 4 (**partitions** for the number 4) :

1) 4
2) 3 + 1
3) 2 + 2
4) 1 + 3
5) 2 + 1 + 1
6) 1 + 2 + 1
7) 1 + 1 + 2
8) 1 + 1 + 1 +1

(b) the number of composition for "7".

(i) one summand : 1

(ii) two summands : $w_1 + w_2 = 7 \Rightarrow x_1 + x_2 = 5,\ x_1, x_2 \geq 0$

$\Rightarrow \binom{2+5-1}{5} = \binom{6}{5}$

(iii) three summands : $y_1 + y_2 + y_3 = 7$

$\Rightarrow z_1 + z_2 + z_3 = 4,\ z_1, z_2, z_3 \geq 0$

$\Rightarrow \binom{3+4-1}{4} = \binom{6}{4}$

⋮

(vii) seven summands $\Rightarrow \binom{7+0-1}{0} = \binom{6}{0}$

$\Rightarrow \sum_{k=0}^{6} \binom{6}{k} = 2^6$ (by corollary 1.1)

§ 1.4 Combinations With Repetition

Note : for each positive integer $m \Rightarrow \sum_{k=0,\, m-1} \binom{m-1}{k} = 2^{m-1}$

EX 1.38 : There are $2^{12-1} = 2^{11} = 2048$ compositions of 12.
How many in those compositions where each summand is even?

Sol.

For instance, $2 + 4 + 6 = 2(1 + 2 + 3)$ $\quad 2 + 8 + 2 = 2(1 + 4 + 1)$
$8 + 2 + 2 = 2(4 + 1 + 1)$ $\quad 6 + 6 = 2(3 + 3)$.

$\Rightarrow$ # of compositions of 12, where each summand is even
= # of compositions of 6
$= 2^{6-1}$
$= 2^5$
$= 32$.

§ 1.4 Combinations With Repetition

EX 1.39 :

```
for i : = 1 to 20 do
    for j : = 1 to i do
        for k : = 1 to j do
            Print (i * j + k)
```

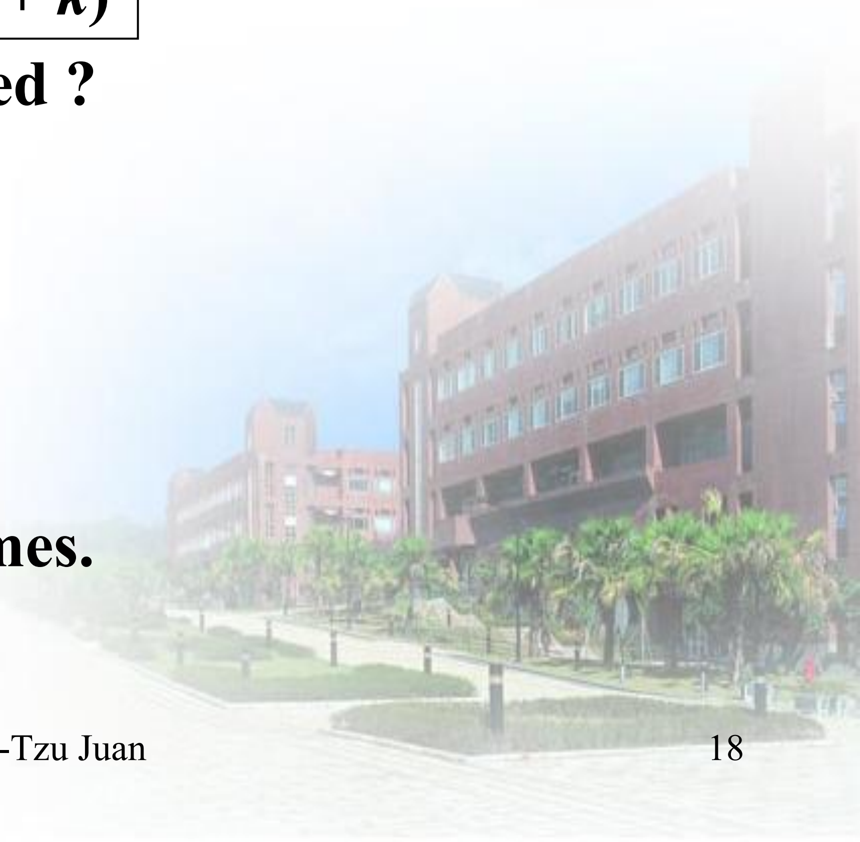
print statement executed ?

Sol.

Since $1 \le k \le j \le i \le 20$,

$\Rightarrow \binom{20+3-1}{3} = \binom{22}{3} = 1540.$

Note : r (≥ 1) for loops : $\binom{20+r-1}{r}$ times.



§ 1.4 Combinations With Repetition

EX 1.40 :

```
counter : = 0
for i : = 1 to n do
      for j : = 1 to i do
            counter : = counter + 1
```

Sol.

As EX 1.39, the statement executed :

$1 \le j \le i \le n, \rightarrow \binom{n+2-1}{2} = \binom{n+1}{2},$

or

$1 + 2 + 3 + \ldots + n.$

Therefore,

$1 + 2 + 3 + \ldots + n = \sum_{i=1}^{n} i = \binom{n+1}{2} = n\,(n+1)\,/\,2.$

§ 1.4 Combinations With Repetition

EX 1.41 : **The counter at Patti and Terri's Bar has 15 bar stools.**

O O E O O O O E E E O O O E O

means 10 occupied (O) stools and 5 empty (E) stools.

Say the occupancy of the 15 stools determine 7 *runs*, as shown:

OO E OOOO EEE OOO E O

run run run run run run run

Find the total number of ways 5 E's and 10 O's can determine seven runs.

Sol. (1/2)

1. Start with E :

Let x_1 count the number of E's in the first run,

x_2 count the number of O's in the second run,

x_3 count the number of E's in the third run, …,

x_7 count the number of E's in the 7th run.

§ 1.4 Combinations With Repetition

Sol. (2/2)

1. Start with E :

$$\begin{cases} x_1 + x_3 + x_5 + x_7 = 5, & x_1, x_3, x_5, x_7 > 0; \\ x_2 + x_4 + x_6 = 10, & x_2, x_4, x_6 > 0. \end{cases}$$

$$\Rightarrow \begin{cases} y_1 + y_3 + y_5 + y_7 = 1, & y_1, y_3, y_5, y_7 \geq 0; \\ y_2 + y_4 + y_6 = 7, & y_2, y_4, y_6 \geq 0. \end{cases}$$

$$\Rightarrow \binom{4+1-1}{1} \cdot \binom{3+7-1}{7} = C(4, 1) \cdot C(9, 7) = 4 \cdot 36 = 144.$$

2. Start with O :

Let w_1 count the number of O's in the first run, …

$$\begin{cases} w_1 + w_3 + w_5 + w_7 = 10, & w_1, w_3, w_5, w_7 > 0; \\ w_2 + w_4 + w_6 = 5, & w_2, w_4, w_6 > 0. \end{cases}$$

$$\Rightarrow \binom{4+6-1}{6} \cdot \binom{3+2-1}{2} = C(9, 6) \cdot C(4, 2) = 84 \cdot 6 = 504.$$

$$\Rightarrow 144 + 504 = 648.$$

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Chapter 2 Fundamentals of Logic

§ 2.1 Basic Connectives and Truth tables

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§ 2.1 Basic Connectives and Truth tables

Def : ***statement*** **(*proposition*) : either *true* or *false*, not both.**

ex :

- ✓ ***p*: Combinatorics is a required course for sophomores.**
- ✓ ***q*: Margaret Mitchell wrote Gone with the Wind.**
- ✓ ***r*: 2 + 3 = 5.**
- ✗ **"What a beautiful evening!"**
- ✗ **"Get up and do your exercises."**
- ✗ **"The number *x* is an integer."**

Def : **1. *primitive statement*: No way to break them down into anything simpler.**

2. 反之: *compound statement*

§ 2.1 Basic Connectives and Truth tables

Def : 1. ***negation***; denoted by $\neg p$; read as "***not p***".

ex : 上ex中p:

$\neg p$ = "Combinatorics is not a required course for sophomores."

Def : 2. ***compound statement***, using the following ***logical connectives***.

a) ***Conjunction***; denoted by $p \wedge q$; read as "***p and q***".

b) ***Disjunction***;

- denoted by $p \vee q$; read as "***p (inclusive) or q***".
- denoted by $p \underline{\vee} q$; read as "***p exclusive or q***".

§ 2.1 Basic Connectives and Truth tables

Def : 2. ***compound statement***.

c) ***Implication***; denoted by $p \to q$; read as "*p implies q*".

$\equiv$ (i) If p, then q

(ii) p is ***sufficient*** for q

(iii) p is a ***sufficient condition*** for q

(iv) p only if q

(v) q is ***necessary*** for p

(vi) q is a ***necessary condition*** for p

(vii) p is called the ***hypothesis*** of the implication.

(viii) q is called the ***conclusion*** of the implication.

d) ***Biconditional***; denoted by $p \leftrightarrow q$; read as "*p if and only if q*".

(i) "p is ***necessary and sufficient*** for q."

(ii) "p ***iff*** q."

§ 2.1 Basic Connectives and Truth tables

Def : ***truth table***: "0" for false and "1" for true.

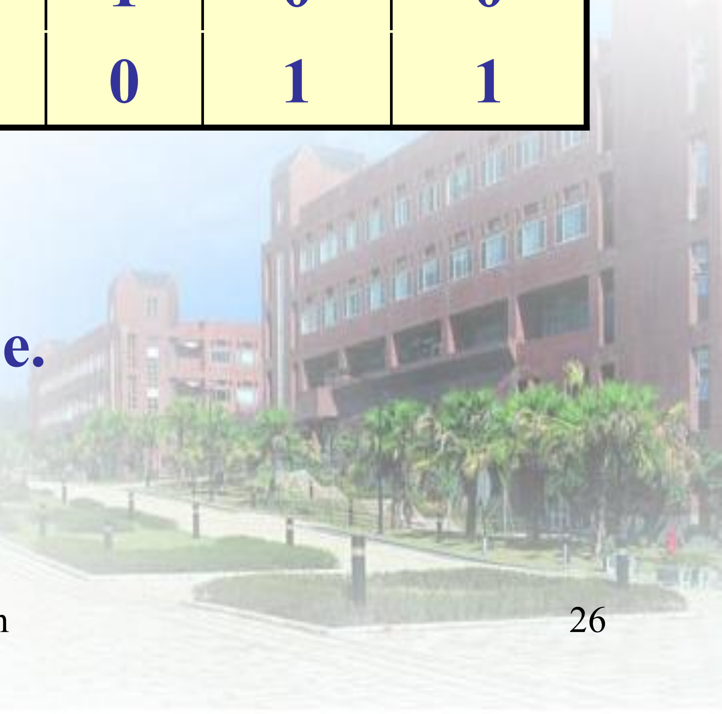
Table2.1:

p	$\neg p$
0	1
1	0

2.2:

p	q	$p \wedge q$	$p \vee q$	$p \underline{\vee} q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

ex : "If 2 + 3 = 6, then 2 + 4 = 7" is true.



§ 2.1 Basic Connectives and Truth tables

EX 2.1 : s: Phyllis goes out for a walk.

t: The moon is out.

u: It is snowing.

a) $(t \wedge \neg u) \rightarrow s$:

b) $t \rightarrow (\neg u \rightarrow s)$: "$\neg u \rightarrow s$" means "$(\neg u) \rightarrow s$", not "$\neg (u \rightarrow s)$"

c) $\neg (s \leftrightarrow (u \vee t))$:

d) "Phyllis will go out walking if and only if the moon is out" : $s \leftrightarrow t$

e) "If it is snowing and the moon is not out, then Phyllis will not go out for a walk" : $(u \wedge \neg t) \rightarrow \neg s$

f) "It is snowing but Phyllis will still go out for a walk" : $u \wedge s$

(where "but" ≡ "and")

§ 2.1 Basic Connectives and Truth tables

EX 2.2 : "If I weigh more than 120 pounds, then I shall enroll in an exercise class".

p: I weigh more than 120 pounds.

q: I shall enroll in an exercise class.

Penny's statement: $p \rightarrow q$

Case 1: $p = 1$ and $q = 1$: > 120 pounds and enrolls :

Case 2: $p = 1$ and $q = 0$: > 120 pounds but not enroll :

Case 3: $p = 0$ and $q = 0$: ≤ 120 pounds and not enroll :

Case 4: $p = 0$ and $q = 1$: ≤ 120 pounds but still enroll :

§ 2.1 Basic Connectives and Truth tables

EX 2.3 : In computer science: if-then, if-then-else.

ex : if $x > 2$ (執行時，給定 "x"值，則 "$x > 2$"為一 "logical statement")
then $y = 2$ ("executable statement", not "logical statement")
else $y = 3$ ("executable statement", not "logical statement")

ex : 生活上的 "→" 與 "↔"

$s \to t$: If you do your homework, then you will get to watch the baseball game.

$t \to s$: You will get to watch the baseball game only if you do your homework.

§ 2.1 Basic Connectives and Truth tables

EX 2.4 : **"Margaret Mitchell wrote Gone with the Wind, and if $2 + 3 \neq 5$, then combinatorics is a required course for sophomores".**

$\equiv q \wedge (\neg r \rightarrow p)$

p	q	r	$\neg r$	$\neg r \rightarrow p$	$q \wedge (\neg r \rightarrow p)$
0	0	0	1	0	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	0	1	1
1	0	0	1	1	0
1	0	1	0	1	0
1	1	0	1	1	1
1	1	1	0	1	1

§ 2.1 Basic Connectives and Truth tables

EX 2.5 : The truth tables for ① $p \vee (q \wedge r)$; ② $(p \vee q) \wedge r$.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$(p \vee q) \wedge r$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	1	0
0	1	1	1	1	1	1
1	0	0	0	1	1	0
1	0	1	0	1	1	1
1	1	0	0	1	1	0
1	1	1	1	1	1	1

∴不可以只寫$p \vee q \wedge r$，需標明為 $p \vee (q \wedge r)$ 或 $(p \vee q) \wedge r$!!

§ 2.1 Basic Connectives and Truth tables

EX 2.6 : $p \rightarrow (p \vee q), p \wedge (\neg p \wedge q)$

p	q	$p \vee q$	$p \rightarrow (p \vee q)$	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0

$p \rightarrow (p \vee q)$ is true for all truth value;
$p \wedge (\neg p \wedge q)$ is false for all truth value.

§ 2.1 Basic Connectives and Truth tables

Def 2.1 : A compound statement is called a ***contradiction*** (***tautology***) if it is false (true) for all truth value assignments for its component statements, denoted by F_0 (T_0).

ex : $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$

only need to prove:

"when $p_1 = p_2 = \ldots = p_n = 1$ and q must $= 1$",

then $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is a tautology

and we have a ***valid argument***.

Def : Where such p_i is called ***given statements*** (***premises***);

q is called ***conclusion***.