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MINIMUM SPAN OF NO-HOLE $(r+1)$-DISTANT COLORINGS*

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Abstract. Given a nonnegative integer r, a no-hole $(r+1)$-distant coloring, called N_r-coloring, of a graph G is a function that assigns a nonnegative integer (color) to each vertex such that the separation of the colors of any pair of adjacent vertices is greater than r, and the set of the colors used must be consecutive. Given r and G, the minimum N_r-span of G, $\mathrm{nsp}_r(G)$, is the minimum difference of the largest and the smallest colors used in an N_r-coloring of G if there exists one; otherwise, define $\mathrm{nsp}_r(G) = \infty$. The values of $\mathrm{nsp}_1(G)$ $(r = 1)$ for bipartite graphs are given by Roberts [*Math. Comput. Modelling*, 17 (1993), pp. 139–144]. Given $r \geq 2$, we determine the values of $\mathrm{nsp}_r(G)$ for all bipartite graph with at least $r-2$ isolated vertices. This leads to complete solutions of $\mathrm{nsp}_2(G)$ for bipartite graphs.

Key words. vertex-coloring, no-hole $(r+1)$-distant coloring, minimum span, bipartite graphs

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1. Introduction.

The T-coloring of graphs models the *channel assignment problem* introduced by Hale [6] in communication networks. In the channel assignment problem, several transmitters and a forbidden set T (called *T-set*) of nonnegative integers with $0 \in T$ are given. We assign a nonnegative integral channel to each transmitter under the constraint that if two transmitters interfere, the difference of their channels does not fall within the given T-set. Two transmitters may interfere due to various reasons such as geographic proximity and meteorological factors. To formulate this problem, we construct a graph G such that each vertex represents a transmitter, and two vertices are adjacent if their corresponding transmitters interfere.

Thus, we have the following definition. Given a T-set and a graph G, a *T-coloring* of G is a function $f : V(G) \to Z^+ \cup \{0\}$ such that

$$|f(x) - f(y)| \notin T \text{ if } xy \in E(G).$$

Note that if $T = \{0\}$, then T-coloring is the same as ordinary vertex-coloring.

A *no-hole T-coloring* of a graph G is a T-coloring f of G such that the set $\{f(v) : v \in V(G)\}$ is consecutive (the no-hole assumption). When $T = \{0, 1\}$ and $T = \{0, 1, 2, \ldots, r\}$, a no-hole T-coloring is also called an *N-coloring* [16] and an *N_r-coloring* (or *no-hole $(r+1)$-distant coloring*) [17], respectively. That is, an N_r-coloring of a graph G is a vertex coloring $f : V(G) \to Z^+ \cup \{0\}$ such that the following two conditions are satisfied:

- $|f(x) - f(y)| \geq r + 1$ if $uv \in E(G)$;

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• the set $\{f(v) : v \in V(G)\}$ is consecutive.

In terms of efficiency of the usage of the channels (colors), the variable T-span has been considered. The *span* of a T-coloring f is the difference of the largest and the smallest colors used in $f(V)$; the *T-span* of a graph G, $\mathrm{sp}_T(G)$, is the minimum span among all T-colorings of G.

The T-spans for different families of graphs and for different T-sets have been studied extensively (see [3, 4, 5, 10, 11, 12, 14, 15, 18]). It is known [3, 10] that if T is an *r-initial set*, that is, $T = \{0, 1, 2, \ldots, r\} \cup A$ where A is a set of integers without multiples of $(r+1)$, then the following holds for all graphs:

$$(*) \qquad \mathrm{sp}_T(G) = (\chi(G) - 1)(r + 1),$$

where $\chi(G)$, the *chromatic number* of G, is the minimum number of colors to properly color vertices of G.

It is known [3] and not difficult to learn that for any given T-set and any graph G, a T-coloring always exists. However, a no-hole T-coloring does not always exist. For instance, as $T = \{0, 1\}$, then K_n, the complete graph with n vertices, does not have a no-hole T-coloring for any $n \geq 2$.

The minimum span of a no-hole T-coloring for a graph G is denoted by $\mathrm{nsp}_T(G)$. If there does not exist a no-hole T-coloring for G, then $\mathrm{nsp}_T(G) = \infty$. If $T = \{0, 1, 2, \ldots, r\}$, denote $\mathrm{nsp}_T(G)$ by $\mathrm{nsp}_r(G)$.

A no-hole T-coloring is also a T-coloring. Hence by $(*)$, a natural lower bound for $\mathrm{nsp}_r(G)$ is $(\chi(G)-1)(r+1)$. Roberts [16] and Sakai and Wang [17] studied N-coloring and N_r-coloring, respectively. Among the findings in [16, 17] are the results about the existence of an N-coloring and an N_r-coloring for several families of graphs including paths, cycles, bipartite graphs, and 1-unit sphere graphs. The authors also compare the span of such a coloring (if there exists one) with the lower bound $(\chi(G)-1)(r+1)$. The N-colorings and N_r-colorings studied in [16, 17] are not necessarily optimal; i.e., the spans are not always the minimum.

This article focuses on the exact values of the minimum N_r-span, $\mathrm{nsp}_r(G)$, especially for bipartite graphs, i.e., graphs with $\chi(G) \leq 2$. In section 2, we give preliminary results for general graphs. In section 3, we explore the values of $\mathrm{nsp}_r(G)$ for bipartite graphs. The solutions of $\mathrm{nsp}_1(G)$ for bipartite graphs are given by Roberts [16]. We determine the values of $\mathrm{nsp}_r(G)$ for any bipartite graph G with at least $r-2$ isolated vertices. This result also leads to a complete description of the values of $\mathrm{nsp}_2(G)$ for all bipartite graphs.

2. Preliminary results. In this section, we present several results about the minimum N_r-span for general graphs. We show a number of upper and lower bounds of $\mathrm{nsp}_r(G)$ for different types of graphs. In order to find the minimum span, without loss of generality, we assume that the color 0 is always used in any N_r-coloring.

If $|V(G)| = n$ and $\mathrm{nsp}_T(G) < \infty$, then by definition a trivial upper bound for $\mathrm{nsp}_T(G)$ is $n-1$. On the other hand, any no-hole T-coloring is also a T-coloring, hence we have the following proposition.

Proposition 2.1. *For any T-set and any graph G with n vertices,* $\mathrm{sp}_T(G) \leq \mathrm{nsp}_T(G)$*; and* $\mathrm{nsp}_T(G) \leq n-1$ *if* $\mathrm{nsp}_T(G) < \infty$.

Combining Proposition 2.1 and $(*)$, we have the following proposition.

Proposition 2.2. *For any $r \in Z^+$ and any graph G with chromatic number $\chi(G)$, $(\chi(G)-1)(r+1) \leq \mathrm{nsp}_r(G)$.*

With the following result, we show a lower bound of $\mathrm{nsp}_r(G)$ in terms of r and the number of isolated vertices in G.

THEOREM 2.3. *Suppose $r \in Z^+$ and G is a graph with i isolated vertices, $i \geq 0$, and at least one edge. Then* $\mathrm{nsp}_r(G) \geq \max\{2r - i + 1, r + 1\}$.

Proof. It suffices to show the result when $\mathrm{nsp}_r(G)$ is finite. Because G has at least one edge, $\mathrm{nsp}_r(G) \geq r + 1$. Thus the lemma holds if $i \geq r$.

Suppose $i < r$. Let f be an optimal N_r-coloring of G. By the no-hole assumption of an N_r-coloring, the colors $r, r-1, \ldots, 2, 1, 0$, must be used by some vertices. Since G has only i isolated vertices and $i < r$, there exists a nonisolated vertex u with $r - i \leq f(u) \leq r$. Because u is nonisolated, there exists some vertex v such that $uv \in E(G)$. Then $f(v) \geq f(u)$, for otherwise $0 \leq f(u) - f(v) \leq r$, a contradiction to $uv \in E(G)$. Therefore, we have

$$f(v) \geq f(u) + r + 1 \geq r - i + r + 1 = \max\{2r - i + 1, r + 1\}.$$

This implies $\mathrm{nsp}_r(G) \geq \max\{2r - i + 1, r + 1\}$. $\square$

The *union* of two *vertex-disjoint* graphs G and H, denoted by $G \cup H$, is the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. For the case in which H has exactly one vertex x, $G \cup H$ is denoted by $G \cup \{x\}$.

The inequality $\mathrm{nsp}_r(G) \leq \mathrm{nsp}_r(G \cup H)$ does not always hold. For instance, if $G = K_2$, then $\mathrm{nsp}_1(G) = \infty$, while $\mathrm{nsp}_1(G \cup \{x\}) = 2$. In the rest of the section, we present several results on unions of graphs.

THEOREM 2.4. *Suppose G is a graph with at least one edge; then* $\mathrm{nsp}_{r+1}(G \cup \{x\}) \geq \mathrm{nsp}_r(G) + 1$.

Proof. It suffices to show the result when $\mathrm{nsp}_{r+1}(G \cup \{x\})$ is finite. Suppose f is an N_{r+1}-coloring of $G \cup \{x\}$. Define a coloring g on $V(G)$ by

$$g(v) = \begin{cases} f(v) & \text{if } f(v) < f(x) \text{ or } f(v) = 0, \\ f(v) - 1 & \text{if } f(v) \geq f(x) \text{ and } f(v) > 0. \end{cases}$$

It is straightforward to verify that g is an N_r-coloring of G and the span of g is one less than the span of f. Therefore, $\mathrm{nsp}_{r+1}(G \cup \{x\}) \geq \mathrm{nsp}_r(G) + 1$. $\square$

THEOREM 2.5. *Suppose G is a graph with* $\mathrm{nsp}_r(G) = q(r + 1) + j$, *where $q \geq 1$ and $0 \leq j \leq r$, and H is a graph with q vertices. Then* $\mathrm{nsp}_{r+1}(G \cup H) \leq \mathrm{nsp}_r(G) + q$.

Proof. It suffices to show the result when $\mathrm{nsp}_r(G) < \infty$. Let f be an optimal N_r-coloring of G and $f(V(G)) = \{0, 1, \ldots, \mathrm{nsp}_r(G)\}$. Suppose $V(H) = \{x_1, x_2, \ldots, x_q\}$. Define a coloring g on $G \cup H$, $g : V(G \cup H) \to Z^+ \cup \{0\}$, by

$$g(v) = \begin{cases} \lfloor \frac{(r+2)f(v)}{r+1} \rfloor & \text{if } v \in V(G), \\ k(r + 2) - 1 & \text{if } v = x_k \in V(H). \end{cases}$$

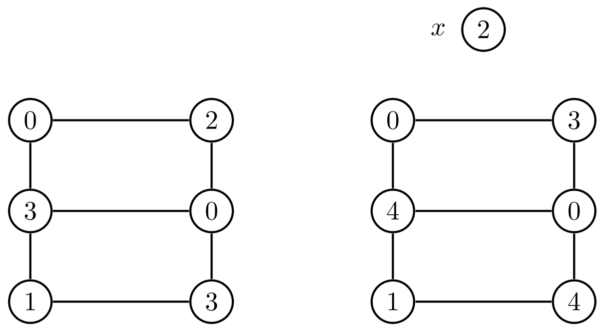
It is enough to show that g is an N_{r+1}-coloring for $G \cup H$. Because f is onto, therefore $g(V(G \cup H))$ is a consecutive set; indeed $g(V(G \cup H)) = \{0, 1, 2, \ldots, \mathrm{nsp}_r(G) + q\}$. If $uv \in E(G \cup H)$, then either $uv \in E(G)$ or $uv \in E(H)$. If $uv \in E(H)$, then $|g(u) - g(v)| \geq r + 2$. If $uv \in E(G)$, without loss of generality, assume $f(u) > f(v)$. Since $f(u) - f(v) \geq r + 1$, we have $\frac{(r+2)f(u)}{r+1} - \frac{(r+2)f(v)}{r+1} \geq r + 2$, so $g(u) - g(v) \geq r + 2$. Hence g is an N_{r+1}-coloring with span $\mathrm{nsp}_r(G) + q$. This completes the proof. $\square$

Note that the result in Theorem 2.5 is not always true if the assumption $\mathrm{nsp}_r(G) = q(r + 1) + j$ does not hold. For instance, let $G = K_2 \cup rK_1$ and $H = K_3$; then $\mathrm{nsp}_r(G) = r + 1$ for any r. However, $\mathrm{nsp}_{r+1}(G \cup H) = \infty$ for any $r \geq 4$.

COROLLARY 2.6. *If G is a graph with $r + 1 \leq \mathrm{nsp}_r(G) \leq 2r + 1$, then* $\mathrm{nsp}_{r+1}(G \cup \{x\}) = \mathrm{nsp}_r(G) + 1$.

Proof. The corollary follows from Theorems 2.4 and 2.5. $\square$

Consider the graph G in Figure 2.1. According to Theorem 2.3, $\mathrm{nsp}_1(G) \geq 3$ and so the labeling in the figure gives that $\mathrm{nsp}_1(G) = 3$. According to Corollary 2.6, we have $\mathrm{nsp}_2(G \cup \{x\}) = \mathrm{nsp}_1(G) + 1 = 4$.



G $\qquad\qquad$ $G \cup \{x\}$

$\mathrm{nsp}_1(G) = 3$ $\qquad\qquad$ $\mathrm{nsp}_2(G \cup \{x\}) = 4$

FIG. 2.1. *Optimal N-coloring for G and optimal N_2-coloring for $G \cup \{x\}$.*

3. Main results. In this section, we explore the minimum N_r-span for bipartite graphs. It turns out that the number of isolated vertices in a bipartite graph plays a key role for this problem. We give the values of $\mathrm{nsp}_r(G)$ for all bipartite graphs G with at least $r-2$ isolated vertices. This result leads to complete solutions of $\mathrm{nsp}_2(G)$ for all bipartite graphs G.

In this section, a bipartite graph is conventionally denoted by $G = (A, B, I, E)$, where I is the set of all isolated vertices and (A, B) is a *bipartition* of all nonisolated vertices such that each edge in G has one end in A and the other in B. A vertex v is called an *A-, B- or I-vertex* if $x \in A, B$, or I, respectively.

The *bipartite-complement* $\widehat{G}$ of a bipartite graph $G = (A, B, I, E)$ with $E \neq \emptyset$ is the bipartite graph $\widehat{G}$ with vertex set $V(\widehat{G}) = A \cup B$ and edge set

$$E(\widehat{G}) = \{ab : a \in A, b \in B, ab \notin E\}.$$

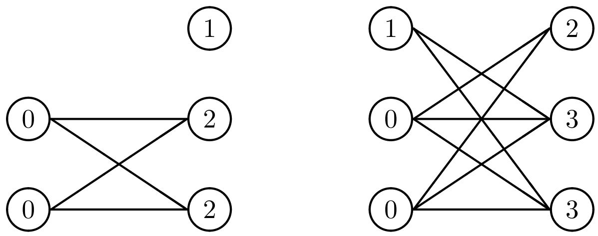
Note that the set of isolated vertices in $\widehat{G}$ is not specified in the notation. Moreover, we shall denote B' the set of all B-vertices not adjacent to any A-vertex in $\widehat{G}$. If G is a bipartite graph, then $\widehat{G}$ is a subgraph of G^c, the *complement* of G (i.e., $V(G^c) = V(G)$ and $E(G^c) = \{uv : u \neq v \text{ and } uv \notin E(G)\}$).

The N_1-coloring for bipartite graphs has been studied by Roberts [16]. Although the concept of the *minimum* N_1-span was not introduced explicitly in [16], the following theorem, which completely determines the values of $\mathrm{nsp}_1(G)$ for bipartite graphs, can be generated from [16].

THEOREM 3.1 (see Roberts [16]). *If $G = (A, B, I, E)$ is a bipartite graph with $E(G) \neq \emptyset$, then*

$$\mathrm{nsp}_1(G) = \begin{cases} 2 & \text{if } |I| \geq 1, \\ 3 & \text{if } |I| = 0 \text{ and } E(\widehat{G}) \neq \emptyset, \\ \infty & \text{if } |I| = 0 \text{ and } E(\widehat{G}) = \emptyset. \end{cases}$$

As examples to Theorem 3.1, consider the graphs G_1 and G_2 in Figure 3.1. As $|I| \geq 1$ for G_1, we have $\mathrm{nsp}_1(G_1) = 2$. For G_2, the facts $|I| = 0$ and $E(\widehat{G}) \neq \emptyset$ imply $\mathrm{nsp}_2(G_2) = 3$.



$G_1 : |I| \geq 1$ $\qquad$ $G_2 : |I| = 0$ and $E(\widehat{G}) \neq \emptyset$

$\mathrm{nsp}_1(G_1) = 2$ $\qquad$ $\mathrm{nsp}_1(G_2) = 3$

FIG. 3.1. *Two examples of optimal N-colorings for bipartite graphs.*

Sakai and Wang [17] characterize the existence of an N_r-coloring by using the Hamiltonian r-path. The *d-path* on n vertices, $v_1, v_2, \ldots, v_n$, has the edge set $\{v_iv_j : 1 \leq |i-j| \leq d\}$. Figure 3.2 shows a 2-path with seven vertices. A 1-path on n vertices is an ordinary path denoted as P_n. A *Hamiltonian d-path* of a graph G is a d-path covering each vertex of G exactly once.

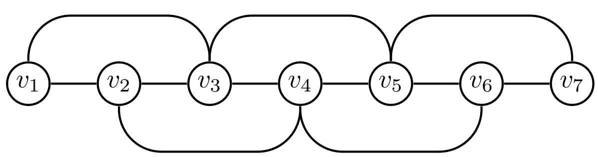


FIG. 3.2. *A 2-path with seven vertices.*

THEOREM 3.2 (see Sakai and Wang [17]). *G has an N_r-coloring if and only if G^c has a Hamiltonian r-path. Indeed, if f is an N_r-coloring such that $f(v_1) \leq f(v_2) \leq \ldots \leq f(v_n)$, then $v_1, v_2, \ldots, v_n$ is a Hamiltonian r-path in G^c.*

If the lower bound of $\mathrm{nsp}_r(G)$ in Theorem 2.3 is attained by some graph G, according to Proposition 2.2, G must be bipartite. Such graphs do exist. In the next theorem, we show a sufficient condition for such graphs.

THEOREM 3.3. *Suppose $G = (A, B, I, E)$ is a bipartite graph with at least one edge. If $|I| \geq r$, then $\mathrm{nsp}_r(G) = r+1$; if $|I| \leq r-1$ and there exist $\{a_1, a_2, \ldots, a_{r-|I|}\} \subseteq A$ and $\{b_1, b_2, \ldots, b_{r-|I|}\} \subseteq B$ such that $a_jb_k \notin E(G)$ for $j + k \geq r - |I| + 1$, then $\mathrm{nsp}_r(G) = 2r - |I| + 1$.*

Proof. It is obvious that $\mathrm{nsp}_r(G) \geq r + 1$, since $E(G) \neq \emptyset$.

If $|I| \geq r$, coloring A-vertices with 0, B-vertices with $r + 1$, and I-vertices with $1, 2, \ldots, r$ gives an N_r-coloring. Therefore, $\mathrm{nsp}_r(G) = r + 1$.

If $|I| \leq r - 1$, by Theorem 2.3, $\mathrm{nsp}_r(G) \geq 2r - |I| + 1$. Hence it suffices to find an N_r-coloring with span at most $2r-|I|+1$. Define a coloring by the following four steps:

(1) color $a_1, a_2, \ldots, a_{r-|I|}$ with $1, 2, \ldots, r-|I|$, respectively;

(2) color I-vertices with $r-|I|+1, r-|I|+2, \ldots, r$;

(3) color $b_{r-|I|}, b_{r-|I|-1}, \ldots, b_1$ with $r+1, r+2, \ldots, 2r-|I|$, respectively; and

(4) color all the remaining A-vertices with 0 and B-vertices with $2r-|I|+1$.

By the assumption that $a_j b_k \notin E(G)$ for $j+k \geq r-|I|+1$, it is easy to verify that the coloring defined above is an N_r-coloring with span at most $2r-|I|+1$. ◻

COROLLARY 3.4. *Let $G=(A,B,I,E)$ be a bipartite graph with at least one edge.*

(a) *If $|I| \leq r-1$ and $E(\widehat{G})=\emptyset$, then $\mathrm{nsp}_r(G)=\infty$.*

(b) *If $|I|=r-1$, then $\mathrm{nsp}_r(G)=r+2$ if and only if $E(\widehat{G}) \neq \emptyset$.*

(c) *If $|I|=r-2$ and there exists a P_4 in $\widehat{G}$, then $\mathrm{nsp}_r(G)=r+3$.*

Proof. We need only to show (a), since (b) and (c) follow from Theorem 3.3.

Suppose $|I| \leq r-1$ and $E(\widehat{G})=\emptyset$. Then, $G-I$ is a complete bipartite graph $K_{|A|,|B|}$. Combining this with the assumption that $|I| \leq r-1$, G does not admit any N_r-coloring, so $\mathrm{nsp}_r(G)=\infty$. ◻

Combining Theorem 3.3 and Corollary 3.4(b), the values of $\mathrm{nsp}_r(G)$ for bipartite graphs with at least $r-1$ isolated vertices are settled. In the rest of the article, we shall focus on the N_r-coloring for bipartite graphs $G=(A,B,I,E)$ with at most $r-2$ isolated vertices. By Corollary 3.4(a), we may assume $2 \leq |A| \leq |B|$. In the rest of the section, we search for the exact value of $\mathrm{nsp}_r(G)$ to complete the case as $|I|=r-2$. By Corollary 3.4(c), it suffices to consider the case that $\widehat{G}$ contains no P_4. We first show a lemma which is a key to settle this problem.

For any real number x, denote $\max\{x,0\}$ by x^+. For any two integers a and b, $a \leq b$, let $[a,b]$ denote the set $\{a, a+1, a+2, \ldots, b\}$.

LEMMA 3.5. *Let $G=(A,B,I,E)$ be a bipartite graph with $2 \leq m=|A| \leq |B|$, $|I| \leq r-2$, and $\widehat{G}$ contains no P_4. If $\mathrm{nsp}_r(G)<\infty$, then the following are all true:*

(a) *In the graph $\widehat{G}$, every B-vertex is adjacent to at most one A-vertex.*

(b) *There exist an arrangement $\Pi=(A_1, A_2, \ldots, A_m)$ of A and nonnegative integers $d_1=0, c_1, d_2, c_2, d_3, \ldots, d_m, c_m=0$ such that $\deg_{\widehat{G}}(A_k)=d_k+c_k$ for $1 \leq k \leq m$ and $|I| \geq q(\Pi) := \sum_{k=1}^{m-1} q_k$, where $q_k=\max\{(r-c_k)^+, (r-d_{k+1})^+\}$.*

(c) $\mathrm{nsp}_r(G) \geq (m-1)(2r+1)-|I|$.

(d) *If $B' \neq \emptyset$, then $|I|-q(\Pi) \geq q'(\Pi) := \min_{1 \leq k \leq m-1} q'_k$, where $q'_k=\min\{(r-c_k)^+, (r-d_{k+1})^+\}$.*

(e) *If $B' \neq \emptyset$, then $\mathrm{nsp}_r(G) \geq \max\{2r+2, (m-1)(2r+1)-|I|+s(\Pi)+1\}$, where $s(\Pi)=\min_{1 \leq k \leq m-1}\{q_k : q'_k \leq |I|-q(\Pi)\}$.*

Proof. Suppose f is an optimal N_r-coloring for G. According to Theorem 3.2, G^c has a Hamiltonian r-path $P=v_1, v_2, \ldots, v_{|V(G)|}$ with $0=f(v_1) \leq f(v_2) \leq \cdots \leq f(v_{|V(G)|})$. Without loss of generality, we assume the order of A-vertices on the r-path P is $\Pi=(A_1, A_2, \ldots, A_m)$. We call this an *arrangement* of A. Hence $f(A_1) \leq f(A_2) \leq \cdots \leq f(A_m)$.

On P, let an A- (or B-) *run* be a maximal interval of consecutive $A \cup I$- (or $B \cup I$-) vertices, starting and ending with A- (or B-) vertices. Note that there may exist some I-vertices within one run or between two consecutive runs; and the runs are alternating between A and B.

It is impossible to have two consecutive runs with at least two vertices in each. For if it is possible, then there exist $x, y \in A$ and $z, w \in B$ whose order in P is (x, y, z, w), and the vertices between x and w, other than y and z, are I-vertices.

Because $|I| \le r-2$, $(x-z-y-w)$ forms a P_4 in $\widehat{G}$, a contradiction.

Analogously it is impossible to have two consecutive singleton runs (except possibly the first run and the last run). For if it is possible, then we get a P_4 in $\widehat{G}$ by connecting the two consecutive singleton A-run and B-run with the B-vertex and A-vertex before and after them.

We conclude that either all A-runs or all B-runs are singletons. As $|A| \le |B|$, all A-runs are singletons and each B-run (except possibly the first run and/or the last run) contains at least two vertices. Therefore between any A_k and A_{k+1} on P, there are only B- or I-vertices. Since $|I| \le r-2$ and P is an Hamiltonian r-path in G^c, there exist at least two B-vertices between A_k and A_{k+1} that are adjacent to A_k.

To prove (a), suppose to the contrary that there exists $v \in B$ such that $vA_k, vA_\ell \in E(\widehat{G})$ for some $k < \ell$. Then between A_k and A_ℓ on P there exists $u \in B - \{v\}$ adjacent to A_k in $\widehat{G}$. Thus $(u - A_k - v - A_\ell)$ forms a P_4 in $\widehat{G}$, a contradiction. This proves (a).

Claim. For all $1 \le k \le m-1$, we have $f(A_{k+1}) - f(A_k) \ge r+2$.

Proof of claim. Suppose $f(A_{k+1}) - f(A_k) \le r+1$ for some k. Then the B-vertices between A_k and A_{k+1} on P are adjacent to both A_k and A_{k+1} in $\widehat{G}$, contradicting (a).

Note that if $A_1 = v_i$, then $P' = v_i, v_{i-1}, \ldots, v_2, v_1, v_{i+1}, v_{i+2}, \ldots, v_{|V(G)|}$ is also a Hamiltonian r-path in G^c, or, equivalently, f' defined by $f'(v_j) = f(v_{1+i-j})$ for $1 \le j \le i$ and $f'(v_j) = f(v_j)$ for $i < j \le |V(G)|$ is also an optimal N_r-coloring of G. Therefore, without loss of generality, we may assume $A_1 = v_1$. Similarly, we may assume that $A_m = v_{|V(G)|}$. Put

$D_1 := \{y \in B : yA_1 \in E(\widehat{G}) \text{ and } f(y) < f(A_1)\}$ and $d_1 := |D_1|$,
$C_1 := \{x \in B : xA_1 \in E(\widehat{G}) \text{ and } f(A_1) \le f(x)\}$ and $c_1 := |C_1|$,
$D_k := \{y \in B : yA_k \in E(\widehat{G}) \text{ and } f(y) \le f(A_k)\}$ and $d_k := |D_k|$ for $2 \le k \le m$,
$C_k := \{x \in B : xA_k \in E(\widehat{G}) \text{ and } f(A_k) < f(x)\}$ and $c_k := |C_k|$ for $2 \le k \le m$,
$I_k := \{z \in I : f(A_k) < f(z) < f(A_{k+1})\}$ and $i_k := |I_k|$ for $1 \le k \le m-1$,
$I'_k := \{z \in I : f(A_k) < f(z) \le f(A_k) + r\}$ and $i'_k := |I'_k|$ for $1 \le k \le m-1$,
$I''_k := \{z \in I : f(A_{k+1}) - r \le f(z) < f(A_{k+1})\}$ and $i''_k := |I''_k|$ for $1 \le k \le m-1$.

Then $d_1 = c_m = 0$ and $\deg_{\widehat{G}}(A_k) = d_k + c_k$ for $1 \le k \le m$. By (a), the C_i's and D_j's are all disjoint. By the claim, for any $1 \le k \le m$, $I'_k \cup I''_k \subseteq I_k$ (while I'_k and I''_k are not necessarily disjoint). Furthermore, it is clear that for any $1 \le k \le m-1$, $f^{-1}[f(A_k)+1, f(A_k)+r] \subseteq C_k \cup I'_k$, since if $f(A_k) < f(x) \le f(A_k)+r$, then $x \in C_k \cup I'_k$. Similarly, $f^{-1}[f(A_{k+1})-r, f(A_{k+1})-1] \subseteq D_{k+1} \cup I''_k$. Hence we have $c_k + i'_k \ge r$ and $d_{k+1} + i''_k \ge r$, implying that $i_k \ge \max\{i'_k, i''_k\} \ge \max\{(r-c_k)^+, (r-d_{k+1})^+\} = q_k$ for $1 \le k \le m-1$. Therefore,

$$(**) \qquad |I| \ge \sum_{k=1}^{m-1} i_k \ge \sum_{k=1}^{m-1} q_k = q(\Pi).$$

This completes the proof of (b).

Now we have $f^{-1}[f(A_k)+1, f(A_k)+r] \subseteq C_k \cup I'_k \subseteq C_k \cup I_k$ and $f^{-1}[f(A_{k+1}) - r, f(A_{k+1})-1] \subseteq D_{k+1} \cup I''_k \subseteq D_{k+1} \cup I_k$. Because $C_k \cap D_{k+1} = \emptyset$, at least $r - i_k$ colors of $[f(A_{k+1}) - r, f(A_{k+1}) - 1]$ are not in $[f(A_k)+1, f(A_k)+r]$. Thus $f(A_{k+1}) - f(A_k) \ge r + (r - i_k) + 1 = 2r + 1 - i_k$ for $1 \le k \le m-1$. Summing up, we get (c): $\mathrm{nsp}_r(G) \ge f(A_m) - f(A_1) \ge (m-1)(2r+1) - |I|$.

Now consider the case that $B' \ne \emptyset$; i.e., there exists some $w \in B$ such that $wA_k \notin E(\widehat{G})$ for all $1 \le k \le m$. Hence $|f(w) - f(A_k)| \ge r+1$ for all $1 \le k \le m$. Assume

$f(A_p) < f(w) < f(A_{p+1})$ for some $1 \le p \le m-1$. Then $f(A_{p+1}) - f(A_p) \ge 2r+2$, so $I'_p \cap I''_p = \emptyset$, implying that $i_p \ge i'_p + i''_p \ge (r-c_p)^+ + (r-d_{p+1})^+ = q_p + q'_p$. Replacing $i_p \ge q_p + q'_p$ to the last summation in $(**)$, we get $|I| \ge q(\Pi) + q'_p \ge q(\Pi) + q'(\Pi)$. This proves (d).

Because $f(A_{p+1}) - f(A_p) \ge 2r+2 \ge 2r+1-i_p+q_p+1$, we have, from the first inequality, $\mathrm{nsp}_r(G) \ge f(A_{p+1}) - f(A_p) \ge 2r+2$. Using the second inequality, similar to the proof of (c), one can get $\mathrm{nsp}_r(G) \ge (m-1)(2r+1) - |I| + q_p + 1 \ge (m-1)(2r+1) - |I| + s(\Pi) + 1$. This proves (e). ∎

In the next result, we complete the solution of $\mathrm{nsp}_r(G)$ for bipartite graphs $G = (A, B, I, E)$ with $|I| = r-2$. Let $s(G) = \min s(\Pi)$, where Π runs over all arrangements of A satisfying Lemma 3.5(b) and (d).

THEOREM 3.6. *Suppose $G = (A, B, I, E)$ is a bipartite graph with $2 \le m = |A| \le |B|$, $0 \le |I| = r-2$, and $\widehat{G}$ has no P_4. Then, $\mathrm{nsp}_r(G) < \infty$ if and only if $\widehat{G}$ satisfies Lemma* 3.5(a), (b), *and* (d). *In this case,*

$$\mathrm{nsp}_r(G) = \begin{cases} (2r+1)(m-1) - r + 2 & \text{if } B' = \emptyset, \\ 2r+2 & \text{if } B' \neq \emptyset \text{ and } m = 2, \\ (2r+1)(m-1) - r + s(G) + 3 & \text{if } B' \neq \emptyset \text{ and } m \ge 3. \end{cases}$$

Proof. The necessity follows from Lemma 3.5. For the sufficiency, suppose $\Pi = (A_1, A_2, \ldots, A_m)$ is an arrangement of A satisfying Lemma 3.5(a), (b), and (d). Moreover, assume $s(\Pi) = s(G)$ when $B' \neq \emptyset$. By Lemma 3.5(a), any two A-vertices have disjoint sets of neighbors in $\widehat{G}$. Then by Lemma 3.5(b), we can label the neighbors of A_k in $\widehat{G}$ by $C_{k,1}, C_{k,2}, \ldots, C_{k,c_k}$ and $D_{k,1}, D_{k,2}, \ldots, D_{k,d_{k+1}}$, respectively, for $1 \le k \le m$. In addition, since $|I| \ge \sum_{k=1}^{m-1} q_k$, there exist distinct I-vertices $I_{k,1}, I_{k,2}, \ldots, I_{k,q_k}$ for all k.

We shall complete the proof by considering the three cases.

Case 1. $B' = \emptyset$. That is, B is the union of all the C-and D-vertices. It suffices to find an N_r-coloring of G with span $(2r+1)(m-1) - r + 2$. (Then we not only prove that $\mathrm{N}_r(G) < \infty$ but also confirm that the span is optimal by Lemma 3.5(c).) We first replace q_{m-1} by $|I| - \sum_{j=1}^{m-2} q_j$. Then $q_{m-1} \ge \max\{(r-c_{m-1})^+, (r-d_m)^+\}$ and $|I| = \sum_{j=1}^{m-1} q_j$. Indeed, letting B represent the C- and D-vertices and I for I-vertices (without indicating the indices), we can line up all vertices of G as an Hamiltonian r-path in G^c as

$$P = A_1 \underbrace{BB\cdots B}_{c_1} \underbrace{II\cdots I}_{q_1} \underbrace{BB\cdots B}_{d_2} A_2 \cdots A_{m-1} \underbrace{BB\cdots B}_{c_{m-1}} \underbrace{II\cdots I}_{q_{m-1}} \underbrace{BB\cdots B}_{d_m} A_m.$$

Note that $d_1 = c_m = 0$. Define a coloring on G by the following three steps. (The idea is to use each I-vertex to reduce the span by 1.)

(1) A-vertices: $f(A_1) = 0$ and $f(A_{k+1}) = f(A_k) + 2r + 1 - q_k$ for $1 \le k \le m-1$.

(2) B-vertices: for all $1 \le k \le m-1$,

$$f(C_{k,j}) = \begin{cases} f(A_k) + j & \text{for } 1 \le j \le r - q_k - 1, \\ f(A_k) + r - q_k & \text{for } r - q_k \le j \le c_k, \end{cases}$$

$$f(D_{k+1,j}) = \begin{cases} f(A_k) + r + j & \text{for } 1 \le j \le r - q_k - 1, \\ f(A_k) + 2r - q_k & \text{for } r - q_k \le j \le d_{k+1}. \end{cases}$$

(3) I-vertices: $f(I_{k,j}) = f(A_k) + r - q_k + j$ for all $q_k > 0$ and $1 \le j \le q_k$.

One can easily verify that f is an N_r-coloring for G with span $(2r+1)(m-1)-|I| = (2r+1)(m-1)-r+2$.

Case 2. $B' \neq \emptyset$ and $m = 2$. Similar to Case 1, by Lemma 3.5(e), it suffices to find an N_r-coloring of G with span $\mathrm{nsp}_r(G) = 2r+2$. Define a coloring by $f(A_1) = 0$, $f(A_2) = 2r+2$, and $f(z) = r+1$ for all vertices z in B'. Since $|I| \geq q(\Pi) + q'(\Pi) = q_1 + q_1' = (r-c_1)^+ + (r-d_2)^+$, there are enough I-vertices to use the colors between 0 and $2r+2$. Thus one can verify that this is an N_r-coloring of G with span $2r+2$.

Case 3. $B' \neq \emptyset$ and $m \geq 3$. Again, by Lemma 3.5(e), it suffices to find an N_r-coloring with span $(2r+1)(m-1)-|I|+s(G)+1$. Suppose $s(\Pi) = q_p$ for some $1 \leq p \leq m-1$ with $q_p' \leq |I| - q(\Pi)$. As before, we replace q_i by $q_i + |I| - q(\Pi) - q_p'$ for some $i \neq p$. Then $|I| = q_1 + q_2 + \cdots + q_{p-1} + (r-c_p)^+ + (r-d_{p+1})^+ + q_{p+1} + \cdots + q_{m-1}$. All the C-, D-, and I-vertices are labeled the same as before, except the I-vertices between A_p and A_{p+1} are labeled as $I_{p,1}', I_{p,2}', \ldots, I_{p,(r-c_p)^+}', I_{p,1}'', I_{p,2}', \ldots, I_{p,(r-d_{p+1})^+}'$. Apply the same three-step coloring method used for the Case 1, except the colors for the vertices between A_p and A_{p+1} are defined by $f(I_{p,j}') = f(A_p) + r - (r-c_p)^+ + j$ for $1 \leq j \leq (r-c_p)^+$; $f(w) = f(A_p) + r + 1$ for all $w \in B'$; $f(I_{p,j}'') = f(A_p) + r + 1 + j$ for $1 \leq j \leq (r-d_{p+1})^+$; $f(A_{p+1}) = f(A_p) + 2r + 2$; and

$$f(C_{p,j}) = \begin{cases} f(A_p) + j & \text{for } 1 \leq j \leq r - (r-c_p)^+ - 1, \\ f(A_p) + r - (r-c_p)^+ & \text{for } r - (r-c_p)^+ \leq j \leq c_p, \end{cases}$$

$$f(D_{k,j}) = \begin{cases} f(A_p) + r + 1 + (r-d_{p+1})^+ + j & \text{for } 1 \leq j \leq r - (r-d_{p+1})^+ - 1, \\ f(A_p) + 2r + 1 & \text{for } r - (r-d_{p+1})^+ \leq j \leq d_{p+1}. \end{cases}$$

This gives an N_r-coloring for G with span $(2r+1)(m-1)-|I|+s(G)+1 = (2r+1)(m-1)-r+s(G)+3$. $\square$

Based on Lemma 3.5, using a similar process in the proof of Theorem 3.6, we can also completely settle the case that $I = \emptyset$ and $r \geq 2$. In this case, Lemma 3.5(b) means that $q_k = 0$ for all k, or, equivalently, that $\widehat{G}$ has two A-vertices of degree at least r and the rest $(m-2)$ A-vertices of degree at least $2r$. Furthermore, Lemma 3.5(d) holds automatically, and $s(\Pi) = 0$. This implies that the lower bound in Lemma 3.5(e) is simply $(m-1)(2r+1)+1$. Hence the same labeling procedure used in Theorem 3.6 gives the following result.

THEOREM 3.7. *Let $G = (A, B, I, E)$ be a bipartite graph with $2 \leq m = |A| \leq |B|$, $I = \emptyset$, and $\widehat{G}$ contains no P_4. If $r \geq 2$, then* $\mathrm{nsp}_r(G) < \infty$ *if and only if Lemma* 3.5(a) *holds and $\widehat{G}$ has two A-vertices of degree at least r and the other $(m-2)$ A-vertices of degree at least $2r$. In this case,*

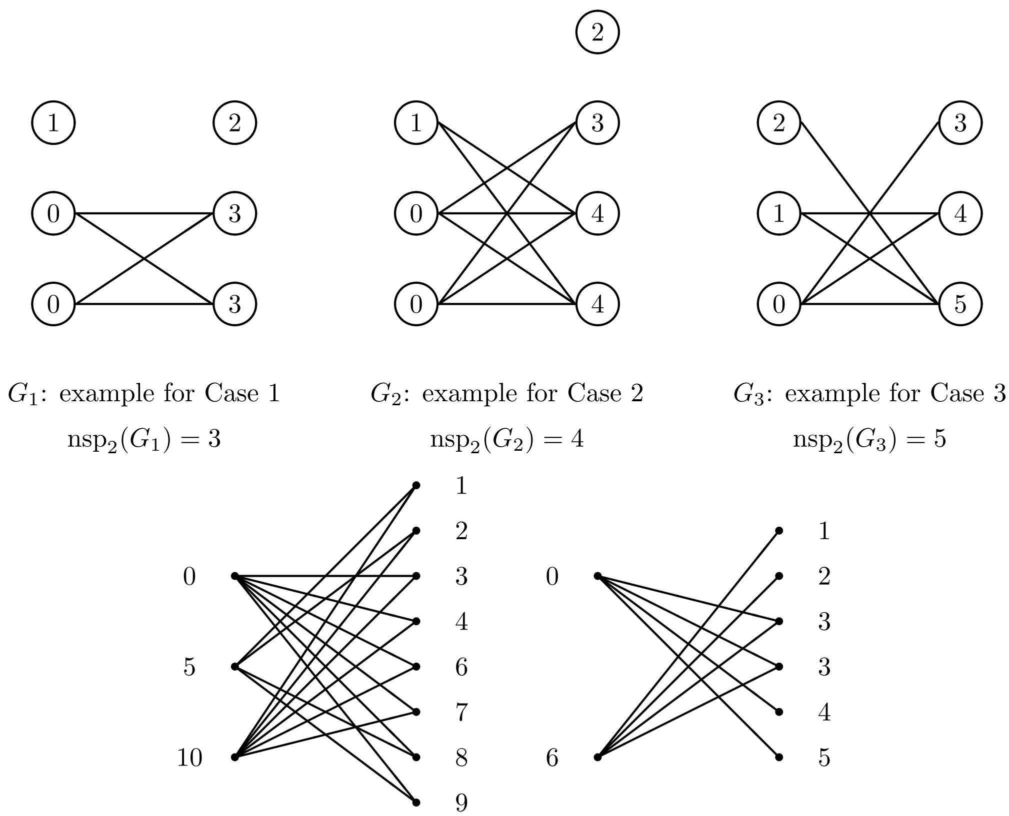
$$\mathrm{nsp}_r(G) = \begin{cases} (2r+1)(m-1) & \text{if } B' = \emptyset, \\ (2r+1)(m-1)+1 & \text{if } B' \neq \emptyset. \end{cases}$$

By Corollary 3.4 and Theorems 3.3 and 3.7, we obtain the complete solutions of $\mathrm{nsp}_2(G)$ for bipartite graphs.

THEOREM 3.8. *If $G = (A, B, I, E)$ is a bipartite graph with at least one edge and $1 \le m = |A| \le |B|$, then*

$$\mathrm{nsp}_2(G) = \begin{cases} 3 & \text{if } |I| \ge 2; \\ 4 & \text{if } |I| = 1 \text{ and } E(\widehat{G}) \ne \emptyset; \\ 5 & \text{if } |I| = 0 \text{ and } \widehat{G} \text{ has a } P_4; \\ 5m-5 & \text{if } |I| = 0,\ B' = \emptyset, \text{ and } \widehat{G} \text{ is a disjoint union of } m \\ & \text{stars, centered at } A \text{ except that two of the stars have} \\ & \text{at least 2 edges, each star has at least 4 edges;} \\ 5m-4 & \text{same as the above, except } B' \ne \emptyset; \\ \infty & \text{other than any of the above.} \end{cases}$$

Figure 3.3 shows examples of Theorem 3.8.



G_1: example for Case 1
$\mathrm{nsp}_2(G_1) = 3$

G_2: example for Case 2
$\mathrm{nsp}_2(G_2) = 4$

G_3: example for Case 3
$\mathrm{nsp}_2(G_3) = 5$

G_4: example for Case 4
$\mathrm{nsp}_2(G_4) = 10$

G_5: example for Case 5
$\mathrm{nsp}_2(G_5) = 6$

FIG. 3.3. *Five examples for Theorem* 3.8.

Remark. This article is aimed at computing the values of $\mathrm{nsp}_T(G)$ for bipartite graphs when $T = \{0, 1, \ldots, r\}$. Another article by Chang, Juan, and Liu [1] deals with the values of $\mathrm{nsp}_T(G)$ for unit-interval graphs when $T = \{0, 1\}$. The no-hole T-colorings for some other T-sets and different families of graphs were studied by Liu and Yeh [13]. It was proved [13] that if T is r-initial or $T = [a, b]$, $1 \le a \le b$, then for any large n, there exists some graph on n vertices such that $\mathrm{nsp}_T(G)$ equals the upper bound $n - 1$.

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